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LIGHTLIKE SUBMANIFOLDS OF AN INDEFINITE LORENTZIAN PARA-SASAKIAN STATISTICAL MANIFOLD

Mobin Ahmad and Mahtab Alam

Faculty of Science, Department of Mathematics and Statistics
P. O. Box 226026, Kursi-Road, Lucknow-226026, India

Abstract. In this paper, we introduce an indefinite Lorentzian para-Sasakian (LP-Sasakian) statistical manifold and study lightlike submanifold of an indefinite LP-Sasakian statistical manifold. We also introduce some relations among induced geometrical objects with respect to dual connections in a lightlike submanifold of an indefinite LP-Sasakian statistical manifold. One example related to this concept is also presented. Finally, we show that an invariant lightlike submanifold of an indefinite LP-Sasakian statistical manifold is an indefinite LP-Sasakian statistical manifold.
Keywords: LP-Sasakian manifold, Lightlike submanifold, Statistical manifold.

1. Introduction

The study of lightlike submanifolds is one of the most important research area in differential geometry, with many applications in physics and mathematics, such as general relativity, electromagnetism and black hole theory (for detail see [12],[13], [14],[17],[15]). The lightlike submanifolds were indroduced and studied by Duggal and Bejancu ([17]). B. Sahin initiated the study of transversal lightlike submanifolds of an indefinite Kaehlar manifold ([19]). Yildirim and Sahin ([18]) defined and studied transversal lightlike submanifolds of an indefinite sasakian manifold. The screen transversal submanifolds of indefinite Kaehlar manifolds were investigated by B. Sahin ([19]).

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Corresponding Author: Mobin Ahmad (mobinahmad68@gmail.com)
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On the other hand, Matsumoto [6] introduced the notion of LP-Sasakian manifold. Mihai and Rosca defined the same notion independently in [20]. LP-Sasakian manifolds were studied by many authors (see [6],[7],[11],[23],[24],[25],[26],[27],[28]).

The geometry of statistical manifolds is an emerging branch of mathematics that generalizes the Riemannian manifold. It uses the tools of differential geometry to study statistical inference, information loss and estimation. Effron [16] first time emphasize the role of differential geometry in statistics. Later, Amari ([1],[2]) used differential geometry tools to develop this idea. Vos [22] initiated the geometry of submanifolds of statistical manifolds. Furuhata [21] studied hypersurfaces of a statistical manifold. Aydin et. al. [3] studied submanifolds of statistical manifolds of constant curvature. Motivated by above circumstance, in the present paper, we initiate the study of lightlike submanifolds of indefinite LP-Sasakian statistical manifolds. The paper is organized as follows.

In Section 2, we define statistical manifolds from differential geometry point of view. Further, an indefinite LP-Sasakian statistical manifold is defined and some results are given for further use. We introduce indefinite LP-Sasakian statistical manifolds and we obtain the characterization theorem of indefinite LP-Sasakian statistical manifolds. Finally, an example is given.

In Section 3, we consider lightlike submanifolds of indefinite LP-Sasakian statistical manifolds. We characterize the parallelness of some distributions, and example is given on screen semi-invariant lightlike hypersurface.

In Section 4, we prove that invariant lightlike submanifolds of an indefinite LP-Sasakian statistical manifold is an indefinite LP-Sasakian statistical manifold.

2. Definition and preliminaries

We follow [17] for the notation and fundamental equations for lightlike submanifolds used in this paper. A submanifold M^m immersed in a semi-Riemannian manifold $(\overline{M}^{m+n}, \overline{g})$ is called a lightlike submanifold if it is a lightlike manifold with respect to the metric g induced from $\overline{g}$ and the radical distribution $Rad\,TM$ is of rank r, where $1 \leq r \leq m$. Let $S(TM)$ be a screen distribution which is a semi-Riemannian complementary distribution of $Rad\,TM$ in TM, that is

$$TM = Rad\,TM \perp S(TM).$$

Consider a screen transversal vector bundle $S(TM^{\perp})$, which is a semi-Riemannian complementary vector bundle of $Rad\,TM$ in $TM^{\perp}$. Since for any local basis $\{\xi_i\}$ of $Rad\,TM$, there exists a local null frame $\{N_i\}$ of section with values in the orthogonal complement of $S(TM^{\perp})$ in $[S(TM)]^{\perp}$ such that $\overline{g}(\xi_i, N_j) = \delta_{ij}$, it follows that there exist a lightlike transversal vector bundle $ltr(TM)$ locally spanned by $\{N_i\}$ [[17], pg-144]. Let $tr(TM)$ be a complementary (but not orthogonal) vector bundle to TM in $T\overline{M}|_M$. Then

$$tr(TM) = ltr(TM) \perp S(TM^{\perp}),$$

$$T\overline{M}|_M = S(TM) \perp [Rad\,(TM) \oplus ltr(TM)] \perp S(TM^{\perp}).$$

Following are four the subcases of a lightlike submanifold $(M, g, S(TM), S(TM^{\perp}))$.

Case 1: r-lightlike if $r < min\{m, n\}$.

Case 2:Co-isotropic if $r = n < m;\ S(TM^{\perp}) = 0$.

Case 3: Isotropic if $r = m < n;\ S(TM) = 0$.

Case 4: Totally lightlike if $r = m = n;\ S(TM) = 0 = S(TM^{\perp})$.

