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HERMITE-HADAMARD TYPE INEQUALITIES FOR CONFORMABLE INTEGRALS VIA η-CONVEX FUNCTIONS

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Abstract. Many recent results have been. This inequality has many applications in the area of pure and applied mathematics. In this paper, our main aim is to give results for conformable integral version of Hermite-Hadamard inequality for η-convex functions. First, we prove an identity associated with the Hermite-Hadamard inequality for conformable integrals using η-convex functions. By using this identity and η-convexity of function and some well-known inequalities, we obtain several results for the inequality.
Keywords: η-convex functions, Hermite-Hadamard inequality, Conformable derivative, Conformable integrals

1. Introduction

Let $I \in \mathbb{R}$ be an interval and $h : I \to \mathbb{R}$ be a convex function defined on I such that $\kappa_1, \kappa_2 \in I$ with $\kappa_1 < \kappa_2$. Then the well known Hermite-Hadamard inequality [15] states that

$$h\left(\frac{\kappa_1+\kappa_2}{2}\right) \leq \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} h(x)dx \leq \frac{h(\kappa_1)+h(\kappa_2)}{2} \tag{1.1}$$

holds. If the function h is concave on I, then both the inequalities in (1.1) hold in the reverse direction.

In the last few years, many researchers have shown their extensive attention on the generalizations, extensions, variations, refinements and applications of the

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Herimte-Hadamard inequality (see [25, 24, 26, 19, 28, 23, 8, 9, 10, 11, 27, 44, 29, 30, 31, 45, 46, 47, 48, 49, 39]).

The idea of η-convex functions was presented by Gordji *et al.* in [13, 14].

Definition 1.1. A function $h : [\kappa_1, \kappa_2] \to \mathbb{R}$ is said to be η-convex (or convex with respect to η) if the inequality

$$h(s\mu_1 + (1-s)\mu_2) \le h(\mu_2) + s\eta(h(\mu_1), h(\mu_2)), \tag{1.2}$$

holds for all $\mu_1, \mu_2 \in [\kappa_1, \kappa_2]$, $s \in [0,1]$ and η is defined by $\eta : h([\kappa_1, \kappa_2]) \times h([\kappa_1, \kappa_2]) \to \mathbb{R}$.

The definition of convex function is obtain by putting $\eta(\mu_1, \mu_2) = \mu_1 - \mu_2$ in the inequality (1.2). From [13, 14], the common η-convex version of $\mathcal{HH}$ inequality is:

Theorem 1.1. *Suppose η-convex function $h : [\kappa_1, \kappa_2] \to \mathbb{R}$, where η is bounded on $h([\kappa_1, \kappa_2]) \times h([\kappa_1, \kappa_2])$. Then the inequality*

$$\begin{aligned} h\left(\frac{\kappa_1+\kappa_2}{2}\right) - \frac{M_\eta}{2} &\le \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} h(x)dx \\ &\le \frac{h(\kappa_1)+h(\kappa_2)}{2} + \frac{\eta(h(\kappa_1),h(\kappa_2)) + \eta(h(\kappa_2),h(\kappa_1))}{4} \\ &\le \frac{h(\kappa_1)+h(\kappa_2)}{2} + \frac{M_\eta}{2} \end{aligned} \tag{1.3}$$

holds for M_η, where M_η is the upper bound for η.

Let $0 < \beta \le 1$, $r > 0$ and $g : [0, \infty) \to \mathbb{R}$ be a real-valued function. Then the conformable derivative $D_\beta(g)(r)$ of g at r [20] is defined by

$$D_\beta(g)(r) := \frac{d_\beta g(r)}{d_\beta r} = \lim_{\epsilon\to 0} \frac{g(r+\epsilon r^{1-\beta}) - g(r)}{\epsilon}. \tag{1.4}$$

g is said to be conformable differentiable if the limit of (1.4) exists. The conformal derivative at 0 is defined by $D_\beta(g)(0) = \lim\limits_{r\to 0^+} D_\beta(g)(r)$.

Let $\kappa_1, \kappa_2, \lambda, c \in \mathbb{R}$ be the constants, and h_1 and h_2 be differentiable at $r > 0$. Then the following formulas can be found in the literature [20]:

$$\frac{d_\beta}{d_\beta r}\left(r^\lambda\right) = \lambda r^{\lambda-\beta}, \quad \frac{d_\beta}{d_\beta r}(c) = 0, \tag{1.5}$$

$$\frac{d_\beta}{d_\beta r}\left(\kappa_1 h_1(r) + \kappa_2 h_2(r)\right) = \kappa_1 \frac{d_\beta}{d_\beta r}(h_1(r)) + \kappa_2 \frac{d_\beta}{d_\beta r}(h_2(r)), \tag{1.6}$$

$$\frac{d_\beta}{d_\beta r}\left(h_1(r) h_2(r)\right) = h_1(r) \frac{d_\beta}{d_\beta r}(h_2(r)) + h_2(r) \frac{d_\beta}{d_\beta r}(h_1(r)), \tag{1.7}$$

$$\frac{d_\beta}{d_\beta r}\left(\frac{h_1(r)}{h_2(r)}\right)=\frac{h_2(r)\frac{d_\beta}{d_\beta r}(h_1(r))-h_1(r)\frac{d_\beta}{d_\beta r}(h_2(r))}{(h_2(r))^2}, \tag{1.8}$$

$$\frac{d_\beta}{d_\beta r}\left(h_1(h_2(r))\right)=h_1'(h_2(r))\frac{d_\beta}{d_\beta r}(h_2(r)) \tag{1.9}$$

if h_1 differentiable at $h_2(r)$. In addition,

$$\frac{d_\beta}{d_\beta r}\left(h_1(r)\right)=r^{1-\beta}\frac{d}{dr}(h_1(r)) \tag{1.10}$$

if h_1 is differentiable.

