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SOME WEIGHTED SIMPSON-LIKE TYPE INEQUALITIES FOR DIFFERENTIABLE β-PREINVEX FUNCTIONS

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Abstract. In this paper, we first prove a new identity based on which we have established some weighted Simpson-type inequalities for functions whose first derivatives are *beta*-preinvex. Some applications of our finding are proposed.
Keywords: Simpson-like type inequalities, beta-preinvex functions, weighted function, P-functions.

1. Introduction

The most important and widely used class of functions is the class of convex functions. This concept plays a fundamental role in various branches of applied mathematics, and particularly in the theory of inequalities.

The fundamental result for convex functions is the so called Hermite-Hadamard's inequality, which can be stated as follows

$$\pounds\left(\tfrac{\tau+b}{2}\right) \le \tfrac{1}{b-\tau}\int_{\tau}^{b} \pounds(x)dx \le \tfrac{\pounds(\tau)+\pounds(b)}{2}, \tag{1.1}$$

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where $\pounds$ is a convex function on the finite interval $[\tau, b]$. If the function $\pounds$ is concave, then (1.1) holds in the reverse direction (see [28]).

Classical convexity has also been generalized in different ways and in several directions. The most important generalizations is the one introduced by Hanson [12] called invexity or preinvexity.

In [27], Noor et al. gave the following Hermite-Hadamard inequality for *beta*-preinvex functions

$$2^{p+q-1}\pounds\left(\frac{2\tau+\eta(b,\tau)}{2}\right) \leq \frac{1}{\eta(b,\tau)} \int\limits_{\tau}^{\tau+\eta(b,\tau)} \pounds(x)dx \leq \frac{\Gamma(p+1)\Gamma(q+1)}{\Gamma(p+q+2)}\left(\pounds(\tau)+\pounds(b)\right). \tag{1.2}$$

The following inequality is well known in the literature as Simpson's inequality

$$\left|\frac{1}{6}\left(\pounds(\tau)+4\pounds\left(\frac{\tau+b}{2}\right)+\pounds(b)\right)-\frac{1}{b-\tau}\int\limits_{\tau}^{b}\pounds(u)\,du\right| \leq \frac{1}{2880}\left\|\pounds^{(4)}\right\|_{\infty}(b-\tau)^{4},$$

where $\pounds$ is four times continuously differentiable function on (τ, b), and $\left\|\pounds^{(4)}\right\|_{\infty} = \sup\limits_{x\in(\tau,b)}\left|\pounds^{(4)}(x)\right|$.

It should be noticed that the above inequalities have attracted a lot of attention. Several papers dealing with their generalizations, improvements and variants have been appeared. For more details regarding integral inequalities, we refer readers to [1, 2, 4, 5, 6, 7, 11, 13, 15, 16, 19, 20, 21, 22, 23, 29, 30, 31, 35], and references therein.

Recently, Luo et al. [18] have established some weighted Simpson-like type inequalities for (α, m, h)-convex functions.

For (m, P)-functions

$$\begin{aligned}
&\left|\frac{1}{8(b-\tau)}\left(\pounds(\tau)+6\pounds\left(\frac{\tau+b}{2}\right)+\pounds(b)\right)\int\limits_{\tau}^{b}w(u)\,du-\frac{1}{b-\tau}\int\limits_{\tau}^{b}w(u)\,\pounds(u)\,du\right| \\
\leq\ & \frac{5(b-\tau)}{64}\|w\|_{[\tau,b],\infty}\left\{\left[\left|\pounds'(\tau)\right|+\left|\pounds'\left(\frac{\tau+b}{2}\right)\right|\right]+m\left[\left|\pounds'\left(\frac{\tau+b}{2m}\right)\right|+\left|\pounds'\left(\frac{b}{m}\right)\right|\right]\right\},
\end{aligned}$$

$$\begin{aligned}
&\left|\frac{1}{8(b-\tau)}\left(\pounds(\tau)+6\pounds\left(\frac{\tau+b}{2}\right)+\pounds(b)\right)\int\limits_{\tau}^{b}w(u)\,du-\frac{1}{b-\tau}\int\limits_{\tau}^{b}w(u)\,\pounds(u)\,du\right| \\
\leq\ & \frac{(b-\tau)}{4}\|w\|_{[\tau,b],\infty}\left(\frac{(q-1)\left(3^{(2q-1)/(q-1)}+1\right)}{(2q-1)2^{2(2q-1)/(q-1)}}\right)^{1-\frac{1}{q}} \\
\times\ & \left\{\left[\left|\pounds'(\tau)\right|^{q}+m\left|\pounds'\left(\frac{\tau+b}{2m}\right)\right|^{q}\right]^{\frac{1}{q}}+\left[\left|\pounds'\left(\frac{\tau+b}{2}\right)\right|^{q}+m\left|\pounds'\left(\frac{b}{m}\right)\right|^{q}\right]^{\frac{1}{q}}\right\},
\end{aligned}$$

and

$$\left| \frac{1}{8(b-\tau)} \left(\pounds(\tau) + 6\pounds\left(\frac{\tau+b}{2}\right) + \pounds(b) \right) \int_{\tau}^{b} w(u)\,du - \frac{1}{b-\tau} \int_{\tau}^{b} w(u)\,\pounds(u)\,du \right|$$

$$\leq \frac{b-\tau}{4} \|w\|_{[\tau,b],\infty} \left(\frac{1+3^{p+1}}{(p+1)4^{p+1}} \right)^{\frac{1}{p}}$$

$$\times \left\{ \left[\frac{|\pounds'(\tau)|^q + |\pounds'(\frac{\tau+b}{2})|^q}{2} \right]^{\frac{1}{q}} + \left[\frac{|\pounds'(\frac{\tau+b}{2})|^q + |\pounds'(b)|^q}{2} \right]^{\frac{1}{q}} \right\}.$$

The study of weighted inequalities is of interest in various fields of applied sciences, such as in functional inequalities where it provides the upper and lower bounds of certain problems seen as the product of two functions. For instance, in probability theory where it establishes the estimation of moments and expectation. It also represents a generalization of classical integral inequalities if the weight function is considered constant. The aim of this paper is to establish some weighted Newton-Cotes type inequalities involving three points under the generalized convexity of the first derivative. To do this, we first prove a new weighted integral identity. Based on this integral equality, we establish some weighted Simpson-like type inequalities for functions whose first derivatives are *beta*-preinvex. Some special cases are discussed. Applications to quadrature formulas and random variables are provided.

2. Preliminaries

Definition 2.1. [28] A function $f : I \to \mathbb{R}$ is said to be convex, if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

holds for all $x, y \in I$ and all $t \in [0, 1]$.

Definition 2.2. [8] A nonnegative function $f : I \to \mathbb{R}$ is said to be P-convex, if

$$f(tx + (1-t)y) \leq f(x) + f(y)$$

holds for all $x, y \in I$ and all $t \in [0, 1]$.

Definition 2.3. [10] A nonnegative function $f : I \to \mathbb{R}$ is said to be Godunova-Levin function, if

$$f(tx + (1-t)y) \leq \frac{f(x)}{t} + \frac{f(y)}{1-t}$$

holds for all $x, y \in I$ and all $t \in (0, 1)$.

Definition 2.4. [9] A nonnegative function $f : I \to \mathbb{R}$ is said to be s-Godunova-Levin function, if

$$f(tx + (1-t)y) \leq \frac{f(x)}{t^s} + \frac{f(y)}{(1-t)^s}$$

holds for all $x, y \in I, t \in (0, 1)$, and $s \in [0, 1]$.

Definition 2.5. [3] A nonnegative function $f : I \subset [0, \infty) \to \mathbb{R}$ is said to be s-convex in the second sense for some fixed $s \in (0, 1]$, if

$$f(tx + (1-t)y) \ \leq t^s f(x) + (1-t)^s f(y)$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

Definition 2.6. [32] Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a nonnegative function. We say that $f : I \to \mathbb{R}$ is tgs-convex function on I if the inequality

$$f\left(tx + (1-t)\,y\right) \leq t\,(1-t)\left[f\left(x\right) + f\left(y\right)\right]$$

holds for all $x, y \in I$, and $t \in (0, 1)$.