The Gauss and Weingarten equations are

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y),\ \forall X,\ Y \in \Gamma(TM), \tag{2.1}$$

$$\overline{\nabla}_X U = -A_U X + \nabla^t_X U,\ U \in \Gamma(tr(TM)), \tag{2.2}$$

where $\{\nabla_X Y,\ A_U X\}$ and $\{h(X, Y),\ \nabla^t_X U\}$ belongs to $\Gamma(TM)$ and $\Gamma(tr(TM))$, respectively, ∇ and ∇^t are linear connections on M and on the vector bundle $tr(TM)$, respectively. Moreover, we have

$$\overline{\nabla}_X Y = \nabla_X Y + h^l(X, Y) + h^s(X, Y), \tag{2.3}$$

$$\overline{\nabla}_X N = -A_N X + \nabla^l_X N + D^s(X, N), \tag{2.4}$$

$$\overline{\nabla}_X W = -A_W X + \nabla^s_X W + D^l(X, W), \tag{2.5}$$

$\forall X,\ Y \in \Gamma(TM)$ and $N \in \Gamma(ltr(TM))$ and $W \in \Gamma(S(TM^{\perp}))$. Then, by using (2.1), (2.3)-(2.5) and the fact that $\overline{\nabla}$ is a metric connection, we get

$$\overline{g}(h^s(X, Y), W) + \overline{g}(Y, D^l(X, W)) = g(A_W X, Y). \tag{2.6}$$

[4] In general, the induced connection ∇ on M is not a metric connection, by using (2.3), we have

$$(\nabla_X g)(Y, Z) = \overline{g}(h^l(X, Y), Z) + \overline{g}(h^l(X, Z), Y) \tag{2.7}$$

for any $X,\ Y \in \Gamma(TM)$, where $\{\nabla_X Y, A_N X, A_W X\} \in \Gamma(TM)$, $\{h^l(X, Y), \nabla^l_X N\} \in \Gamma(ltr(TM))$ and $\{h^s(X, Y), \nabla^s_X N\} \in \Gamma(S(TM^{\perp}))$.

If we set $B^l(X, Y) = \overline{g}(h^l(X, Y), \xi),\ B^s(X, Y) = \overline{g}(h^s(X, Y), \xi),\ \tau^l(X) = \overline{g}(\nabla^l_X N, \xi)$ and $\tau^s(X) = \overline{g}(\nabla^s_X N, \xi)$. Then equation (2.3), (2.4) and (2.5) become

$$\overline{\nabla}_X Y = \nabla_X Y + B^l(X, Y)N + B^s(X, Y)N, \tag{2.8}$$

$$\overline{\nabla}_X N = -A_N X + \tau^l(X)N + E^s(X, N), \tag{2.9}$$

$$\overline{\nabla}_X W = -A_W X + \tau^s(X)W + E^l(X, W), \tag{2.10}$$

respectively. Here, B and A are called second fundamental form and shape operator of the lightlike submanifold M. On the other hand, if we take the vector field $\xi \in \Gamma(Rad\ TM)$ and $X \in \Gamma(TM)$, we have the following relation like Weingarten formula

$$D_X \xi = -A^*_\xi X + \nabla_X \xi, \tag{2.11}$$

$$D^*_X\xi = -A_\xi X + \nabla^*_X\xi, \tag{2.12}$$

where $\{D_X\xi, D^*_X\xi\}$ and $\{A_\xi X,\ A^*_\xi X\}$ are the shape operators on $\Gamma(S(TM))$ and linear connections on $\Gamma(Rad(TM))$, respectively [9].

Now we define some statistical basic concepts:

Definition 2.1. [21] Let $\widetilde{M}$ be a smooth manifold. Let $\widetilde{D}$ be an affine connection with the torsion tensor $T^{\widetilde{D}}$ and $\widetilde{g}$ a semi- Riemannian metric on $\widetilde{M}$. Then the pair $(\widetilde{D}, \widetilde{g})$ is called statistical structure on $\widetilde{M}$ if

(1) $(\widetilde{D}_X\widetilde{g})(Y,Z) - (\widetilde{D}_Y\widetilde{g})(X,Z) = \widetilde{g}(T^{\widetilde{D}}(X,Y),Z)$

for all $X, Y, Z \in \Gamma(T\widetilde{M})$, and

(2) $T^{\widetilde{D}} = 0$.

Definition 2.2. [21] Let $(\widetilde{M}, \widetilde{g})$ be a semi-Riemannian manifold. Two affine connections $\widetilde{D}$ and $\widetilde{D}^*$ on $\widetilde{M}$ are said to be dual with respect to the metric $\widetilde{g}$, if

$$Z\widetilde{g}(X,Y) = \widetilde{g}(\widetilde{D}_Z X, Y) + \widetilde{g}(X, \widetilde{D}^*_Z Y) \tag{2.13}$$

for all $X, Y, Z \in \Gamma(T\widetilde{M})$.

A statistical manifold will be represented by $(\widetilde{M}, \widetilde{g}, \widetilde{D}, \widetilde{D}^*)$. If $\widetilde{\nabla}$ is Levi-Civita connection of $\widetilde{g}$, then

$$\widetilde{\nabla} = \frac{1}{2}(\widetilde{D} + \widetilde{D}^*). \tag{2.14}$$

In (2.13), if we choose $\widetilde{D}^* = \widetilde{D}$, then Levi-Civita connection is obtained.