Let $\beta\in(0,1]$ and $0\leq\kappa_1<\kappa_2$. Then the function $g:[\kappa_1,\kappa_2]\to\mathbb{R}$ is said to be conformable integrable if

$$\int_{\kappa_1}^{\kappa_2} g(x)d_\beta x:=\int_{\kappa_1}^{\kappa_2} g(x)x^{\beta-1}dx \tag{1.11}$$

exists and finite. All conformable integrable functions on $[\kappa_1,\kappa_2]$ is denoted by $L_\beta([\kappa_1,\kappa_2])$. Note that

$$I_\beta^{\kappa_1}(h_1)(r)=I_1^{\kappa_1}(r^{\beta-1}h_1)=\int_{\kappa_1}^{r}\frac{h_1(x)}{x^{1-\beta}}dx \tag{1.12}$$

for all $\beta\in(0,1]$, where the integral is the usual Riemann improper integral.

The theory and applications for the conformable integrals and derivatives we recommend the readers to refer the article[1, 2, 3, 21, 18, 22, 40, 6, 41, 32, 33, 37, 38].

Anderson [4] established the conformable integral version of the Hermite-Hadamard inequality

$$\frac{\beta}{\kappa_2^\beta-\kappa_1^\beta}\int_{\kappa_1}^{\kappa_2} h(x)d_\beta x\leq\frac{h(\kappa_1)+h(\kappa_2)}{2}, \tag{1.13}$$

for the conformable differentiable function $h:[\kappa_1,\kappa_2]\to\mathbb{R}$ with $\beta\in(0,1]$ and $D_\beta(h)$ is increasing. Moreover, if h is decreasing on $[\kappa_1,\kappa_2]$, then

$$h\left(\frac{\kappa_1+\kappa_2}{2}\right)\leq\frac{\beta}{\kappa_2^\beta-\kappa_1^\beta}\int_{\kappa_1}^{\kappa_2} h(x)d_\beta x. \tag{1.14}$$

Theorem 1.2. [36] *Let $\kappa_1,\kappa_2>0$ and $h:[\kappa_1,\kappa_2]\to(0,\infty)$ is a η-convex function and symmetric with respect to $\frac{\kappa_1+\kappa_2}{2}$, then the following conformable fractional integrals inequality*

$$h\left(\frac{\kappa_1+\kappa_2}{2}\right)-\frac{M_\eta}{2}\leq\frac{\beta}{\kappa_2^\beta-\kappa_1^\beta}\int_{\kappa_1}^{\kappa_2} h(x)d_\beta x \tag{1.15}$$

$$\leq\frac{h(\kappa_1)+h(\kappa_2)}{2}+\frac{\eta(h(\kappa_1),h(\kappa_2))+\eta(h(\kappa_2),h(\kappa_1))}{4} \tag{1.16}$$

$$\leq\frac{h(\kappa_1)+h(\kappa_2)}{2}+\frac{M_\eta}{2}. \tag{1.17}$$

holds for any $\beta \in (0,1]$.

In recent years, many results are devoted to the well-known Hermite-Hadamard inequality. This inequality has many applications in the area of pure and applied mathematics. In this paper, our main aim is to give results for conformable integral version of Hermite-Hadamard inequality for η-convex functions. First, we prove an identity associated with the Hermite-Hadamard inequality for conformable integrals using η-convex functions. By using this identity and η-convexity of function and some well-known inequalities, we obtain several results for the inequality.

2. Results Connected With Left Part of Hadamard's Type Inequality

In this section, first we prove the following lemma associated with the inequality (1.15), which will be used in the derivation of our main results.

Lemma 2.1. *Let* $\kappa_1, \kappa_2 \in \mathbb{R}^+$ *with* $\kappa_1 < \kappa_2$, $\beta \in (0,1]$ *and* $h : [\kappa_1, \kappa_2] \to \mathbb{R}$ *be a differentiable function on* (κ_1, κ_2). *Then the identity*

$$\begin{aligned} & h\left(\frac{\kappa_1+\kappa_2}{2}\right) - \frac{\beta}{\kappa_2^\beta - \kappa_1^\beta}\int_{\kappa_1}^{\kappa_2} h(x)d_\beta x \\ = & \frac{\kappa_2-\kappa_1}{\kappa_2^\beta - \kappa_1^\beta}\Bigg[\int_0^{\frac{1}{2}} \big(((1-s)\kappa_1+s\kappa_2)^{2\beta-1} - \kappa_1^\beta((1-s)\kappa_1+s\kappa_2)^{\beta-1}\big) \\ & \times D_\beta(h)((1-s)\kappa_1+s\kappa_2)s^{1-\beta}d_\beta s + \int_{\frac{1}{2}}^1 \big(((1-s)\kappa_1+s\kappa_2)^{2\beta-1} - \kappa_2^\beta \\ & \times((1-s)\kappa_1+s\kappa_2)^{\beta-1}\big)D_\beta(h)((1-s)\kappa_1+s\kappa_2)s^{1-\beta}d_\beta s\Bigg]. \end{aligned} \tag{2.1}$$

holds if $D_\beta(h) \in L_\beta^1([\kappa_1,\kappa_2])$.