Definition 2.7. [33] A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is said to be *beta*-convex on I, if

$$f\left(tx + (1-t)\,y\right) \leq t^p\,(1-t)^q\,f\left(x\right) + t^q\,(1-t)^p\,f\left(y\right)$$

holds for all $x, y \in I$, and $t \in [0, 1]$, where $p, q > -1$.

Definition 2.8. [34] A set $K \subseteq \mathbb{R}^n$ is said an invex with respect to the bifunction $\eta : K \times K \to \mathbb{R}^n$, if for all $x, y \in K$, we have

$$x + t\eta\left(y, x\right) \in K.$$

Definition 2.9. [34] A function $f : K \to \mathbb{R}$ is said to be preinvex with respect to η, if

$$f\left(x + t\eta\left(y, x\right)\right) \leq (1-t)\,f\left(x\right) + tf(y)$$

holds for all $x, y \in K$ and all $t \in [0, 1]$.

Definition 2.10. [25] A nonnegative function $f : K \to \mathbb{R}$ is said to be P-preinvex function with respect to η, if

$$f\left(x + t\eta\left(y, x\right)\right) \leq f\left(x\right) + f(y)$$

holds for all $x, y \in K$ and all $t \in [0, 1]$.

Definition 2.11. [25] A nonnegative function $f : K \to \mathbb{R}$ is said to be Godunova-Levin preinvex function with respect to η, if

$$f\left(x + t\eta\left(y, x\right)\right) \leq \tfrac{f(x)}{1-t} + \tfrac{f(y)}{t}$$

holds for all $x, y \in K$ and all $t \in (0, 1)$.

Definition 2.12. [24] A nonnegative function $f : K \to \mathbb{R}$ is said to be s-Godunova-Levin preinvex function with respect to η, if

$$f\left(x + t\eta\left(y, x\right)\right) \leq \tfrac{1}{(1-t)^s} f\left(x\right) + \tfrac{1}{t^s} f(y)$$

holds for all $x, y \in K$ and all $t \in (0, 1)$, and $s \in [0, 1]$.

Definition 2.13. [17] A nonnegative function $f : K \subset [0, \infty) \to \mathbb{R}$ is said to be s-preinvex in the second sense with respect to η, if

$$f\left(x + t\eta\left(y, x\right)\right) \ \leq (1-t)^s f(x) + t^s f(y)$$

holds for all $x, y \in K$ and $t \in [0,1]$, and $s \in [0,1]$.

Definition 2.14. [26] A function $f : K \to \mathbb{R}$ is said to be tgs-preinvex with respect to η, if

$$f\left(x + t\eta\left(y, x\right)\right) \leq (1-t)\, t\left(f\left(x\right) + f(y)\right)$$

holds for all $x, y \in K$ and all $t \in [0,1]$.

Definition 2.15. [27] A function $f : K \to \mathbb{R}$ is said to be *beta*-preinvex with respect to η, if

$$f\left(x + t\eta\left(y, x\right)\right) \leq (1-t)^p\, t^q f\left(x\right) + (1-t)^q\, t^p f(y)$$

holds for all $x, y \in K, p, q > -1$ and all $t \in [0,1]$.

Remark 2.1. Definition 2.15 includes all the definitions quoted above with the exception of Definition 2.8 according to the values of p, q and $\eta\left(y, x\right)$.

Now, we recall some special functions

Definition 2.16. [14] For any complex numbers and nonpositive integers x, y such that $\operatorname{Re}(x) > 0$ and $\operatorname{Re}(y) > 0$. The beta function is defined by

$$B\left(x, y\right) = \int_0^1 t^{x-1}\left(1-t\right)^{y-1} dt = \tfrac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Here $\Gamma\left(.\right)$ is the Euler gamma function.

Definition 2.17. [14] For any complex numbers and nonpositive integers x, y such that $\operatorname{Re}(x) > 0$ and $\operatorname{Re}(y) > 0$. The incomplete beta function is given by

$$B_a\left(x, y\right) = \int_0^a t^{x-1}\left(1-t\right)^{y-1} dt, \quad a < 1.$$

Clearly from Definition 2.17, we have

$$B_a\left(x, y\right) + B_{1-a}\left(y, x\right) = B\left(x, y\right). \tag{2.1}$$

3. Main results

In order to prove our results, we need the following lemma

Lemma 3.1. *Let $\pounds : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, $\tau, b \in I^\circ$ with $\eta(b,\tau) > 0$, and let $w : [\tau, \tau + \eta(b,\tau)] \to \mathbb{R}$ be continuous and symmetric with respect to $\frac{2\tau+\eta(b,\tau)}{2}$. If $\pounds', w \in L^1[\tau, \tau+\eta(b,\tau)]$, then*

$$\frac{\pounds(\tau)+6\pounds\left(\frac{2\tau+\eta(b,\tau)}{2}\right)+\pounds(\tau+\eta(b,\tau))}{8}\int_{\tau}^{\tau+\eta(b,\tau)} w(u)\,du - \int_{\tau}^{\tau+\eta(b,\tau)} w(u)\,\pounds(u)\,du$$

$$= \frac{(\eta(b,\tau))^2}{4}\left(\int_0^1 p_1(t)\,\pounds'\left(\tau+\tfrac{t}{2}\eta(b,\tau)\right)dt + \int_0^1 p_2(t)\,\pounds'\left(\tau+\tfrac{1+t}{2}\eta(b,\tau)\right)dt\right),$$

where

$$p_1(t) = \frac{3}{4}\int_0^1 w\left(\frac{2\tau+(1-s)\eta(b,\tau)}{2}\right)ds - \int_0^{1-t} w\left(\frac{2\tau+(1-s)\eta(b,\tau)}{2}\right)ds$$

and

$$p_2(t) = \frac{1}{4}\int_0^1 w\left(\frac{2\tau+(2-s)\eta(b,\tau)}{2}\right)ds - \int_0^{1-t} w\left(\frac{2\tau+(2-s)\eta(b,\tau)}{2}\right)ds.$$

Proof. Integrating by parts, changing the variables and using the symmetry of w, we obtain

$$\int_0^1 p_1(t)\,\pounds'\left(\frac{2\tau+t\eta(b,\tau)}{2}\right)dt$$

$$= \int_0^1\left(\frac{3}{4}\int_0^1 w\left(\frac{2\tau+(1-s)\eta(b,\tau)}{2}\right)ds - \int_0^{1-t} w\left(\frac{2\tau+(1-s)\eta(b,\tau)}{2}\right)ds\right)\pounds'\left(\frac{2\tau+t\eta(b,\tau)}{2}\right)dt$$

$$= \frac{2}{\eta(b,\tau)}\left(\frac{3}{4}\int_0^1 w\left(\frac{2\tau+(1-s)\eta(b,\tau)}{2}\right)ds - \int_0^{1-t} w\left(\frac{2\tau+(1-s)\eta(b,\tau)}{2}\right)ds\right)\pounds\left(\frac{2\tau+t\eta(b,\tau)}{2}\right)\Bigg|_{t=0}^{t=1}$$

$$-\frac{2}{\eta(b,\tau)}\int_0^1 w\left(\frac{2\tau+t\eta(b,\tau)}{2}\right)\pounds\left(\frac{2\tau+t\eta(b,\tau)}{2}\right)dt$$

$$= \left(\frac{3}{2\eta(b,\tau)}\int_0^1 w\left(\frac{2\tau+(1-s)\eta(b,\tau)}{2}\right)ds\right)\pounds\left(\frac{2\tau+\eta(b,\tau)}{2}\right)$$