Lemma 2.1. For statistical manifold $(\widetilde{M}, \widetilde{g}, \widetilde{D}, \widetilde{D}^*)$, we set

$$\overline{\mathbb{K}} = \widetilde{D} - \widetilde{\nabla}, \tag{2.15}$$

then we have

$$\overline{\mathbb{K}}(X,Y) = \overline{\mathbb{K}}(Y,X),\ \widetilde{g}(\overline{\mathbb{K}}((X,Y),Z) = \widetilde{g}(\overline{\mathbb{K}}((X,Z),Y) \tag{2.16}$$

for all $X, Y, Z \in \Gamma(TM)$.

Conversely, for a Riemannian metric g, if $\overline{\mathbb{K}}$ satisfies (2.21), the pair $(\widetilde{D} = \widetilde{\nabla} + \overline{\mathbb{K}}, \widetilde{g})$ is statistical structure on $\widetilde{M}$ [9].

Let (M, g) be a submanifold of $(\widetilde{M}, \widetilde{g})$. If (M, g, D, D^*) is statistical manifold, then (M, g, D, D^*) is called a statistical submanifold of $(\widetilde{M}, \widetilde{g}, \widetilde{D}, \widetilde{D}^*)$, where D, D^* are affine dual connections on M and $\widetilde{D}, \widetilde{D}^*$ are affine dual connections on $\widetilde{M}$ [1],[21],[22].

Let (M, g) be a lightlike submanifold of a statistical manifold $(\widetilde{M}, \widetilde{g}, \widetilde{D}, \widetilde{D}^*)$ then Gauss and Wiengarten formulas with respect to the dual connections are given by

$$\widetilde{D}_X Y = D_X Y + B^l(X,Y)N + B^s(X,Y)N, \tag{2.17}$$

(2.18) $$\widetilde{D}_X N = -A_N X + \tau^l(X)N + E^s(X,N),$$

(2.19) $$\widetilde{D}_X W = -A_W X + \tau^s(X)W + E^l(X,W),$$

(2.20) $$\widetilde{D}^*_X Y = D^*_X Y + B^{l^*}(X,Y)N + B^{s^*}(X,Y)N,$$

(2.21) $$\widetilde{D}^*_X N = -A^*_N X + \tau^{l^*}(X)N + E^{s^*}(X,N),$$

(2.22) $$\widetilde{D}^*_X W = -A^*_W X + \tau^{s^*}(X)W + E^{l^*}(X,W)$$

for all $X, Y \in \Gamma(TM)$, $N \in \Gamma(ltrTM)$ and $W \in \Gamma(S(TM^{\perp}))$, Here, D , D^*, B, B^{l^*}, B^s, B^{s^*}, A_N, and A^*_N are called the induced connections on M, the second fundamental forms and the Weingarten mappings with respect to $\widetilde{D}$ and $\widetilde{D}^*$, respectively. Using Gauss formulas and the equation (2.13), we obtain

(2.23) $$Xg(Y,Z) = g(\widetilde{D}_X Y, Z) + g(Y, \widetilde{D}^*_X Z) = g(D_X Y, Z) + g(X, D^*_X Z)$$
$$+B^l(X,Y)\eta(Z) + B^{l^*}(X,Z)\eta(Y) + B^s(X,Y)\eta(Z) + B^{s^*}(X,Z)\eta(Y).$$

From the equation (2.23), we have the following results.

Proposition 2.1. [10] *Let (M,g) be a lightlike submanifold of a statistical manifold $(\widetilde{M}, \widetilde{g}, \widetilde{D}, \widetilde{D}^*)$. Then the following assertions are true:*

(i) Induced connection D and D^* are symmetric connections.

(ii) The second fundamental forms B^l, B^s, B^* and B^{s^*} are symmetric.

Proof. We know that $T^{\widetilde{D}} = 0$. Moreover,

(2.24) $$T^{\widetilde{D}}(X,Y) = \widetilde{D}_X Y - \widetilde{D}_Y X - [X,Y] = D_X Y - D_Y X - [X,Y] +$$
$$B^l(X,Y)N + B^s(X,Y)N - B^l(Y,X)N - B^s(Y,X)N = 0.$$

Comparing the tangential and transversal components of (2.24), we obtain

$$B^l(X,Y) = B^s(X,Y), \quad B^l(Y,X) = B^s(Y,X), \quad T^D = 0,$$

where T^D is the tensor field of D. Thus, second fundamental form B is symmetric and the induced connection D is symmetric connection.

Similarly, it can be shown that the second fundamental form B^* is symmetric and the induced connection D^* is a symmetric connection. □

([10],[8],[9]) In order to call a differentiable semi-Riemannian manifold $(\widetilde{M}, \widetilde{g})$ of dimension $n = 2m+1$ as practically contact metric one, a (1,1) tensor field $\widetilde{\phi}$, a contravariant vector field v, a 1-form η and a Lorentzian metric $\widetilde{g}$ should be admitted, which satisfy

(2.25) $$\widetilde{\phi} v = 0, \ \eta(\widetilde{\phi} X) = 0, \ \eta(v) = \epsilon,$$

(2.26) $$\widetilde{\phi}^2(X) = X + \eta(X)v, \ \widetilde{g}(X,v) = \epsilon\eta(X),$$

$$\widetilde{g}(\widetilde{\phi}X, \widetilde{\phi}Y) = \widetilde{g}(X, Y) - \epsilon\eta(Y)\eta(X), \quad \epsilon = \mp 1 \tag{2.27}$$

for all the vector field X, Y on $\widetilde{M}$. When a Lorentzian metric manifold $\widetilde{g}$ performs

$$(\widetilde{\nabla}_X\widetilde{\phi})Y = -\epsilon\eta(Y)\phi^2 X + \widetilde{g}(\widetilde{\phi}X, \widetilde{\phi}Y)v, \tag{2.28}$$

$$\widetilde{\nabla}_X v = \widetilde{\phi}X, \tag{2.29}$$

$\widetilde{M}$ is regarded as an indefinite LP-Sasakian manifold [11]. In this study, we assume that the vector field v is spacelike.