Proof. Integrating by parts, we have

$$\begin{aligned} I = & \int_0^{\frac{1}{2}} \big((((1-s)\kappa_1+s\kappa_2)^{2\beta-1} - \kappa_1^\beta((1-s)\kappa_1+s\kappa_2)^{\beta-1}\big)D_\beta(h)((1-s)\kappa_1+s\kappa_2)ds \\ & + \int_{\frac{1}{2}}^1 \big(((1-s)\kappa_1+s\kappa_2)^{2\beta-1} - \kappa_2^\beta((1-s)\kappa_1+s\kappa_2)^{\beta-1}\big)D_\beta(h)((1-s)\kappa_1+s\kappa_2)ds \\ = & \int_0^{\frac{1}{2}} \big(((1-s)\kappa_1+s\kappa_2)^\beta - \kappa_1^\beta\big)h'((1-s)\kappa_1+s\kappa_2)ds \\ & + \int_{\frac{1}{2}}^1 \big(((1-s)\kappa_1+s\kappa_2)^\beta - \kappa_2^\beta\big)h'((1-s)\kappa_1+s\kappa_2)ds \\ = & \big(((1-s)\kappa_1+s\kappa_2)^\beta - \kappa_1^\beta\big)\frac{h((1-s)\kappa_1+s\kappa_2)}{\kappa_2-\kappa_1}\Bigg|_0^{\frac{1}{2}} \end{aligned}$$

$$
\begin{aligned}
&-\int_0^{\frac{1}{2}} \beta((1-s)\kappa_1+s\kappa_2)^{\beta-1}(\kappa_2-\kappa_1)\frac{h((1-s)\kappa_1+s\kappa_2)}{\kappa_2-\kappa_1}ds\\
&+\big(((1-s)\kappa_1+s\kappa_2)^{\beta}-\kappa_2^{\beta}\big)\frac{h((1-s)\kappa_1+s\kappa_2)}{\kappa_2-\kappa_1}\Bigg|_{\frac{1}{2}}^{1}\\
&-\int_{\frac{1}{2}}^{1} \beta((1-s)\kappa_1+s\kappa_2)^{\beta-1}(\kappa_2-\kappa_1)\frac{h((1-s)\kappa_1+s\kappa_2)}{\kappa_2-\kappa_1}ds
\end{aligned}
$$

$$
\begin{aligned}
= &\frac{1}{\kappa_2-\kappa_1}\left[\left(\left(\frac{\kappa_1+\kappa_2}{2}\right)^{\beta}-\kappa_1^{\beta}\right)h\left(\frac{\kappa_1+\kappa_2}{2}\right)-\beta\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}h(x)d_\beta x\right]\\
&+\frac{1}{\kappa_2-\kappa_1}\left[\left((\kappa_2^{\beta}-\left(\frac{\kappa_1+\kappa_2}{2}\right)^{\beta}\right)h\left(\frac{\kappa_1+\kappa_2}{2}\right)-\beta\int_{\frac{\kappa_1+\kappa_2}{2}}^{\kappa_2}h(x)d_\beta x\right]\\
= &\frac{\kappa_2^{\beta}-\kappa_1^{\beta}}{\kappa_2-\kappa_1}h\left(\frac{\kappa_1+\kappa_2}{2}\right)-\frac{\beta}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}h(x)d_\beta x,
\end{aligned}
$$

where, we have used the change of variable $x=(1-s)\kappa_1+s\kappa_2$ and then multiplying both sides by $\frac{\kappa_2-\kappa_1}{\kappa_2^{\beta}-\kappa_1^{\beta}}$ to get the desired result in (2.1). □

Theorem 2.1. *Let $\kappa_1,\kappa_2\in\mathbb{R}^+$ with $\kappa_1<\kappa_2$, $\beta\in(0,1]$ and $h:[\kappa_1,\kappa_2]\to\mathbb{R}$ be a differentiable function on (κ_1,κ_2). Then the inequality*

$$
\begin{aligned}
&\left|h\left(\frac{\kappa_1+\kappa_2}{2}\right)-\frac{\beta}{\kappa_2^{\beta}-\kappa_1^{\beta}}\int_{\kappa_1}^{\kappa_2}h(x)d_\beta x\right|\\
\leq\ &\frac{(\kappa_2-\kappa_1)}{2(\kappa_2^{\beta}-\kappa_1^{\beta})}\Bigg[\left(\frac{7\kappa_1^{\beta}+2\kappa_1^{\beta-1}\kappa_2+2\kappa_2^{\beta-1}\kappa_1+\kappa_2^{\beta}-12\kappa_1^{\beta}}{24}\right)|h'(\kappa_1)|\\
&+\eta(|h'(\kappa_2)|,|h'(\kappa_1)|)\left(\frac{11\kappa_1^{\beta}+5\kappa_1^{\beta-1}\kappa_2+5\kappa_2^{\beta-1}\kappa_1+3\kappa_2^{\beta}-24\kappa_1^{\beta}}{192}\right)\\
&+|h'(\kappa_1)|\left(\frac{\kappa_2^{\beta}-\kappa_1^{\beta}}{8}\right)+\eta(|h'(\kappa_2)|,|h'(\kappa_1)|)\left(\frac{\kappa_2^{\beta}-\kappa_1^{\beta}}{12}\right)\Bigg].
\end{aligned}
\tag{2.2}
$$

holds if $D_\beta(h)\in L^1_\beta([\kappa_1,\kappa_2])$ and $|h'|$ is η-convex on $[\kappa_1,\kappa_2]$.