$$
\begin{aligned}
& +\left(\frac{1}{2\eta(b,\tau)}\int_0^1 w\left(\frac{2\tau+(1-s)\eta(b,\tau)}{2}\right)ds\right)\pounds(\tau)-\frac{2}{\eta(b,\tau)}\int_0^1 w\left(\frac{2\tau+t\eta(b,\tau)}{2}\right)\pounds\left(\frac{2\tau+t\eta(b,\tau)}{2}\right)dt \\
= \; & \frac{1}{2\eta(b,\tau)}\left(3\pounds\left(\frac{2\tau+\eta(b,\tau)}{2}\right)+\pounds(\tau)\right)\int_0^1 w\left(\frac{2\tau+(1-s)\eta(b,\tau)}{2}\right)ds \\
& -\frac{4}{(\eta(b,\tau))^2}\int_\tau^{\frac{2\tau+\eta(b,\tau)}{2}} w(u)\,\pounds(u)\,du \\
= \; & \frac{1}{(\eta(b,\tau))^2}\left(3\pounds\left(\frac{2\tau+\eta(b,\tau)}{2}\right)+\pounds(\tau)\right)\int_\tau^{\frac{2\tau+\eta(b,\tau)}{2}} w(u)\,du-\frac{4}{(\eta(b,\tau))^2}\int_\tau^{\frac{2\tau+\eta(b,\tau)}{2}} w(u)\,\pounds(u)\,du \\
= \; & \frac{1}{2(\eta(b,\tau))^2}\left(\pounds(\tau)+3\pounds\left(\frac{2\tau+\eta(b,\tau)}{2}\right)\right)\int_\tau^{\tau+\eta(b,\tau)} w(u)\,du-\frac{4}{(\eta(b,\tau))^2}\int_\tau^{\frac{2\tau+\eta(b,\tau)}{2}} w(u)\,\pounds(u)\,du.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
& \int_0^1 p_2(t)\,\pounds'\left(\frac{2\tau+(1+t)\eta(b,\tau)}{2}\right)dt \\
= \; & \frac{2}{\eta(b,\tau)}\left(\frac{1}{4}\int_0^1 w\left(\frac{2\tau+(2-s)\eta(b,\tau)}{2}\right)ds-\int_0^{1-t} w\left(\frac{2\tau+(2-s)\eta(b,\tau)}{2}\right)ds\right)\pounds\left(\frac{2\tau+(1+t)\eta(b,\tau)}{2}\right)\Bigg|_{t=0}^{t=1} \\
& -\frac{2}{\eta(b,\tau)}\int_0^1 w\left(\frac{2\tau+(2-(1-t))\eta(b,\tau)}{2}\right)\pounds\left(\frac{2\tau+(1+t)\eta(b,\tau)}{2}\right)dt \\
= \; & \frac{2}{\eta(b,\tau)}\left(\frac{1}{4}\int_0^1 w\left(\frac{2\tau+(2-s)\eta(b,\tau)}{2}\right)ds\right)\pounds(\tau+\eta(b,\tau)) \\
& +\frac{2}{\eta(b,\tau)}\left(\frac{3}{4}\int_0^1 w\left(\frac{2\tau+(2-s)\eta(b,\tau)}{2}\right)ds\right)\pounds\left(\frac{2\tau+\eta(b,\tau)}{2}\right) \\
& -\frac{2}{\eta(b,\tau)}\int_0^1 w\left(\frac{2\tau+(1+t)\eta(b,\tau)}{2}\right)\pounds\left(\frac{2\tau+(1+t)\eta(b,\tau)}{2}\right)dt \\
= \; & \frac{1}{2\eta(b,\tau)}\left(\pounds(\tau+\eta(b,\tau))+3\pounds\left(\frac{2\tau+\eta(b,\tau)}{2}\right)\right)\int_0^1 w\left(\frac{2\tau+(2-s)\eta(b,\tau)}{2}\right)ds
\end{aligned}
$$

$$-\frac{4}{(\eta(b,\tau))^2}\int_{\frac{2\tau+\eta(b,\tau)}{2}}^{\tau+\eta(b,\tau)} w(u)\,\pounds(u)\,du$$

$$= \frac{1}{(\eta(b,\tau))^2}\left(\pounds(\tau+\eta(b,\tau))+3\pounds\left(\frac{2\tau+\eta(b,\tau)}{2}\right)\right)\int_{\frac{2\tau+\eta(b,\tau)}{2}}^{\tau+\eta(b,\tau)} w(u)\,du$$

$$-\frac{4}{(\eta(b,\tau))^2}\int_{\frac{2\tau+\eta(b,\tau)}{2}}^{\tau+\eta(b,\tau)} w(u)\,\pounds(u)\,du$$

$$= \frac{1}{2(\eta(b,\tau))^2}\left(3\pounds\left(\frac{2\tau+\eta(b,\tau)}{2}\right)+\pounds(\tau+\eta(b,\tau))\right)\int_{\tau}^{\tau+\eta(b,\tau)} w(u)\,du$$

$$-\frac{4}{(\eta(b,\tau))^2}\int_{\frac{2\tau+\eta(b,\tau)}{2}}^{\tau+\eta(b,\tau)} w(u)\,\pounds(u)\,du.$$

Summing the two above inequalities, and then multiplying the resulting equality by $\frac{(\eta(b,\tau))^2}{4}$, we get the desired result. The proof is completed. □

Theorem 3.1. *Let $\pounds : [\tau,\tau+\eta(b,\tau)] \to \mathbb{R}$ be a differentiable function on $(\tau,\tau+\eta(b,\tau))$ with $\eta(b,\tau) > 0$ and $|\pounds'| \in L^1[\tau,\tau+\eta(b,\tau)]$. Let $w : [\tau,\tau+\eta(b,\tau)] \to [0,\infty)$ be continuous and symmetric function with respect to $\frac{2\tau+\eta(b,\tau)}{2}$. If $|\pounds'|$ is β-preinvex function for $p,q > -1$, we have*

$$\left|\frac{\pounds(\tau)+6\pounds\left(\frac{2\tau+\eta(b,\tau)}{2}\right)+\pounds(\tau+\eta(b,\tau))}{8}\int_{\tau}^{\tau+\eta(b,\tau)} w(u)\,du-\int_{\tau}^{\tau+\eta(b,\tau)} w(u)\,\pounds(u)\,du\right|$$

$$\leq \frac{(\eta(b,\tau))^2\|w\|_{[\tau,\tau+\eta(b,\tau)],\infty}}{2^{2+p+q}}\left(\Upsilon(p,q)+\Upsilon(q,p)+\Lambda(p,q)+\Lambda(q,p)\right)\left(|\pounds'(\tau)|+|\pounds'(b)|\right),$$

with

$$\Lambda(x,y) = 2^{x+y-1}B_{\frac{1}{8}}(y+1,x+1)-2^{x+y+2}B_{\frac{1}{8}}(y+2,x+1) \tag{3.1}$$

and

$$\begin{aligned}\Upsilon(x,y) &= 2^{x+y+2}\left(B_{\frac{1}{2}}(y+2,x+1)-B_{\frac{1}{8}}(y+2,x+1)\right)\\ &\quad -2^{x+y-1}\left(B_{\frac{1}{2}}(y+1,x+1)-B_{\frac{1}{8}}(y+1,x+1)\right),\end{aligned} \tag{3.2}$$

where $B_a(.,.)$ is the incomplete beta function.