Definition 2.3. *Let $(\widetilde{g}, \widetilde{\phi}, v)$ be an indefinite LP-Sasakian structure on $\widetilde{M}$. A quadruplet $(\widetilde{D} = \widetilde{\nabla} + \mathbb{K}, \widetilde{g}, \widetilde{\phi}, v)$ is called an indefinite LP-Sasakian statistical structure on $\widetilde{M}$ if $(\widetilde{D}, \widetilde{g})$ is a statistical structure on $\widetilde{M}$ and the formula*

$$\overline{\mathbb{K}}(X, \widetilde{\phi}Y) = -\widetilde{\phi}\overline{\mathbb{K}}(X, Y) \tag{2.30}$$

holds for any $X, Y \in \Gamma(T\widetilde{M})$. Then $(\widetilde{M}, \widetilde{D}, \widetilde{g}, \widetilde{\phi}, v)$ is said to be an indefinite LP-Sasakian statistical manifold.

An indefinite LP-Sasakian statistical manifold will be represented by $(\widetilde{M}, \widetilde{D}, \widetilde{g}, \widetilde{\phi}, v)$.

Theorem 2.1. *Let $(\widetilde{M}, \widetilde{D}, \widetilde{g},)$ be a statistical manifold and $(\widetilde{g}, \widetilde{\phi}, v)$ an almost contact metric structure on $\widetilde{M}$. $(\widetilde{D}, \widetilde{g}, \widetilde{\phi}, v)$ is an indefinite LP-Sasakian statistical structure if and only if the following conditions hold:*

$$\widetilde{D}_X\widetilde{\phi}Y - \widetilde{\phi}\widetilde{D}^*_X Y = \widetilde{g}(\widetilde{\phi}X, \widetilde{\phi}Y)v - \eta(Y)\phi^2 X, \tag{2.31}$$

$$\widetilde{D}_X v = \widetilde{\phi}X + \widetilde{g}(\widetilde{D}_X v, v)v \tag{2.32}$$

for all the vector field X, Y on $\widetilde{M}$.

Proof. Using (2.20), we get

$$\widetilde{D}_X\widetilde{\phi}Y - \widetilde{\phi}\widetilde{D}^*_X Y = (\widetilde{\nabla}_X\widetilde{\phi})Y + \overline{\mathbb{K}}(X, \widetilde{\phi}Y) + \widetilde{\phi}\overline{\mathbb{K}}(X, Y) \tag{2.33}$$

for all vector fields X, Y on $\widetilde{M}$. If we consider Definition 2. and the equation (2.28), we have the formula (2.31). If we write $\widetilde{D}^*$ instead of $\widetilde{D}$ in (2.31), we have

$$\widetilde{D}^*_X\widetilde{\phi}Y - \widetilde{\phi}\widetilde{D}_X Y = \widetilde{g}(\widetilde{\phi}X, \widetilde{\phi}Y)v - \widetilde{g}(Y, v)\widetilde{\phi}^2 X. \tag{2.34}$$

Substituting v for Y in (2.34), we have the equation (2.32).

$$\widetilde{\phi}(\widetilde{D}_X\widetilde{\phi}^2 Y - \widetilde{\phi}\widetilde{D}^*_X\widetilde{\phi}Y) = 0.$$

Assume (2.26) and (2.32) as well, we get

$$0 = -\widetilde{\phi}\widetilde{D}_X Y + \widetilde{g}(Y, v)\widetilde{\phi}^2 X + \widetilde{D}^*_X\widetilde{\phi}Y - \widetilde{g}(\widetilde{\phi}X, \widetilde{\phi}Y)v.$$

From (2.27), this equation gives us (2.34). Now, we will prove (2.28) and (2.30) by using (2.31) and (2.34), we have the following equations

$$(\widetilde{\nabla}_X\widetilde{\phi})Y + \widetilde{g}(Y, v)\widetilde{\phi}^2 X - \widetilde{g}(\widetilde{\phi}X, \widetilde{\phi}Y)v = \overline{\mathbb{K}}(X, \widetilde{\phi}Y) + \widetilde{\phi}\overline{\mathbb{K}}(X, Y).$$

and

$$(\widetilde{\nabla}_X\widetilde{\phi})Y+\widetilde{g}(Y,v)\widetilde{\phi}^2X-\widetilde{g}(\widetilde{\phi}X,\widetilde{\phi}Y)v=-\overline{\mathbb{K}}(X,\widetilde{\phi}Y)-\widetilde{\phi}\overline{\mathbb{K}}(X,Y).$$

The last two equations satisfy (2.28) and (2.30). □

Example 2.1. Recall example 1 from [5] as follows:

Let R^3 be 3-dimensional Euclidean space with rectangular coordinates (x, y, z). In R^3, we define

$$\eta=-dz-ydx,\qquad v=\frac{\partial}{\partial z}$$

$$\phi\frac{\partial}{\partial x}=\frac{\partial}{\partial y},\quad \phi\frac{\partial}{\partial y}=\frac{\partial}{\partial x}-y\frac{\partial}{\partial z},\quad \phi\frac{\partial}{\partial z}=0.$$

The Lorentzian metric $\widetilde{g}$ is defined by the matrix: $\begin{pmatrix} -\epsilon y^2 & 0 & \epsilon y \\ 0 & 0 & 0 \\ \epsilon y & 0 & -\epsilon \end{pmatrix}.$

Then it can be easily seen that $(\widetilde{\phi}, v, \eta, \widetilde{g})$ forms an indefinite LP-Sasakian structure in R^3. If we choose difference tensor $\overline{\mathbb{K}}(X,Y)=\widetilde{g}(Y,v)\widetilde{g}(X,v)v$, then $(\widetilde{D},\widetilde{g},\widetilde{\phi},v)$ is an indefinite LP-Sasakian statistical structure on $\widetilde{M}$.