Proof. Let $\varphi_1(y)=y^{\beta-1}$ and $\varphi_2(y)=-y^{\beta}, y>0,\beta\in(0,1]$ clearly the functions φ_1 and φ_2 are convex. Now using Lemma 2.1 and the convexity of φ_1, φ_2 and η-convexity of $|h'|$, we have

$$
\begin{aligned}
&\left|h\left(\frac{\kappa_1+\kappa_2}{2}\right)-\frac{\beta}{\kappa_2^{\beta}-\kappa_1^{\beta}}\int_{\kappa_1}^{\kappa_2}h(x)d_\beta x\right|\\
\leq\ &\frac{\kappa_2-\kappa_1}{\kappa_2^{\beta}-\kappa_1^{\beta}}\left[\int_0^{\frac{1}{2}}\big(((1-s)\kappa_1+s\kappa_2)^{\beta}-\kappa_1^{\beta}\big)|h'((1-s)\kappa_1+s\kappa_2)|ds\right.
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{\frac{1}{2}}^{1}\left(\kappa_2^{\beta}-((1-s)\kappa_1+s\kappa_2)^{\beta}\right)|h'((1-s)\kappa_1+s\kappa_2)|ds\Bigg]\\
=\ &\frac{\kappa_2-\kappa_1}{\kappa_2^{\beta}-\kappa_1^{\beta}}\Bigg[\int_0^{\frac{1}{2}}\left(((1-s)\kappa_1+s\kappa_2)^{\beta+1-1}-\kappa_1^{\beta}\right)|h'((1-s)\kappa_1+s\kappa_2)|ds\\
&+\int_{\frac{1}{2}}^{1}\left(\kappa_2^{\beta}-((1-s)\kappa_1+s\kappa_2)^{\beta}\right)|h'((1-s)\kappa_1+s\kappa_2)|ds\Bigg]\\
\leq\ &\frac{\kappa_2-\kappa_1}{\kappa_2^{\beta}-\kappa_1^{\beta}}\Bigg[\int_0^{\frac{1}{2}}\left(((1-s)\kappa_1+s\kappa_2)^{\beta-1}((1-s)\kappa_1+s\kappa_2)-\kappa_1^{\beta}\right)|h'(((1-s)\kappa_1+s\kappa_2))|ds\\
&+\int_{\frac{1}{2}}^{1}\left(\kappa_2^{\beta}-((1-s)\kappa_1^{\beta}+s\kappa_2^{\beta})\right)|h'((1-s)\kappa_1+s\kappa_2)|ds\Bigg]\\
\leq\ &\frac{\kappa_2-\kappa_1}{\kappa_2^{\beta}-\kappa_1^{\beta}}\Bigg[\int_0^{\frac{1}{2}}\left((((1-s)\kappa_1^{\beta-1}+s\kappa_2^{\beta-1}))(((1-s)\kappa_1+s\kappa_2))-\kappa_1^{\beta}\right)\\
&\times|h'(((1-s)\kappa_1+s\kappa_2))|ds+\int_{\frac{1}{2}}^{1}\left(\kappa_2^{\beta}-\left(((1-s)\kappa_1^{\beta}+s\kappa_2^{\beta})\right)\right)\\
&\times|h'((1-s)\kappa_1+s\kappa_2)|ds\Bigg]\\
\leq\ &\frac{\kappa_2-\kappa_1}{\kappa_2^{\beta}-\kappa_1^{\beta}}\Bigg[\int_0^{\frac{1}{2}}\left((((1-s)\kappa_1^{\beta-1}+s\kappa_2^{\beta-1}))(((1-s)\kappa_1+s\kappa_2))-\kappa_1^{\beta}\right)\\
&\times\big[|h'(\kappa_1)|+s\eta(|h'(\kappa_2)|,|h'(\kappa_1)|)\big]ds+\int_{\frac{1}{2}}^{1}\left(\kappa_2^{\beta}-\left(((1-s)\kappa_2^{\beta}+s\kappa_1^{\beta})\right)\right)\\
&\times\big[|h'(\kappa_1)|+s\eta(|h'(\kappa_2)|,|h'(\kappa_1)|)\big]ds\Bigg].
\end{aligned}
$$