Proof. From Lemma 3.1 and *beta*-convexity of $|\pounds'|$, we have

$$\left|\frac{\pounds(\tau)+6\pounds\left(\frac{2\tau+\eta(b,\tau)}{2}\right)+\pounds(\tau+\eta(b,\tau))}{8}\int_{\tau}^{\tau+\eta(b,\tau)}w(u)\,du-\int_{\tau}^{\tau+\eta(b,\tau)}w(u)\,\pounds(u)\,du\right|$$

$$\leq \frac{(\eta(b,\tau))^2}{4}\left(\int_0^1|p_1(t)|\left|\pounds'\left(\tau+\tfrac{1}{2}t\eta(b,\tau)\right)\right|dt\right.$$
$$\left.+\int_0^1|p_2(t)|\left|\pounds'\left(\tau+\tfrac{1+t}{2}\eta(b,\tau)\right)\right|dt\right)$$

$$\leq \frac{(\eta(b,\tau))^2}{4}\|w\|_{[\tau,\tau+\eta(b,\tau)],\infty}\left(\int_0^1\left|\frac{3}{4}\int_0^1 ds-\int_0^{1-t}ds\right|\left|\pounds'\left(\tau+\tfrac{1}{2}t\eta(b,\tau)\right)\right|dt\right.$$
$$\left.+\int_0^1\left|\frac{1}{4}\int_0^1 ds-\int_0^{1-t}ds\right|\left|\pounds'\left(\tau+\tfrac{1+t}{2}\eta(b,\tau)\right)\right|dt\right)$$

$$= \frac{(\eta(b,\tau))^2}{4}\|w\|_{[\tau,\tau+\eta(b,\tau)],\infty}\left(\int_0^1\left|t-\tfrac{1}{4}\right|\left|\pounds'\left(\tau+\tfrac{1}{2}t\eta(b,\tau)\right)\right|dt\right.$$
$$\left.+\int_0^1\left|t-\tfrac{3}{4}\right|\left|\pounds'\left(\tau+\tfrac{1+t}{2}\eta(b,\tau)\right)\right|dt\right)$$

$$(1-t)^p\,t^q f(x)+(1-t)^q\,t^p f(y)$$

$$\leq \frac{(\eta(b,\tau))^2}{4}\|w\|_{[\tau,\tau+\eta(b,\tau)],\infty}$$
$$\times\left(\int_0^1\left|t-\tfrac{1}{4}\right|\left(\left(1-\tfrac{1}{2}t\right)^p\left(\tfrac{1}{2}t\right)^q|\pounds'(\tau)|+\left(1-\tfrac{1}{2}t\right)^q\left(\tfrac{1}{2}t\right)^p|\pounds'(b)|\right)dt\right.$$
$$\left.+\int_0^1\left|t-\tfrac{3}{4}\right|\left(\left(1-\tfrac{1+t}{2}\right)^p\left(\tfrac{1+t}{2}\right)^q|\pounds'(\tau)|+\left(1-\tfrac{1+t}{2}\right)^q\left(\tfrac{1+t}{2}\right)^p|\pounds'(b)|\right)dt\right)$$

$$= \frac{(\eta(b,\tau))^2}{2^{2+p+q}}\|w\|_{[\tau,\tau+\eta(b,\tau)],\infty}$$
$$\times\left(\int_0^{\frac{1}{4}}\left(\tfrac{1}{4}-t\right)\left((2-t)^p\,t^q\,|\pounds'(\tau)|+(2-t)^q\,t^p\,|\pounds'(b)|\right)dt\right.$$
$$+\int_{\frac{1}{4}}^1\left(t-\tfrac{1}{4}\right)\left((2-t)^p\,t^q\,|\pounds'(\tau)|+(2-t)^q\,t^p\,|\pounds'(b)|\right)dt$$

$$
\begin{aligned}
&+\int_0^{\frac{3}{4}}\left(\tfrac{3}{4}-t\right)\left((1-t)^p(1+t)^q\left|\pounds'(\tau)\right|+(1-t)^q(1+t)^p\left|\pounds'(b)\right|\right)dt\\
&\left.+\int_{\frac{3}{4}}^{1}\left(t-\tfrac{3}{4}\right)\left((1-t)^p(1+t)^q\left|\pounds'(\tau)\right|+(1-t)^q(1+t)^p\left|\pounds'(b)\right|\right)dt\right)\\
= &\ \tfrac{(\eta(b,\tau))^2}{2^{2+p+q}}\|w\|_{[\tau,\tau+\eta(b,\tau)],\infty}\\
&\times\left(\left|\pounds'(\tau)\right|\left(\int_0^{\frac{1}{4}}\left(\tfrac{1}{4}-t\right)(2-t)^p t^q dt+\int_{\frac{1}{4}}^{1}\left(t-\tfrac{1}{4}\right)(2-t)^p t^q dt\right.\right.\\
&\left.+\int_0^{\frac{3}{4}}\left(\tfrac{3}{4}-t\right)(1-t)^p(1+t)^q dt+\int_{\frac{3}{4}}^{1}\left(t-\tfrac{3}{4}\right)(1-t)^p(1+t)^q dt\right)\\
&+\left|\pounds'(b)\right|\left(\int_0^{\frac{1}{4}}\left(\tfrac{1}{4}-t\right)(2-t)^q t^p dt+\int_{\frac{1}{4}}^{1}\left(t-\tfrac{1}{4}\right)(2-t)^q t^p dt\right.\\
&\left.\left.+\int_0^{\frac{3}{4}}\left(\tfrac{3}{4}-t\right)(1-t)^q(1+t)^p dt+\int_{\frac{3}{4}}^{1}\left(t-\tfrac{3}{4}\right)(1-t)^q(1+t)^p dt\right)\right)\\
= &\ \tfrac{(\eta(b,\tau))^2\|w\|_{[\tau,\tau+\eta(b,\tau)],\infty}}{2^{2+p+q}}\left(\Upsilon(p,q)+\Upsilon(q,p)+\Lambda(p,q)+\Lambda(q,p)\right)\left(\left|\pounds'(\tau)\right|+\left|\pounds'(b)\right|\right),
\end{aligned}
$$

where we have used

$$
\begin{aligned}
\int_0^{\frac{1}{4}}\left(\tfrac{1}{4}-t\right)(2-t)^p t^q dt &= \int_{\frac{3}{4}}^{1}\left(t-\tfrac{3}{4}\right)(1-t)^q(1+t)^p dt\\
&= 2^{p+q-1}\int_0^{\frac{1}{8}}x^q(1-x)^p dt-2^{p+q+2}\int_0^{\frac{1}{8}}x^{q+1}(1-x)^p dt\\
&= 2^{p+q-1}B_{\frac{1}{8}}(q+1,p+1)-2^{p+q+2}B_{\frac{1}{8}}(q+2,p+1)\\
&= \Lambda(p,q),\\
\int_{\frac{1}{4}}^{1}\left(t-\tfrac{1}{4}\right)(2-t)^p t^q dt &= \int_0^{\frac{3}{4}}\left(\tfrac{3}{4}-t\right)(1-t)^q(1+t)^p dt
\end{aligned}
$$

$$\begin{aligned}
&= 2^{p+q+2}\int_{\frac{1}{8}}^{\frac{1}{2}} x^{q+1}(1-x)^p\,dt - 2^{p+q-1}\int_{\frac{1}{8}}^{\frac{1}{2}} x^{q}(1-x)^p\,dt \\
&= 2^{p+q+2}\left(\int_0^{\frac{1}{2}} x^{q+1}(1-x)^p\,dt - \int_0^{\frac{1}{8}} x^{q+1}(1-x)^p\,dt\right) \\
&\quad -2^{p+q-1}\left(\int_0^{\frac{1}{2}} x^{q}(1-x)^p\,dt - \int_0^{\frac{1}{8}} x^{q}(1-x)^p\,dt\right) \\
&= 2^{p+q+2}\left(B_{\frac{1}{2}}(q+2,p+1) - B_{\frac{1}{8}}(q+2,p+1)\right) \\
&\quad -2^{p+q-1}\left(B_{\frac{1}{2}}(q+1,p+1) - B_{\frac{1}{8}}(q+1,p+1)\right) \\
&= \Upsilon(p,q),
\end{aligned}$$

$$\int_{\frac{1}{4}}^{1}\left(t-\tfrac{1}{4}\right)(2-t)^q t^p dt = \int_0^{\frac{3}{4}}\left(\tfrac{3}{4}-t\right)(1-t)^p(1+t)^q\,dt = \Upsilon(q,p),$$

$$\int_{\frac{3}{4}}^{1}\left(t-\tfrac{3}{4}\right)(1-t)^p(1+t)^q\,dt = \int_0^{\frac{1}{4}}\left(\tfrac{1}{4}-t\right)(2-t)^q t^p dt = \Lambda(q,p).$$