3. Lightlike submanifolds of indefinite LP-Sasakian statistical manifolds

Definition 3.1. *Let (M, g, D, D^*) be a submanifold of indefinite LP-Sasakian statistical manifold $(\widetilde{M},\widetilde{D},\widetilde{g},\widetilde{\phi},v)$. The quadruplet (M, g, D, D^*) is called lightlike submanifolds of indefinite LP-Sasakian statistical manifold $(\widetilde{M},\widetilde{D},\widetilde{g},\widetilde{\phi},v)$ if the induced metric g is degenerate.*

Let $(\widetilde{M},\widetilde{D},\widetilde{g},\widetilde{\phi},v)$ be a $(2m+1)$- dimensional LP-Sasakian statistical manifold and (M, g) be a lightlike submanifold of $\widetilde{M}$, such that the structure vector field v is tangent to M. For any $\xi\in\Gamma(Rad\,TM)$ *and* $N\in\Gamma(ltr(TM))$, in view of (2.25)-(2.27), we have

(3.1) $$\widetilde{g}(\xi,v)=0,\quad \widetilde{g}(N,v)=0,$$

(3.2) $$\widetilde{\phi}^2\xi=-\xi,\quad \widetilde{\phi}^2N=-N.$$

Also, using (2.17) and (2.32), we obtain

(3.3) $$B(\xi,v)=0,\quad B(v,v)=0,$$

(3.4) $$B^*(\xi,v)=0,\quad B^*(v,v)=0.$$

(3.5) $$S(TM)=\{\widetilde{\phi}Rad\,TM\oplus\widetilde{\phi}ltr(TM)\}\perp L_0\perp<v>,$$

where L_0 is non-degenerate and $\widetilde{\phi}$ - invariant distribution of rank $2m-4$ on M. If we denote the following distributions on M

$$L = Rad\,TM \perp \widetilde{\phi} Rad\,TM \perp L_0, \quad L^{'} = \widetilde{\phi} ltr(TM), \tag{3.6}$$

then L is invariant and $L^{'}$ is anti-invariant distributions under $\widetilde{\phi}$. Also, we have

$$TM = L \oplus L^{'} \perp < v > . \tag{3.7}$$

Now, we consider two null vector fields U and W and their 1-forms u and w as follows:

$$U = -\widetilde{\phi} N, \; u(X) = \widetilde{g}(X, W), \tag{3.8}$$

$$W = -\widetilde{\phi}\xi, \; w(X) = \widetilde{g}(X, U). \tag{3.9}$$

Then, for any$X \in \Gamma(T\widetilde{M})$, we have

$$X = SX + u(X)U, \tag{3.10}$$

where S is projection morphism of $T\widetilde{M}$ on the distribution L. Applying $\widetilde{\phi}$ to last equation, we obtain

$$\widetilde{\phi}X = \widetilde{\phi}SX + u(X)\widetilde{\phi}U,$$

$$\widetilde{\phi}X = \phi X + u(X)N, \tag{3.11}$$

where ϕ is a tensor field of type (1,1) defined on M by $\phi X = \widetilde{\phi}SX$. Again, applying $\widetilde{\phi}$ to the equation (3.11) and using (2.25)-(2.27), we have

$$\widetilde{\phi}^2 X = \widetilde{\phi}\phi X + u(X)\widetilde{\phi}N,$$

$$X + g(X, v)v = \phi^2 X - u(X)U,$$

which means that

$$\phi^2 X = X + g(X, v)v + u(X)U. \tag{3.12}$$

Now, applying ϕ to the equation (3.12) and then, since ϕU=0, we have $\phi^3 - \phi$=0, which gives that ϕ is an f-structure.

Definition 3.2. *Let (M, g, D, D^*) be submanifolds of an indefinite LP-Sasakian statistical manifold $(\widetilde{M}, \widetilde{D}, \widetilde{g}, \widetilde{\phi}, v)$. The quadruplet (M, g, D, D^*) is called screen semi-invariant lightlike submanifolds of an indefinite LP-Sasakian statistical manifold $(\widetilde{M}, \widetilde{D}, \widetilde{g}, \widetilde{\phi}, v)$ if*

$$\widetilde{\phi}(ltrTM) \subset S(TM),$$

$$\widetilde{\phi}(RadTM) \subset S(TM).$$

We remark that a submanifold of indefinite LP-Sasakian statistical manifold is screen semi-invariant lightlike submanifold.

Example 3.1. Let $\widetilde{M}$ be the 9-dimensional manifold with respect to the canonical basis $\{\frac{\partial}{\partial l_1}, \frac{\partial}{\partial l_2}, \frac{\partial}{\partial l_3}, \frac{\partial}{\partial l_4}, \frac{\partial}{\partial m_1}, \frac{\partial}{\partial m_2}, \frac{\partial}{\partial m_3}, \frac{\partial}{\partial m_4}, \frac{\partial}{\partial z}\}$.