Evaluating all the above integrals, we have the following

$$
\begin{aligned}
&\left|h\left(\frac{\kappa_1+\kappa_2}{2}\right)-\frac{\beta}{\kappa_2^{\beta}-\kappa_1^{\beta}}\int_{\kappa_1}^{\kappa_2}h(x)d_{\beta}x\right|\\
&\leq\ \frac{\kappa_2-\kappa_1}{\kappa_2^{\beta}-\kappa_1^{\beta}}\Bigg[\left(\frac{7\kappa_1^{\beta}+2\kappa_1^{\beta-1}\kappa_2+2\kappa_2^{\beta-1}\kappa_1+\kappa_2^{\beta}-12\kappa_1^{\beta}}{24}\right)|h'(\kappa_1)|\\
&\quad+\eta(|h'(\kappa_2)|,|h'(\kappa_1)|)\left(\frac{11\kappa_1^{\beta}+5\kappa_1^{\beta-1}\kappa_2+5\kappa_2^{\beta-1}\kappa_1+3\kappa_2^{\beta}-24\kappa_1^{\beta}}{192}\right)\\
&\quad+|h'(\kappa_1)|\left(\frac{\kappa_2^{\beta}-\kappa_1^{\beta}}{8}\right)+\eta(|h'(\kappa_2)|,|h'(\kappa_1)|)\left(\frac{\kappa_2^{\beta}-\kappa_1^{\beta}}{12}\right)\Bigg].
\end{aligned}
$$

□

Corollary 2.1. *If we replace $\eta(\kappa_2,\kappa_1)=\kappa_2-\kappa_1$, we get*

$$\begin{aligned}
&\left|h\left(\frac{\kappa_1+\kappa_2}{2}\right)-\frac{\beta}{\kappa_2^\beta-\kappa_1^\beta}\int_{\kappa_1}^{\kappa_2}h(x)d_\beta x\right|\\
\leq\ &\frac{\kappa_2-\kappa_1}{\kappa_2^\beta-\kappa_1^\beta}\Bigg[\left(\frac{-27\kappa_1^\beta+11\kappa_1^{\beta-1}\kappa_2+11\kappa_2^{\beta-1}\kappa_1+5\kappa_2^\beta}{192}\right)|h'(\kappa_1)|\\
&+|h'(\kappa_2)|\left(\frac{11\kappa_1^\beta+5\kappa_1^{\beta-1}\kappa_2+5\kappa_2^{\beta-1}\kappa_1+3\kappa_2^\beta-24\kappa_1^\beta}{192}\right)\\
&+|h'(\kappa_1)|\left(\frac{\kappa_2^\beta-\kappa_1^\beta}{24}\right)+|h'(\kappa_2)|\left(\frac{\kappa_2^\beta-\kappa_1^\beta}{12}\right)\Bigg].
\end{aligned}$$

Theorem 2.2. *Let $\beta\in(0,1]$, $p,q>1$ with $p^{-1}+q^{-1}=1$, $\kappa_1,\kappa_2>0$ with $\kappa_2>\kappa_1$ and $h:[(\kappa_1,\kappa_2)]\to\mathbb{R}$ be a differentiable function on (κ_1,κ_2) such that $D_\beta(h)\in L_\beta([\kappa_1,\kappa_2])$. Then*

$$\left|h\left(\frac{\kappa_1+\kappa_2}{2}\right)-\frac{\beta}{\kappa_2^\beta-\kappa_1^\beta}\int_{\kappa_1}^{\kappa_2}h(x)d_\beta x\right|$$

$$\begin{aligned}
\leq\ &\frac{\kappa_2-\kappa_1}{\kappa_2^\beta-\kappa_1^\beta}\Bigg[\left(\overline{\mathcal{A}}_1(\beta,p)\right)^{\frac{1}{p}}\left(\frac{4|h'(\kappa_1)|^q+\eta(|h'(\kappa_2)|^q,|h'(\kappa_1)|^q)}{8}\right)^{\frac{1}{q}}\\
&+\left(\overline{\mathcal{A}}_2(\beta,p)\right)^{\frac{1}{p}}\left(\frac{4|h'(\kappa_1)|^q+3\eta(|h'(\kappa_2)|^q,|h'(\kappa_1)|^q)}{8}\right)^{\frac{1}{q}}\Bigg],
\end{aligned}\tag{2.3}$$

if $|h'|^q$ is η-convex, where

$$\begin{aligned}
\overline{\mathcal{A}}_1(\beta,p) &= \int_0^{\frac{1}{2}}\left((((1-s)\kappa_1^{\beta-1}+s\kappa_2^{\beta-1}))(((1-s)\kappa_1+s\kappa_2))-\kappa_1^\beta\right)^p ds\\
\overline{\mathcal{A}}_2(\beta,p) &= \int_{\frac{1}{2}}^1\left(\kappa_2^\beta-((1-s)\kappa_1^\beta+s\kappa_2^\beta)\right)^p ds.
\end{aligned}$$

Proof. We clearly see that

$$\begin{aligned}
&\left|h\left(\frac{\kappa_1+\kappa_2}{2}\right)-\frac{\beta}{\kappa_2^\beta-\kappa_1^\beta}\int_{\kappa_1}^{\kappa_2}h(x)d_\beta x\right|\\
\leq\ &\frac{\kappa_2-\kappa_1}{\kappa_2^\beta-\kappa_1^\beta}\Bigg[\int_0^{\frac{1}{2}}\left(((1-s)\kappa_1+s\kappa_2)^{\beta-1}((1-s)\kappa_1+s\kappa_2)-\kappa_1^\beta\right)|h'(((1-s)\kappa_1+s\kappa_2))|ds\\
&+\int_{\frac{1}{2}}^1\left(\kappa_2^\beta-((1-s)\kappa_1^\beta+s\kappa_2^\beta)\right)|h'((1-s)\kappa_1+s\kappa_2)|ds\Bigg].
\end{aligned}$$