The proof is completed. □

Corollary 3.1. *Consider the assumptions of Theorem 3.1. If $|\pounds'|$ is preinvex function, then we have*

$$\left|\frac{\pounds(\tau)+6\pounds\left(\frac{2\tau+\eta(b,\tau)}{2}\right)+\pounds(\tau+\eta(b,\tau))}{8}\int_{\tau}^{\tau+\eta(b,\tau)} w(u)\,du - \int_{\tau}^{\tau+\eta(b,\tau)} w(u)\,\pounds(u)\,du\right|$$

$$(3.3)\quad \leq \frac{5(\eta(b,\tau))^2}{64}\|w\|_{[\tau,\tau+\eta(b,\tau)],\infty}\left(|\pounds'(\tau)|+|\pounds'(b)|\right).$$

Moreover, if we take $\eta(b,\tau)=b-\tau$ and $w(u)=\frac{1}{b-\tau}$, we get

$$\left|\frac{1}{8}\left(\pounds(\tau)+6\pounds\left(\frac{\tau+b}{2}\right)+\pounds(b)\right)-\frac{1}{b-\tau}\int_{\tau}^{b}\pounds(u)\,du\right| \leq \frac{5(b-\tau)}{64}\left(|\pounds'(\tau)|+|\pounds'(b)|\right).$$

Corollary 3.2. *Let us consider the assumptions of Theorem 3.1. If $|\pounds'|$ is tgs-preinvex function, then we have*

$$\left|\frac{\pounds(\tau)+6\pounds\left(\frac{2\tau+\eta(b,\tau)}{2}\right)+\pounds(\tau+\eta(b,\tau))}{8}\int_{\tau}^{\tau+\eta(b,\tau)} w(u)\,du - \int_{\tau}^{\tau+\eta(b,\tau)} w(u)\,\pounds(u)\,du\right|$$

$$\leq \frac{133(\eta(b,\tau))^2}{2048}\|w\|_{[\tau,\tau+\eta(b,\tau)],\infty}\left(|£'(\tau)|+|£'(b)|\right).$$

If we take $\eta(b,\tau)=b-\tau$ *and* $w(u)=\frac{1}{\eta(b,\tau)}$, *we get*

$$\left|\frac{1}{8}\left(£(\tau)+6£\left(\frac{\tau+b}{2}\right)+£(b)\right)-\frac{1}{b-\tau}\int_{\tau}^{b}£(u)\,du\right|\leq\frac{133(b-\tau)}{2048}\left(|£'(\tau)|+|£'(b)|\right).$$

Theorem 3.2. *Let* $£:[\tau,\tau+\eta(b,\tau)]\to\mathbb{R}$ *be a differentiable function on* $(\tau,\tau+\eta(b,\tau))$ *with* $\eta(b,\tau)>0$ *and* $|£'|\in L^1[\tau,\tau+\eta(b,\tau)]$. *Let* $w:[\tau,\tau+\eta(b,\tau)]\to[0,\infty)$ *be continuous and symmetric function with respect to* $\frac{2\tau+\eta(b,\tau)}{2}$. *If* $|£'|^{\lambda}$ *is* β*-preinvex function for* $p,q>-1$ *and* $k>1$ *with* $\frac{1}{k}+\frac{1}{\lambda}=1$, *then we have*

$$\begin{aligned}&\left|\frac{£(\tau)+6£\left(\frac{2\tau+\eta(b,\tau)}{2}\right)+£(\tau+\eta(b,\tau))}{8}\int_{\tau}^{\tau+\eta(b,\tau)}w(u)\,du-\int_{\tau}^{\tau+\eta(b,\tau)}w(u)£(u)\,du\right|\\
\leq&\ \frac{(\eta(b,\tau))^2}{8}\|w\|_{[\tau,\tau+\eta(b,\tau)],\infty}\left(\frac{3^{k+1}+1}{8(k+1)}\right)^{\frac{1}{k}}\\
&\times\left(\left(B_{\frac{1}{2}}(q+1,p+1)|£'(\tau)|^{\lambda}+B_{\frac{1}{2}}(p+1,q+1)|£'(b)|^{\lambda}\right)^{\frac{1}{\lambda}}\right.\\
&\left.+\left(B_{\frac{1}{2}}(p+1,q+1)|£'(\tau)|^{\lambda}+B_{\frac{1}{2}}(q+1,p+1)|£'(b)|^{\lambda}\right)^{\frac{1}{\lambda}}\right),\end{aligned}$$

where $B_{\frac{1}{2}}(.,.)$ *is the incomplete beta function.*

Proof. From Lemma 3.1 and properties of modulus, Hölder's inequality and β-preinvexity of $|£'|^{\lambda}$, we have

$$\begin{aligned}&\left|\frac{£(\tau)+6£\left(\frac{2\tau+\eta(b,\tau)}{2}\right)+£(\tau+\eta(b,\tau))}{8}\int_{\tau}^{\tau+\eta(b,\tau)}w(u)\,du-\int_{\tau}^{\tau+\eta(b,\tau)}w(u)£(u)\,du\right|\\
\leq&\ \frac{(\eta(b,\tau))^2}{4}\left(\int_0^1|p_1(t)|\left|£'\left(\tau+\tfrac{t}{2}\eta(b,\tau)\right)\right|dt+\int_0^1|p_2(t)|\left|£'\left(\tau+\tfrac{1+t}{2}\eta(b,\tau)\right)\right|dt\right)\\
\leq&\ \frac{(\eta(b,\tau))^2}{4}\left(\left(\int_0^1|p_1(t)|^k dt\right)^{\frac{1}{k}}\left(\int_0^1\left|£'\left(\tau+\tfrac{1}{2}t\eta(b,\tau)\right)\right|^{\lambda}dt\right)^{\frac{1}{\lambda}}\right.\\
&\left.+\left(\int_0^1|p_2(t)|^k dt\right)^{\frac{1}{k}}\left(\int_0^1\left|£'\left(\tau+\tfrac{1+t}{2}\eta(b,\tau)\right)\right|^{\lambda}dt\right)^{\frac{1}{\lambda}}\right)\end{aligned}$$

$$\begin{aligned}
&\leq \frac{(\eta(b,\tau))^2}{4}\|w\|_{[\tau,\tau+\eta(b,\tau)],\infty}\left(\left(\int_0^1\left|\frac{3}{4}\int_0^1 ds-\int_0^{1-t}ds\right|^k dt\right)^{\frac{1}{k}}\right.\\
&\times\left(\int_0^1\left(\left(1-\tfrac{1}{2}t\right)^p\left(\tfrac{1}{2}t\right)^q\left|\pounds'(\tau)\right|^\lambda+\left(1-\tfrac{1}{2}t\right)^q\left(\tfrac{1}{2}t\right)^p\left|\pounds'(b)\right|^\lambda\right)dt\right)^{\frac{1}{\lambda}}\\
&+\left(\int_0^1\left|\frac{1}{4}\int_0^1 ds-\int_0^{1-t}ds\right|^k dt\right)^{\frac{1}{k}}\\
&\times\left.\left(\int_0^1\left(\left(1-\tfrac{1+t}{2}\right)^p\left(\tfrac{1+t}{2}\right)^q\left|\pounds'(\tau)\right|^\lambda+\left(1-\tfrac{1+t}{2}\right)^q\left(\tfrac{1+t}{2}\right)^p\left|\pounds'(b)\right|^\lambda\right)dt\right)^{\frac{1}{\lambda}}\right)\\
&\leq \frac{(\eta(b,\tau))^2}{4}\|w\|_{[\tau,\tau+\eta(b,\tau)],\infty}\left(\left(\int_0^1\left|t-\tfrac{1}{4}\right|^k dt\right)^{\frac{1}{k}}\right.\\
&\times\left(2\int_0^{\frac{1}{2}}\left(u^q(1-u)^p\left|\pounds'(\tau)\right|^\lambda+u^p(1-u)^q\left|\pounds'(b)\right|^\lambda\right)du\right)^{\frac{1}{\lambda}}\\
&+\left(\int_0^1\left|t-\tfrac{3}{4}\right|^k dt\right)^{\frac{1}{k}}\\
&\times\left.\left(2\int_{\frac{1}{2}}^1\left(u^q(1-u)^p\left|\pounds'(\tau)\right|^\lambda+u^p(1-u)^q\left|\pounds'(b)\right|^\lambda\right)du\right)^{\frac{1}{\lambda}}\right)\\
&= \frac{2^{\frac{1}{\lambda}}(\eta(b,\tau))^2}{4}\|w\|_{[\tau,\tau+\eta(b,\tau)],\infty}\left(\frac{1}{k+1}\right)^{\frac{1}{k}}\left(\left(\tfrac{1}{4}\right)^{k+1}+\left(\tfrac{3}{4}\right)^{k+1}\right)^{\frac{1}{k}}\\
&\times\left(\left(\left|\pounds'(\tau)\right|^\lambda\int_0^{\frac{1}{2}}u^q(1-u)^p du+\left|\pounds'(b)\right|^\lambda\int_0^{\frac{1}{2}}u^p(1-u)^q du\right)^{\frac{1}{\lambda}}\right.\\
&+\left.\left(\left|\pounds'(\tau)\right|^\lambda\int_0^{\frac{1}{2}}u^p(1-u)^q du+\left|\pounds'(b)\right|^\lambda\int_0^{\frac{1}{2}}u^q(1-u)^p du\right)^{\frac{1}{\lambda}}\right)\\
&= \frac{(\eta(b,\tau))^2}{8}\|w\|_{[\tau,\tau+\eta(b,\tau)],\infty}\left(\frac{3^{k+1}+1}{8(k+1)}\right)^{\frac{1}{k}}
\end{aligned}$$