Defining $\widetilde{\phi}\frac{\partial}{\partial l_i} = \frac{\partial}{\partial m_i}$, $\widetilde{\phi}\frac{\partial}{\partial m_i} = \frac{\partial}{\partial l_i}$, $\widetilde{\phi}\frac{\partial}{\partial z} = 0$, $v = \frac{\partial}{\partial z}$, $\eta = dz$. By choosing the difference tensor $\widetilde{\mathbb{K}}(X,Y) = \widetilde{g}(Y,v)\widetilde{g}(X,v)v$, then $(\widetilde{D} = \widetilde{\nabla} + \widetilde{\mathbb{K}}, \widetilde{g}, \widetilde{\phi}, v)$ is an indefinite LP-Sasakian statistical manifold on $\widetilde{M}$.

Suppose M is a submanifold of $\widetilde{M}$ defined by

$$l_1 = m_3$$

then $Rad\,TM$ and $ltr(TM)$ are spanned by

$$\xi = \frac{\partial}{\partial l_1} + \frac{\partial}{\partial m_3}, \qquad N = \frac{1}{2}\{\frac{\partial}{\partial l_3} - \frac{\partial}{\partial m_1}\}.$$

Applying $\widetilde{\phi}$ to this vector field, we have

$$\widetilde{\phi}\xi = \frac{\partial}{\partial m_1} + \frac{\partial}{\partial l_3}, \qquad \widetilde{\phi}N = \frac{1}{2}\{\frac{\partial}{\partial m_3} - \frac{\partial}{\partial l_1}\}.$$

This shows that M is a screen semi-invariant lightlike submanifold of an indfinite LP-Sasakian statistical manifold.

Lemma 3.1. *Let (M, g, D, D^*) be a lightlike submanifold of indefinite LP-Sasakian statistical manifold $(\widetilde{M}, \widetilde{D}, \widetilde{g}, \widetilde{\phi}, v)$. For any $X, Y \in \Gamma(TM)$, we have the following identities:*

$$D_X\phi Y - \phi D^*_X Y = -B^{l^*}(X,Y)U - B^{s^*}(X,Y)U + u(Y)A_N X + g(\phi X, \phi Y)v - g(Y,v)\phi^2(X), \tag{3.13}$$

$$D_X(u(Y)) - u(D^*_X Y) = -B^l(X,\phi Y) - B^s(X,\phi Y) - u(Y)\tau^l(X) - u(Y)E^s(X,N). \tag{3.14}$$

Proof. Using Gauss and Weingarten formulas in (2.31), we have

$$D_X\phi Y + B^l(X,\phi Y)N + B^s(X,\phi Y)N - u(Y)A_N X + u(Y)\tau^l(X)N + u(Y)E^s(X,N) + D_X(u(Y))N - \phi D^*_X Y - u(D^*_X Y)N + B^{l^*}(X,Y)U + B^{s^*}(X,Y)U = -g(Y,v)\phi^2 X + g(\phi X,\phi Y)v. \tag{3.15}$$

If we take the tangential and transversal parts of (3.15), we get (3.13) and (3.14). □

Lemma 3.2. *Let (M, g, D, D^*) be a lightlike submanifolds of indefinite LP-Sasakian statistical manifold $(\widetilde{M}, \widetilde{D}, \widetilde{g}, \widetilde{\phi}, v)$. For any $X, Y \in \Gamma(TM)$, we have the following identities:*

$$D^*_X\phi Y - \phi D_X Y = B^l(X,Y)U + B^s(X,Y)U + u(Y)A^*_N X - g(Y,v)\phi(X) - g(X,\phi Y)v, \tag{3.16}$$

$$D^*_X(u(Y)) - u(D_X Y) = -B^{l^*}(X,\phi Y) - B^{s^*}(X,\phi Y) \tag{3.17}$$

$$-u(Y)\tau^{l^*}(X) - u(Y)E^{s^*}(X,N).$$

Proof. Using Gauss and Weingarten formulas in (2.34), we have

$$D^*_X\phi Y + B^{l^*}(X,\phi Y)N + B^{s^*}(X,\phi Y)N - u(Y)A^*_N X + u(Y)\tau^{l^*}(X)N$$
$$+u(Y)E^{s^*}(X,N) + D^*_X(u(Y))N - u(D_X Y)N - \phi D_X Y - B^l(X,Y)U$$
$$-B^s(X,Y)U = g(\phi X,\phi Y)v - g(Y,v)\phi^2 X$$

If we take the tangential and transversal parts of last equation, we get (3.16) and (3.17). □

Proposition 3.1. *Let (M,g,D,D^*) be a lightlike submanifolds of indefinite LP-sasakian statistical manifold $(\widetilde{M},\widetilde{D},\widetilde{g},\widetilde{\phi},v)$. For any $X,Y\in\Gamma(TM)$, we have the following expressions:*

(i) If the vector field U is parallel with respect to ∇, then

(3.18) $$A_N X = u(A_N X)U - \eta(A_N X)v,\ \tau^l(X) = 0 \text{ and } E^s(X,N) = 0.$$

(ii) If the vector field U is parallel with respect to ∇^*, then

(3.19) $$A^*_N X = u(A^*_N X)U - \eta(A^*_N X)v\ ,\ \tau^{l^*}(X) = 0 \text{ and } E^{s^*}(X,N) = 0.$$