Now by the Hölder's inequality

$$\begin{aligned}
&\int_0^{\frac{1}{2}} \left((((1-s)\kappa_1^{\beta-1}+s\kappa_2^{\beta-1}))(((1-s)\kappa_1+s\kappa_2))-\kappa_1^{\beta}\right)|h'(((1-s)\kappa_1+s\kappa_2))|ds \\
\leq\ & \left(\int_0^{\frac{1}{2}} \left((((1-s)\kappa_1^{\beta-1}+s\kappa_2^{\beta-1}))(((1-s)\kappa_1+s\kappa_2))-\kappa_1^{\beta}\right)^p ds\right)^{\frac{1}{p}} \\
&\times\left(\int_0^{\frac{1}{2}} |h'(((1-s)\kappa_1+s\kappa_2))|^q ds\right)^{\frac{1}{q}} \\
\leq\ & \left(\int_0^{\frac{1}{2}} \left((((1-s)\kappa_1^{\beta-1}+s\kappa_2^{\beta-1}))(((1-s)\kappa_1+s\kappa_2))-\kappa_1^{\beta}\right)^p ds\right)^{\frac{1}{p}} \\
&\times\left(\int_0^{\frac{1}{2}} |h'(\kappa_1)|^q + s\eta(|h'(\kappa_2)|^q,|h'(\kappa_1)|^q)ds\right)^{\frac{1}{q}} \\
=\ & \left(\overline{\mathcal{A}}_1(\beta,p)\right)^{\frac{1}{p}}\left(\frac{4|h'(\kappa_1)|^q+\eta(|h'(\kappa_2)|^q,|h'(\kappa_1)|^q)}{8}\right)^{\frac{1}{q}}
\end{aligned}$$

and similarly, we have

$$\begin{aligned}
&\int_{\frac{1}{2}}^1 \left(\kappa_2^{\beta}-((1-s)\kappa_1^{\beta}+s\kappa_2^{\beta})\right)\left|h'((1-s)\kappa_1+s\kappa_2)\right|ds \\
\leq\ & \left(\int_{\frac{1}{2}}^1 \left(\kappa_2^{\beta}-((1-s)\kappa_1^{\beta}+s\kappa_2^{\beta})\right)^p ds\right)^{\frac{1}{p}}\left(\left|h'((1-s)\kappa_1+s\kappa_2)\right|^q ds\right)^{\frac{1}{q}} \\
\leq\ & \left(\int_{\frac{1}{2}}^1 \left(\kappa_2^{\beta}-((1-s)\kappa_1^{\beta}+s\kappa_2^{\beta})\right)^p ds\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1 |h'(\kappa_1)|^q+s\eta(|h'(\kappa_2)|^q,|h'(\kappa_1)|^q)ds\right)^{\frac{1}{q}} \\
=\ & \left(\overline{\mathcal{A}}_2(\beta,p)\right)^{\frac{1}{p}}\left(\frac{4|h'(\kappa_1)|^q+3\eta(|h'(\kappa_2)|^q,|h'(\kappa_1)|^q)}{8}\right)^{\frac{1}{q}}.
\end{aligned}$$

□

Corollary 2.2. *If we replace* $\eta(\kappa_2,\kappa_1)=\kappa_2-\kappa_1$*, we get*

$$\begin{aligned}
&\left|h\left(\frac{\kappa_1+\kappa_2}{2}\right)-\frac{\beta}{\kappa_2^{\beta}-\kappa_1^{\beta}}\int_{\kappa_1}^{\kappa_2} h(x)d_{\beta}x\right| \\
\leq\ & \frac{\kappa_2-\kappa_1}{\kappa_2^{\beta}-\kappa_1^{\beta}}\left[\left(\overline{\mathcal{A}}_1(\beta,p)\right)^{\frac{1}{p}}\left(\frac{3|h'(\kappa_1)|^q+|h'(\kappa_2)|^q}{8}\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\overline{\mathcal{A}}_2(\beta,p)\right)^{\frac{1}{p}}\left(\frac{|h'(\kappa_1)|^q+3|h'(\kappa_2)|^q}{8}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

where

$$\begin{aligned}\overline{\mathcal{A}}_1(\beta,p) &= \int_0^{\frac{1}{2}} \left((((1-s)\kappa_1^{\beta-1}+s\kappa_2^{\beta-1}))(((1-s)\kappa_1+s\kappa_2))-\kappa_1^{\beta}\right)^p ds\\ \overline{\mathcal{A}}_2(\beta,p) &= \int_{\frac{1}{2}}^1 \left(\kappa_2^{\beta}-((1-s)\kappa_1^{\beta}+s\kappa_2^{\beta})\right)^p ds.\end{aligned}$$