$$\times\left(\left(B_{\frac{1}{2}}\left(q+1,p+1\right)\left|\pounds'\left(\tau\right)\right|^{\lambda}+B_{\frac{1}{2}}\left(p+1,q+1\right)\left|\pounds'\left(b\right)\right|^{\lambda}\right)^{\frac{1}{\lambda}}\right.$$
$$\left.+\left(B_{\frac{1}{2}}\left(p+1,q+1\right)\left|\pounds'\left(\tau\right)\right|^{\lambda}+B_{\frac{1}{2}}\left(q+1,p+1\right)\left|\pounds'\left(b\right)\right|^{\lambda}\right)^{\frac{1}{\lambda}}\right).$$

The proof is completed. □

Corollary 3.3. *Consider the assumptions of Theorem 3.2. If $|\pounds'|^{\lambda}$ is preinvex function, then we have*

$$\left|\frac{\pounds(\tau)+6\pounds\left(\frac{2\tau+\eta(b,\tau)}{2}\right)+\pounds(\tau+\eta(b,\tau))}{8}\int\limits_{\tau}^{\tau+\eta(b,\tau)} w\left(u\right)du-\int\limits_{\tau}^{\tau+\eta(b,\tau)} w\left(u\right)\pounds\left(u\right)du\right|$$
$$\leq \frac{(\eta(b,\tau))^2}{16}\left\|w\right\|_{[\tau,\tau+\eta(b,\tau)],\infty}\left(\frac{3^{k+1}+1}{4(k+1)}\right)^{\frac{1}{k}}$$
$$\times\left(\left(\frac{3\left|\pounds'(\tau)\right|^{\lambda}+\left|\pounds'(b)\right|^{\lambda}}{4}\right)^{\frac{1}{\lambda}}+\left(\frac{\left|\pounds'(\tau)\right|^{\lambda}+3\left|\pounds'(b)\right|^{\lambda}}{4}\right)^{\frac{1}{\lambda}}\right).$$

If we take $\eta\left(b,\tau\right)=b-\tau$ and $w\left(u\right)=\frac{1}{b-\tau}$, we get

$$\left|\frac{1}{8}\left(\pounds\left(\tau\right)+6\pounds\left(\frac{\tau+b}{2}\right)+\pounds\left(b\right)\right)-\frac{1}{b-\tau}\int\limits_{\tau}^{b}\pounds\left(u\right)du\right|$$
$$\leq \frac{b-\tau}{16}\left(\frac{3^{k+1}+1}{4(k+1)}\right)^{\frac{1}{k}}\left(\left(\frac{3\left|\pounds'(\tau)\right|^{\lambda}+\left|\pounds'(b)\right|^{\lambda}}{4}\right)^{\frac{1}{\lambda}}+\left(\frac{\left|\pounds'(\tau)\right|^{\lambda}+3\left|\pounds'(b)\right|^{\lambda}}{4}\right)^{\frac{1}{\lambda}}\right).$$

Corollary 3.4. *Let us consider the assumptions of Theorem 3.2. If $|\pounds'|^{\lambda}$ is s-preinvex function, then we have*

$$\left|\frac{\pounds(\tau)+6\pounds\left(\frac{2\tau+\eta(b,\tau)}{2}\right)+\pounds(\tau+\eta(b,\tau))}{8}\int\limits_{\tau}^{\tau+\eta(b,\tau)} w\left(u\right)du-\int\limits_{\tau}^{\tau+\eta(b,\tau)} w\left(u\right)\pounds\left(u\right)du\right|$$
$$\leq \frac{(\eta(b,\tau))^2}{8}\left\|w\right\|_{[\tau,\tau+\eta(b,\tau)],\infty}\left(\frac{3^{k+1}+1}{8(k+1)}\right)^{\frac{1}{k}}\left(\frac{1}{1+s}\right)^{\frac{1}{\lambda}}$$
$$\times\left(\left(\frac{\left(2^{s+1}-1\right)\left|\pounds'(\tau)\right|^{\lambda}+\left|\pounds'(b)\right|^{\lambda}}{2^{s+1}}\right)^{\frac{1}{\lambda}}+\left(\frac{\left|\pounds'(\tau)\right|^{\lambda}+\left(2^{s+1}-1\right)\left|\pounds'(b)\right|^{\lambda}}{2^{s+1}}\right)^{\frac{1}{\lambda}}\right).$$

If we take $\eta\left(b,\tau\right)=b-\tau$ and $w\left(u\right)=\frac{1}{b-\tau}$, we get

$$\left|\frac{1}{8}\left(\pounds\left(\tau\right)+6\pounds\left(\frac{\tau+b}{2}\right)+\pounds\left(b\right)\right)-\frac{1}{b-\tau}\int\limits_{\tau}^{b}\pounds\left(u\right)du\right|$$

$$\leq \frac{b-\tau}{8}\left(\frac{3^{k+1}+1}{8(k+1)}\right)^{\frac{1}{k}}\left(\frac{1}{1+s}\right)^{\frac{1}{\lambda}}$$
$$\times\left(\left(\frac{\left(2^{s+1}-1\right)\left|£'(\tau)\right|^{\lambda}+\left|£'(b)\right|^{\lambda}}{2^{s+1}}\right)^{\frac{1}{\lambda}}+\left(\frac{\left|£'(\tau)\right|^{\lambda}+\left(2^{s+1}-1\right)\left|£'(b)\right|^{\lambda}}{2^{s+1}}\right)^{\frac{1}{\lambda}}\right).$$