Proof. On replacing Y in (3.13) by U, we have

$$-\phi D^*_X U = A_N X - B^{l^*}(X,U)U - B^{s^*}(X,U)U.$$

Applying ϕ in last equation and using (3.12), we obtain

$$-\{D^*_X U + g(D^*_X U,v)v + u(D^*_X U)U\} = \widetilde{\phi}A_N X.$$

If U is parallel to ∇^*, then $\phi A_N X = 0$. From (3.11), we have $\widetilde{\phi}A_N X = u(A_N X)N$. Applying $\widetilde{\phi}$ on last equation and using (2.26), we obtain

$$A_N X = u(A_N X)U - \eta(A_N X)v.$$

Now, if we take U instead of Y in the equation (3.14), we get

$$D_X(u(U)) - u(D^*_X U) = -B^l(X,\phi U) - B^s(X,\phi U) - u(U)\tau^l(X) - E^s(X,N).$$
$$\tau^l(X) + E^s(X,N) = 0,$$

this shows that

$$\tau^l(X) = 0, \quad E^s(X,N) = 0.$$

Similarly, we can also obtained (3.19) by same procedure. □

Proposition 3.2. *Let (M,g,D,D^*) be a lightlike submanifolds of indefinite LP-Sasakian statistical manifold $(\widetilde{M},\widetilde{D},\widetilde{g},\widetilde{\phi},v)$. For any $X,Y\in\Gamma(TM)$, we have the following expressions:*

(i) If the vector field W is parallel with respect to ∇^*, then

$$\overline{A_\xi}^* X = -u(\overline{A_\xi}^* X)U - g(\overline{A_\xi}^* X, v)v,\ \tau^{l^*}(X) = 0. \tag{3.20}$$

(ii) If the vector field W is parallel with respect to ∇, then

$$\overline{A_\xi} X = -u(\overline{A_\xi} X)U - g(\overline{A_\xi} X, v)v,\ \tau^{l}(X) = 0. \tag{3.21}$$

Proof. If we take ξ instead of Y in (3.16), we get

$$D_X\phi\xi - \phi D^*_X\xi = -B^{l^*}(X,\xi)U - B^{s^*}(X,\xi)U + u(\xi)A_N X - g(\xi,v)\phi^2 X + g(\phi X,\phi\xi)v.$$

$$-\phi D^*_X\xi = -B^{l^*}(X,\xi)U - B^{s^*}(X,\xi)U.$$

The relation of induced dual objects on $S(TM)$

$$D^*_X\xi = -\overline{A}_\xi X - \tau^{l^*}(X)\xi + E^{s^*}(X,\xi),\quad \forall X,\ Y \in \Gamma(TM). \tag{3.22}$$

If W is parallel with respect to D, using 3.22 and (3.12) in the above equation, we get

$$-\phi[-\overline{A}^*_\xi X - \tau^{l^*}(X)\xi + E^{s^*}(X,\xi)] = -B^{l^*}(X,\xi)U - B^{s^*}(X,\xi)U.$$

$$-\phi\overline{A}^*_\xi X - \tau^{l^*}(X)\phi\xi = -B^{l^*}(X,\xi)U - B^{s^*}(X,\xi)U.$$

Applying $\widetilde{\phi}$ and using (3.12), we get

$$\overline{A^*_\xi} X + u(\overline{A^*_\xi} X)U + g(\overline{A^*_\xi} X, v)v = \tau^{l^*}(X)\phi\xi$$

Comparing screen and radical parts of last equation we obtain (3.20). Similarly, we can obtain (3.21). □

4. Invariant lightlike submanifolds

Let (M, g, D, D^*) be an invariant lightlike submanifolds of an indefinite LP-Sasakian statistical manifold $(\widetilde{M}, \widetilde{D}, \widetilde{g}, \widetilde{\phi}, v)$. If M is tangent to the structure vector field v, then v belongs to $S(TM)$. For invariant lightlike submanifolds, we have the following expressions:

$$\widetilde{\phi}(S(TM)) = S(TM),\quad \widetilde{\phi}(RadTM) = RadTM. \tag{4.1}$$

Proposition 4.1. *Let (M, g, D, D^*) be an invariant lightlike submanifolds of indefinite LP-Sasakian statistical manifold $(\widetilde{M}, \widetilde{D}, \widetilde{g}, \widetilde{\phi}, v)$. If M is tangent to the structure vector field v, then v belongs to $S(TM)$ for $X, Y \in \Gamma(TM)$, we have the following identities:*

$$D_X\phi Y - \phi D^*_X Y = g(\phi X, \phi Y)v + g(Y, v)\phi^2 X, \tag{4.2}$$

$$h(X, \widetilde{\phi}Y) = \widetilde{\phi}h^*(X, Y), \tag{4.3}$$

where h and h^* are second fundamental forms for affine dual connections $\widetilde{D}$ and $\widetilde{D}^*$, respectively.

Proof. In (2.31), using (3.11) and Gauss formula, we have

$$D_X\phi Y - \phi D^*_X Y + h(X, \widetilde{\phi}Y) - \widetilde{\phi}h^*(X, Y) = g(\phi X, \phi Y)v + g(Y, v)\phi^2 X.$$

On taking tangential and transversal parts of the last equation, we get (4.2) and (4.3), respectively. □

Proposition 4.2. *Let (M, g, D, D^*) be an invariant lightlike submanifold of an indefinite LP-Sasakian statistical manifold $(\widetilde{M}, \widetilde{D}, \widetilde{g}, \widetilde{\phi}, v)$. If M is tangent to the structure vector field v, then v belongs to $S(TM)$ for $X, Y \in \Gamma(TM)$, we have the following identities:*

$$D^*_X\phi Y - \phi D_X Y = g(\phi X, \phi Y)v + g(Y, v)\phi^2 X, \tag{4.4}$$

$$h^*(X, \widetilde{\phi}Y) = \widetilde{\phi}h(X, Y), \tag{4.5}$$

where h and h^* are second fundamental forms for affine dual connections $\widetilde{D}$ and $\widetilde{D}^*$, respectively.