Theorem 2.3. *Let $\beta\in(0,1]$, $q>1$, $\kappa_1,\kappa_2>0$ with $\kappa_2>\kappa_1$ and $h:[(\kappa_1,\kappa_2)]\to\mathbb{R}$ be a differentiable function on (κ_1,κ_2) such that $D_\beta(h)\in L_\beta([\kappa_1,\kappa_2])$. Then*

$$\begin{aligned}&\left|h\left(\frac{\kappa_1+\kappa_2}{2}\right)-\frac{\beta}{\kappa_2^{\beta}-\kappa_1^{\beta}}\int_{\kappa_1}^{\kappa_2}h(x)d_\beta x\right|\\ \leq\ &\frac{\kappa_2-\kappa_1}{\kappa_2^{\beta}-\kappa_1^{\beta}}\Bigg[\left(\overline{A}_1(\beta)\right)^{1-\frac{1}{q}}\Bigg(|h'(\kappa_1)|^q\left(\frac{7\kappa_1^{\beta}+2\kappa_1^{\beta-1}\kappa_2+2\kappa_2^{\beta-1}\kappa_1+\kappa_2^{\beta}-12\kappa_1^{\beta}}{24}\right)\\ &+\eta(|h'(\kappa_2)|^q,|h'(\kappa_1)|^q)\left(\frac{11\kappa_1^{\beta}+5\kappa_1^{\beta-1}\kappa_2+5\kappa_2^{\beta-1}\kappa_1+3\kappa_2^{\beta}-24\kappa_1^{\beta}}{192}\right)\Bigg)\\ &+\left(\overline{B}_1(\beta)\right)^{1-\frac{1}{q}}\left(|h'(\kappa_1)|^q\left(\frac{\kappa_2^{\beta}-\kappa_1^{\beta}}{8}\right)+\eta(|h'(\kappa_2)|^q,|h'(\kappa_1)|^q)\left(\frac{\kappa_2^{\beta}-\kappa_1^{\beta}}{12}\right)\right)\Bigg]\end{aligned}\tag{2.4}$$

if $|h'|^q$ is η-convex, where

$$\overline{A}_1(\beta) = \frac{7\kappa_1^{\beta}+2\kappa_1^{\beta-1}\kappa_2+2\kappa_2^{\beta-1}\kappa_1+\kappa_2^{\beta}-12\kappa_1^{\beta}}{24},\quad \overline{B}_1(\beta)=\frac{\kappa_2^{\beta}-\kappa_1^{\beta}}{8}.$$

Proof. We clearly see that

$$\begin{aligned}&\left|h\left(\frac{\kappa_1+\kappa_2}{2}\right)-\frac{\beta}{\kappa_2^{\beta}-\kappa_1^{\beta}}\int_{\kappa_1}^{\kappa_2}h(x)d_\beta x\right|\\ \leq\ &\frac{\kappa_2-\kappa_1}{\kappa_2^{\beta}-\kappa_1^{\beta}}\Bigg[\int_0^{\frac{1}{2}}\left(((1-s)\kappa_1+s\kappa_2)^{\beta-1}((1-s)\kappa_1+s\kappa_2)-\kappa_1^{\beta}\right)|h'(((1-s)\kappa_1+s\kappa_2))|ds\\ &+\int_{\frac{1}{2}}^1\left(\kappa_2^{\beta}-((1-s)\kappa_1^{\beta}+s\kappa_2^{\beta})\right)|h'((1-s)\kappa_1+s\kappa_2)|ds\Bigg].\end{aligned}$$

Now by the power-mean inequality

$$\begin{aligned}&\int_0^{\frac{1}{2}}\left((((1-s)\kappa_1^{\beta-1}+s\kappa_2^{\beta-1}))(((1-s)\kappa_1+s\kappa_2))-\kappa_1^{\beta}\right)|h'(((1-s)\kappa_1+s\kappa_2))|ds\\ \leq\ &\left(\int_0^{\frac{1}{2}}\left((((1-s)\kappa_1^{\beta-1}+s\kappa_2^{\beta-1}))(((1-s)\kappa_1+s\kappa_2))-\kappa_1^{\beta}\right)ds\right)^{1-\frac{1}{q}}\\ &\times\left(\int_0^1\left((((1-s)\kappa_1^{\beta-1}+s\kappa_2^{\beta-1}))(((1-s)\kappa_1+s\kappa_2))-\kappa_1^{\beta}\right)|h'(((1-s)\kappa_1+s\kappa_2))|^q ds\right)^{\frac{1}{q}}\end{aligned}$$

and similarly, we have

$$\begin{aligned}
&\int_{\frac{1}{2}}^{1}\left(\kappa_2^{\beta}-((1-s)\kappa_1^{\beta}+s\kappa_2^{\beta})\right)\left|h'((1-s)\kappa_1+s\kappa_2)\right|ds\\
&\leq \left(\int_{\frac{1}{2}}^{1}\left(\kappa_2^{\beta}-((1-s)\kappa_1^{\beta}+s\kappa_2^{\beta})\right)ds\right)^{1-\frac{1}{q}}\\
&\quad\times\left(\int_{\frac{1}{2}}^{1}\left(\kappa_2^{\beta}-((1-s)\kappa_1^{\beta}+s\kappa_2^{\beta})\right)\left|h'((1-s)\kappa_2+s\kappa_1)\right|^q ds\right)^{\frac{1}{q}}.
\end{aligned}$$