Theorem 3.3. *Let $£ : [\tau, \tau+\eta(b,\tau)] \to \mathbb{R}$ be a differentiable function on $(\tau, \tau+\eta(b,\tau))$ with $\eta(b,\tau) > 0$ and $|£'| \in L^1[\tau, \tau+\eta(b,\tau)]$. Let $w : [\tau, \tau+\eta(b,\tau)] \to [0,\infty)$ be continuous and symmetric function with respect to $\frac{2\tau+\eta(b,\tau)}{2}$. If $|£'|^{\lambda}$ is β-preinvex function for $p, q > -1$ and $\lambda \geq 1$, then we have*

$$\left|\frac{£(\tau)+6£\left(\frac{2\tau+\eta(b,\tau)}{2}\right)+£(\tau+\eta(b,\tau))}{8}\int_{\tau}^{\tau+\eta(b,\tau)} w(u)\,du-\int_{\tau}^{\tau+\eta(b,\tau)} w(u)\,£(u)\,du\right|$$
$$\leq \frac{(\eta(b,\tau))^2}{4}\|w\|_{[\tau,\tau+\eta(b,\tau)],\infty}\left(\frac{5}{16}\right)^{1-\frac{1}{\lambda}}\left(\frac{1}{2^{p+q}}\right)^{\frac{1}{\lambda}}$$
$$\times\left(\left((\Lambda(p,q)+\Upsilon(p,q))\left|£'(\tau)\right|^{\lambda}+(\Lambda(q,p)+\Upsilon(q,p))\left|£'(b)\right|^{\lambda}\right)^{\frac{1}{\lambda}}\right.$$
$$\left.+\left((\Lambda(q,p)+\Upsilon(q,p))\left|£'(\tau)\right|^{\lambda}+(\Lambda(p,q)+\Upsilon(p,q))\left|£'(b)\right|^{\lambda}\right)^{\frac{1}{\lambda}}\right),$$

where $\Lambda(.,.)$ and $\Upsilon(.,.)$ are defined as in (3.1) *and* (3.2) *respectively.*

Proof. From Lemma 3.1 and properties of modulus, power mean inequality and β-preinvexity of $|£'|^{k}$, we have

$$\left|\frac{£(\tau)+6£\left(\frac{2\tau+\eta(b,\tau)}{2}\right)+£(\tau+\eta(b,\tau))}{8}\int_{\tau}^{\tau+\eta(b,\tau)} w(u)\,du-\int_{\tau}^{\tau+\eta(b,\tau)} w(u)\,£(u)\,du\right|$$
$$\leq \frac{(\eta(b,\tau))^2}{4}\left(\int_0^1|p_1(t)|\left|£'\left(\tau+\tfrac{t}{2}\eta(b,\tau)\right)\right|dt+\int_0^1|p_2(t)|\left|£'\left(\tau+\tfrac{1+t}{2}\eta(b,\tau)\right)\right|dt\right)$$
$$\leq \frac{(\eta(b,\tau))^2}{4}\left(\left(\int_0^1|p_1(t)|\,dt\right)^{1-\frac{1}{\lambda}}\left(\int_0^1|p_1(t)|\left|£'\left(\tau+\tfrac{1}{2}t\eta(b,\tau)\right)\right|^{\lambda}dt\right)^{\frac{1}{\lambda}}\right.$$
$$\left.+\left(\int_0^1|p_2(t)|\,dt\right)^{1-\frac{1}{\lambda}}\left(\int_0^1|p_2(t)|\left|£'\left(\tau+\tfrac{1+t}{2}\eta(b,\tau)\right)\right|^{\lambda}dt\right)^{\frac{1}{\lambda}}\right)$$
$$\leq \frac{(\eta(b,\tau))^2}{4}\|w\|_{[\tau,\tau+\eta(b,\tau)],\infty}\left(\left(\int_0^1\left|t-\tfrac{1}{4}\right|dt\right)^{1-\frac{1}{\lambda}}\right.$$

$$
\begin{aligned}
& \times\left(\int_0^1\left|t-\tfrac{1}{4}\right|\left(\left(1-\tfrac{1}{2}t\right)^p\left(\tfrac{1}{2}t\right)^q\left|\pounds'(\tau)\right|^\lambda+\left(1-\tfrac{1}{2}t\right)^q\left(\tfrac{1}{2}t\right)^p\left|\pounds'(b)\right|^\lambda\right)dt\right)^{\frac{1}{\lambda}} \\
& +\left(\int_0^1\left|t-\tfrac{3}{4}\right|dt\right)^{1-\frac{1}{\lambda}} \\
& \times\left(\int_0^1\left|t-\tfrac{3}{4}\right|\left(\left(1-\tfrac{1+t}{2}\right)^p\left(\tfrac{1+t}{2}\right)^q\left|\pounds'(\tau)\right|^\lambda+\left(1-\tfrac{1+t}{2}\right)^q\left(\tfrac{1+t}{2}\right)^p\left|\pounds'(b)\right|^\lambda\right)dt\right)^{\frac{1}{\lambda}} \\
\leq\ & \tfrac{(\eta(b,\tau))^2}{4}\|w\|_{[\tau,\tau+\eta(b,\tau)],\infty}\left(\int_0^1\left|t-\tfrac{1}{4}\right|dt\right)^{1-\frac{1}{\lambda}}\left(\tfrac{1}{2^{p+q}}\right)^{\frac{1}{\lambda}} \\
& \times\left(\left(\left|\pounds'(\tau)\right|^\lambda\left(\int_0^{\frac{1}{4}}\left(\tfrac{1}{4}-t\right)(2-t)^p t^q dt+\int_{\frac{1}{4}}^1\left(t-\tfrac{1}{4}\right)(2-t)^p t^q dt\right)\right.\right. \\
& \left.+\left|\pounds'(b)\right|^\lambda\left(\int_0^{\frac{1}{4}}\left(\tfrac{1}{4}-t\right)(2-t)^q t^p dt+\int_{\frac{1}{4}}^1\left(t-\tfrac{1}{4}\right)(2-t)^q t^p dt\right)\right)^{\frac{1}{\lambda}} \\
& +\left(\left|\pounds'(\tau)\right|^\lambda\left(\int_0^{\frac{3}{4}}\left(\tfrac{3}{4}-t\right)(1-t)^p(1+t)^q dt+\int_{\frac{3}{4}}^1\left(t-\tfrac{3}{4}\right)(1-t)^p(1+t)^q dt\right)\right. \\
& \left.\left.+\left|\pounds'(b)\right|^\lambda\left(\int_0^{\frac{3}{4}}\left(\tfrac{3}{4}-t\right)(1-t)^q(1+t)^p dt+\int_{\frac{3}{4}}^1\left(t-\tfrac{3}{4}\right)(1-t)^q(1+t)^p dt\right)\right)^{\frac{1}{\lambda}}\right) \\
=\ & \tfrac{(\eta(b,\tau))^2}{4}\|w\|_{[\tau,\tau+\eta(b,\tau)],\infty}\left(\tfrac{5}{16}\right)^{1-\frac{1}{\lambda}}\left(\tfrac{1}{2^{p+q}}\right)^{\frac{1}{\lambda}} \\
& \times\left(\left(\left(\Lambda(p,q)+\Upsilon(p,q)\right)\left|\pounds'(\tau)\right|^\lambda+\left(\Lambda(q,p)+\Upsilon(q,p)\right)\left|\pounds'(b)\right|^\lambda\right)^{\frac{1}{\lambda}}\right. \\
& \left.+\left(\left(\Lambda(q,p)+\Upsilon(q,p)\right)\left|\pounds'(\tau)\right|^\lambda+\left(\Lambda(p,q)+\Upsilon(p,q)\right)\left|\pounds'(b)\right|^\lambda\right)^{\frac{1}{\lambda}}\right),
\end{aligned}
$$

where $\Lambda(.,.)$ and $\Upsilon(.,.)$ are defined as in (3.1) and (3.2) respectively. The proof is completed. □

Corollary 3.5. *Let us consider the assumptions of Theorem 3.2. If* $|\pounds'|^\lambda$ *is prein-*

vex function, then we have

$$\begin{aligned}&\left|\frac{\pounds(\tau)+6\pounds\left(\frac{2\tau+\eta(b,\tau)}{2}\right)+\pounds(\tau+\eta(b,\tau))}{8}\int\limits_{\tau}^{\tau+\eta(b,\tau)}w(u)\,du-\int\limits_{\tau}^{\tau+\eta(b,\tau)}w(u)\,\pounds(u)\,du\right|\\ \leq\;&\frac{5(\eta(b,\tau))^2}{64}\|w\|_{[\tau,\tau+\eta(b,\tau)],\infty}\\ &\times\left(\left(\frac{79\left|\pounds'(\tau)\right|^{\lambda}+41\left|\pounds'(b)\right|^{\lambda}}{120}\right)^{\frac{1}{\lambda}}+\left(\frac{41\left|\pounds'(\tau)\right|^{\lambda}+79\left|\pounds'(b)\right|^{\lambda}}{120}\right)^{\frac{1}{\lambda}}\right).\end{aligned}$$