Proof. In (2.34), using (3.11) and Gauss formula, we have

$$D^*_X\phi Y - \phi D_X Y + h^*(X, \widetilde{\phi}Y) - \widetilde{\phi}h(X, Y) = g(\phi X, \phi Y)v + g(Y, v)\phi^2 X.$$

On taking tangential and transversal parts of this last equation, we get (4.4) and (4.5). □

Theorem 4.1. *An invariant lightlike subnmanifold of an indefinite LP-Sasakian statistical manifold is an indefinite LP-Sasakian statistical manifold.*

Proof. For any $X \in \Gamma(TM)$, $u(X) = 0$ in a invariant lightlike submanifold, then from (3.11), we have

$$\phi^2(X) = X + g(X, v)v.$$

Since, $\widetilde{\phi}X = \phi X$, using (2.25), (2.26) and (2.27), we obtain

$$\phi v = 0,\ \eta(\phi X) = 0,$$

$$\widetilde{g}(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y).$$

Then, (g, ϕ, v) is an almost contact metric structure.

Using(2.23), we get

$$Xg(\phi Y, \phi Z) = g(D_X\phi Y, \phi Z) + g(\phi Y, D^*_X\phi Z).$$

This equation says that D and D^* are dual connections. Moreover, torsion tensor of the connection D is equal to zero. Then, the equations (2.23) and definition 2. tell us that (D, g) is a statistical structure, that is

$$T^{\widetilde{D}} = 0,$$

$$\widetilde{D}_X Y - \widetilde{D}_Y X - [X,Y] = 0,$$

$$D_X Y - D_Y X - [X,Y] + B(X,Y)N - B(Y,X)N = 0.$$

Comparing tangential and transversal parts of last equation, we obtain

$$T^D = 0.$$

If we consider Gauss formula and (2.32),

$$\widetilde{D}_X v = \widetilde{\phi} X + \widetilde{g}(\widetilde{D}_X v, v)v,$$

$$D_X v + B(X,v)N = \phi X + g(D_X v + B(X,v)N, v)v,$$

we have

$$D_X v = \phi X + g(D_X v, v)v.$$

Our assertions is proved. □

Example 4.1. Let $\widetilde{M}$ be the 9-dimensional manifold with respect to the canonical basis $\{\frac{\partial}{\partial p_1}, \frac{\partial}{\partial p_2}, \frac{\partial}{\partial p_3}, \frac{\partial}{\partial p_4}, \frac{\partial}{\partial q_1}, \frac{\partial}{\partial q_2}, \frac{\partial}{\partial q_3}, \frac{\partial}{\partial q_4}, \frac{\partial}{\partial z}\}$.

Defining $\widetilde{\phi}\frac{\partial}{\partial p_i} = \frac{\partial}{\partial q_i}$, $\widetilde{\phi}\frac{\partial}{\partial q_i} = \frac{\partial}{\partial p_i}$, $\widetilde{\phi}\frac{\partial}{\partial z} = 0$, $v = \frac{\partial}{\partial z}$, $\eta = dz$. By choosing the difference tensor $\widetilde{\mathbb{K}}(X,Y) = \widetilde{g}(Y,v)\widetilde{g}(X,v)v$, then $(\widetilde{D} = \widetilde{\nabla} + \widetilde{\mathbb{K}}, \widetilde{g}, \widetilde{\phi}, v)$ is an indefinite LP-Sasakian statistical manifold on $\widetilde{M}$.

Suppose M is a submanifold of $\widetilde{M}$ defined by $p_1 = q_3,\ p_3 = q_1,\ p_2 = q_4,\ p_4 = q_2$ Then the tangent bundle TM of M is spanned by

$\xi_1 = \frac{\partial}{\partial p_1} + \frac{\partial}{\partial q_3}$, $\quad \xi_2 = \frac{\partial}{\partial p_3} + \frac{\partial}{\partial q_1}$

$Z_1 = \frac{\partial}{\partial p_2} - \frac{\partial}{\partial q_4}$, $\quad Z_2 = \frac{\partial}{\partial q_2} - \frac{\partial}{\partial p_4}$.

Moreover, one can show that $Rad\ (TM) = Span\{\xi_1, \xi_2\}$ and $S(TM) = Span\{Z_1, Z_2, v\}$. Furthermore, we note that $\widetilde{\phi}\xi_2 = \xi_1$ and $\widetilde{\phi}Z_2 = Z_1$. It follows that that $Rad\ (TM)$ and $S(TM)$ are invariant under $\widetilde{\phi}$. On the other hand, $ltr(TM)$ is spanned by N_1 and N_2, where

$N_1 = \frac{\partial}{\partial p_1} - \frac{\partial}{\partial q_3}$, $\quad N_2 = \frac{\partial}{\partial q_1} - \frac{\partial}{\partial p_3}$. Note that $\widetilde{\phi}N_2 = N_1$; hence, $ltr(TM)$ is invariant under $\widetilde{\phi}$. Therefore, M is an invariant lightlike submanifold of indefinite LP-Sasakian statistical manifold $\widetilde{M}$ and M is an indefinite LP-Sasakian statistical manifold.

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