Now by the η-convexity of $|h'|^q$ from above, we have

$$\begin{aligned}
&\int_0^{\frac{1}{2}}\left((((1-s)\kappa_1^{\beta-1}+s\kappa_2^{\beta-1}))(((1-s)\kappa_1+s\kappa_2))-\kappa_1^{\beta}\right)|h'(((1-s)\kappa_1+s\kappa_2))|^q ds\\
&\leq \int_0^{\frac{1}{2}}\left((((1-s)\kappa_1^{\beta-1}+s\kappa_2^{\beta-1}))(((1-s)\kappa_1+s\kappa_2))-\kappa_1^{\beta}\right)\\
&\quad\times\left[|h'(\kappa_1)|^q+s\eta(|h'(\kappa_2)|^q,|h'(\kappa_1)|^q)\right]ds\\
&= |h'(\kappa_1)|^q\int_0^{\frac{1}{2}}\left((((1-s)\kappa_1^{\beta-1}+s\kappa_2^{\beta-1}))(((1-s)\kappa_1+s\kappa_2))-\kappa_1^{\beta}\right)ds\\
&\quad+\eta(|h'(\kappa_2)|^q,|h'(\kappa_1)|^q)\int_0^{\frac{1}{2}}\left((((1-s)\kappa_1^{\beta-1}+s\kappa_2^{\beta-1}))(((1-s)\kappa_1+s\kappa_2))-\kappa_1^{\beta}\right)s\,ds\\
&= |h'(\kappa_1)|^q\left(\frac{7\kappa_1^{\beta}+2\kappa_1^{\beta-1}\kappa_2+2\kappa_2^{\beta-1}\kappa_1+\kappa_2^{\beta}-12\kappa_1^{\beta}}{24}\right)+\eta(|h'(\kappa_2)|^q,|h'(\kappa_1)|^q)\\
&\quad\times\left(\frac{11\kappa_1^{\beta}+5\kappa_1^{\beta-1}\kappa_2+5\kappa_2^{\beta-1}\kappa_1+3\kappa_2^{\beta}-24\kappa_1^{\beta}}{192}\right)
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\frac{1}{2}}^{1}\left(\kappa_2^{\beta}-((1-s)\kappa_1^{\beta}+s\kappa_2^{\beta})\right)|h'((1-s)\kappa_1+s\kappa_2)|^q ds\\
&\leq \int_{\frac{1}{2}}^{1}\left(\kappa_2^{\beta}-((1-s)\kappa_1^{\beta}+s\kappa_2^{\beta})\right)\left[|h'(\kappa_1)|^q+s\eta(|h'(\kappa_2)|^q,|h'(\kappa_1)|^q)\right]ds\\
&= |h'(\kappa_1)|^q\left(\frac{\kappa_2^{\beta}-\kappa_1^{\beta}}{8}\right)+\eta(|h'(\kappa_2)|^q,|h'(\kappa_1)|^q)\left(\frac{\kappa_2^{\beta}-\kappa_1^{\beta}}{12}\right).
\end{aligned}$$

Where, we have also used the facts that

$$\begin{aligned}
&\int_0^{\frac{1}{2}}\left((1-s)\kappa_1^{\beta-1}\ +\ s\kappa_2^{\beta-1})((1-s)\kappa_1+s\kappa_2)-\kappa_1^{\beta}\right)dt\\
&\qquad= \overline{A}_1(\beta)=\frac{7\kappa_1^{\beta}+2\kappa_1^{\beta-1}\kappa_2+2\kappa_2^{\beta-1}\kappa_1+\kappa_2^{\beta}-12\kappa_1^{\beta}}{24}
\end{aligned}$$

$$\int_{\frac{1}{2}}^{1}\left(\kappa_2^{\beta}-((1-s)\kappa_2^{\beta}+s\kappa_1^{\beta})\right)ds=\overline{B}_1(\beta)=\frac{\kappa_2^{\beta}-\kappa_1^{\beta}}{8}.$$
(2.5)

Hence, we have the result in (2.6). □

Corollary 2.3. *If we replace* $\eta(\kappa_2,\kappa_1)=\kappa_2-\kappa_1$*, we get*

$$\begin{aligned}
&\left|h\left(\frac{\kappa_1+\kappa_2}{2}\right)-\frac{\beta}{\kappa_2^{\beta}-\kappa_1^{\beta}}\int_{\kappa_1}^{\kappa_2}h(x)d_{\beta}x\right| \\
\leq\ & \frac{\kappa_2-\kappa_1}{\kappa_2^{\beta}-\kappa_1^{\beta}}\Bigg[\left(\overline{A}_1(\beta)\right)^{1-\frac{1}{q}}\left(|h'(\kappa_1)|^q\left(\frac{-27\kappa_1^{\beta}+11\kappa_1^{\beta-1}\kappa_2+11\kappa_2^{\beta-1}\kappa_1+5\kappa_2^{\beta}}{192}\right)\right. \\
&\left.+|h'(\kappa_2)|^q\left(\frac{11\kappa_1^{\beta}+5\kappa_1^{\beta-1}\kappa_2+5\kappa_2^{\beta-1}\kappa_1+3\kappa_2^{\beta}-24\kappa_1^{\beta}}{192}\right)\right) \\
&+\left(\overline{B}_1(\beta)\right)^{1-\frac{1}{q}}\left(|h'(\kappa_1)|^q\left(\frac{\kappa_2^{\beta}-\kappa_1^{\beta}}{24}\right)+|h'(\kappa_2)|^q\left(\frac{\kappa_2^{\beta}-\kappa_1^{\beta}}{12}\right)\right)\Bigg],
\end{aligned}$$

where

$$\overline{A}_1(\beta)=\frac{7\kappa_1^{\beta}+2\kappa_1^{\beta-1}\kappa_2+2\kappa_2^{\beta-1}\kappa_1+\kappa_2^{\beta}-12\kappa_1^{\beta}}{24},\quad \overline{B}_1(\beta)=\frac{\kappa_2^{\beta}-\kappa_1^{\beta}}{8}. \tag{2.6}$$

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