If we take $\eta(b,\tau)=b-\tau$ and $w(u)=\frac{1}{b-\tau}$, we get

$$\begin{aligned}&\left|\frac{1}{8}\left(\pounds(\tau)+6\pounds\left(\frac{\tau+b}{2}\right)+\pounds(b)\right)-\frac{1}{b-\tau}\int\limits_{\tau}^{b}\pounds(u)\,du\right|\\ \leq\;&\frac{5(b-\tau)}{64}\left(\left(\frac{79\left|\pounds'(\tau)\right|^{\lambda}+41\left|\pounds'(b)\right|^{\lambda}}{120}\right)^{\frac{1}{\lambda}}+\left(\frac{41\left|\pounds'(\tau)\right|^{\lambda}+79\left|\pounds'(b)\right|^{\lambda}}{120}\right)^{\frac{1}{\lambda}}\right).\end{aligned}$$

Corollary 3.6. *Let us consider the assumptions of Theorem 3.3. If $|\pounds'|^{\lambda}$ is P-function, then we have*

$$\begin{aligned}&\left|\frac{\pounds(\tau)+6\pounds\left(\frac{2\tau+\eta(b,\tau)}{2}\right)+\pounds(\tau+\eta(b,\tau))}{8}\int\limits_{\tau}^{\tau+\eta(b,\tau)}w(u)\,du-\int\limits_{\tau}^{\tau+\eta(b,\tau)}w(u)\,\pounds(u)\,du\right|\\ \leq\;&\frac{5(\eta(b,\tau))^2}{32}\|w\|_{[\tau,\tau+\eta(b,\tau)],\infty}\left(\left|\pounds'(\tau)\right|^{\lambda}+\left|\pounds'(b)\right|^{\lambda}\right)^{\frac{1}{\lambda}}.\end{aligned}$$

If we take $\eta(b,\tau)=b-\tau$ and $w(u)=\frac{1}{b-\tau}$, we get

$$\begin{aligned}&\left|\frac{1}{8}\left(\pounds(\tau)+6\pounds\left(\frac{\tau+b}{2}\right)+\pounds(b)\right)-\frac{1}{b-\tau}\int\limits_{\tau}^{b}\pounds(u)\,du\right|\\ \leq\;&\frac{5(b-\tau)}{32}\left(\left|\pounds'(\tau)\right|^{\lambda}+\left|\pounds'(b)\right|^{\lambda}\right)^{\frac{1}{\lambda}}.\end{aligned}$$

4. Applications

4.1. Application to quadrature formula

Let Ξ be the partition of the points $a=x_0<x_1<...<x_n=b$ of the interval $[a,b]$, and consider the quadrature formula

$$\int\limits_{a}^{b}w(u)\,\pounds(u)\,du=\lambda_w(\pounds,\Xi)+R_w(\pounds,\Xi),$$

where

$$\lambda_w(\pounds,\Xi)=\sum_{i=0}^{n-1}\frac{1}{8}\left(\pounds(x_i)+6\pounds\left(\frac{x_i+x_{i+1}}{2}\right)+\pounds(x_{i+1})\right)\int_{x_i}^{x_{i+1}}w(u)\,du$$

and $R_w(\pounds,\Xi)$ denotes the associated approximation error.

Proposition 4.1. *Let $n\in\mathbb{N}$ and $\pounds:[\tau,b]\to\mathbb{R}$ be a differentiable function on (τ,b) with $0\le\tau<b$ such that $\pounds'\in L^1[\tau,b]$, and let w be continuous and symmetric functions with respect to $\frac{\tau+b}{2}$. If $|\pounds'|^k$ is convex function for a certain $k\ge1$, we have*

$$|R_w(\pounds,\Xi)|\le\|w\|_{[\tau,b],\infty}\sum_{i=0}^{n-1}\frac{5(x_{i+1}-x_i)^2}{32}\left(\frac{|\pounds'(x_i)|^k+|\pounds'(x_{i+1})|^k}{2}\right)^{\frac{1}{k}}.$$

Proof. Applying the first inequality of Corollary 3.5 on the subintervals $[x_i,x_{i+1}]$ $(i=0,1,...,n-1)$ of the partition Ξ, we obtain

$$\begin{aligned}&\left|\frac{\pounds(x_i)+6\pounds\left(\frac{x_i+x_{i+1}}{2}\right)+\pounds(x_{i+1})}{8}\int_{x_i}^{x_{i+1}}w(u)\,du-\int_{x_i}^{x_{i+1}}w(u)\pounds(u)\,du\right|\\ \le\;&\frac{5(x_{i+1}-x_i)^2}{64}\|w\|_{[x_i,x_{i+1}],\infty}\\ &\times\left(\left(\frac{79|\pounds'(x_i)|^k+41|\pounds'(x_{i+1})|^k}{120}\right)^{\frac{1}{k}}+\left(\frac{41|\pounds'(x_i)|^k+79|\pounds'(x_{i+1})|^k}{120}\right)^{\frac{1}{k}}\right).\end{aligned}\tag{4.1}$$

Using the discrete power mean inequality to (4.1), we get

$$\begin{aligned}&\left|\frac{\pounds(x_i)+6\pounds\left(\frac{x_i+x_{i+1}}{2}\right)+\pounds(x_{i+1})}{8}\int_{x_i}^{x_{i+1}}w(u)\,du-\int_{x_i}^{x_{i+1}}w(u)\pounds(u)\,du\right|\\ \le\;&\frac{5(x_{i+1}-x_i)^2}{32}\|w\|_{[x_i,x_{i+1}],\infty}\left(\frac{|\pounds'(x_i)|^k+|\pounds'(x_{i+1})|^k}{2}\right)^{\frac{1}{k}}.\end{aligned}\tag{4.2}$$

Summing the inequalities (4.2) for all $i=0,1,...,n-1$ and using the triangular inequality, we get the desired result. □

4.2. Application to random variables

Let X be a random variable taking its values in the finite interval $[v_1,v_2]$ where $v_1<v_2$. Let a probability density $w:[v_1,v_2]\to[0,1]$ be a continuous function and symmetric to $\frac{v_1+v_2}{2}$. The n-moment is defined

$$E_n[X]=\int_{v_1}^{v_2}t^n w(t)\,dt.$$

Proposition 4.2. *Let X be a random variable taking its values in the finite interval $[v_1, v_2]$, where $0 \leq v_1 < v_2$. If a probability density function $w : [v_1, v_2] \to [0, 1]$ which is symmetric to $\frac{v_1+v_2}{2}$, then for $n \geq 2$ we have*

$$\left|A\left(v_1^n, v_2^n\right) + 3A\left(v_1, v_2\right) - 4E_n\left[X\right]\right| \leq \frac{5n(v_2-v_1)^2}{16}\left(v_1^{n-1} + v_2^{n-1}\right),$$

where A is the arithmetic mean i.e. $A\left(v_1, v_2\right) = \frac{v_1+v_2}{2}$.

Proof. The assertion follows from inequality (3.3) of Corollary 3.1 with $\eta\left(b, \tau\right) = b-\tau$, applied to the function $\pounds\left(t\right) = t^n$, and taking into account that w is probability density. □

5. Conclusion

In this study, we have considered the weighted Simpson-like type integral inequalities for functions whose first derivatives are β-preinvex. By proving a new weighted identity, we have established some new weighted Simpson-like type inequalities. Some special cases which can be derived from the main results are discussed. We have furnished some applications of our findings. We hope that the ideas of this paper will inspire researchers working in field of inequalities to generalize our results for different kinds of classical and generalized convexity.

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