FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 37, No 5 (2022), 1007–1020
https://doi.org/10.22190/FUMI220924069O
Original Scientific Paper

SOME PROPERTIES OF COMPLEX GOLDEN CONJUGATE CONNECTIONS

Mustafa Özkan and Tuğba Tamirci

Department of Mathematics, Faculty of Science
Gazi University, 06500 Ankara, Turkey

Abstract. In this study, we mainly aim to investigate the fundamentals of complex golden conjugate connections. We study the relation between complex conjugate connections and complex golden conjugate connections. We first investigate the similarity of structural tensors and virtual tensors that are defined in almost complex geometry with almost complex golden geometry and then describe complex golden conjugate connections through the tensors stated here.
Key words: complex golden conjugate, structural tensors, virtual tensors.

1. Introduction

In [17, 18], it is well known that many polynomial structures on an n–dim a differentiable manifold M^n arise from $\mathcal{J}_M$ which are C^∞–tensor fields of type $(1,1)$ and are obtained by finding the roots of the equation

$$Q\left(\mathcal{J}_M\right) = \mathcal{J}_M^m + b_m \mathcal{J}_M^{m-1} + ... + b_2 \mathcal{J}_M + b_1 I = 0,$$

where I is the identity $(1,1)$ tensor field on M^n and $b_1, b_2, b_3, \ldots, b_m$ are real numbers, at every point of M^n. Almost tangent, almost product, almost complex, golden, silver and metallic structure are some examples. Several authors have recently studied on polynomial structures (see, [10, 2, 3, 20, 21, 24, 25, 26, 27, 19, 12, 23]).

Received September 24, 2022, accepted: November 10, 2022
Communicated by Uday Chand De
Corresponding Author: Mustafa Özkan, Department of Mathematics, Faculty of Science, Gazi University, 06500 Ankara, Turkey | E-mail: ozkanm@gazi.edu.tr
2010 *Mathematics Subject Classification.* Primary 53C07; Secondary 53C15, 53C25

The solution of the equation $Q(\mathcal{J}_M) = \mathcal{J}_M^2 - \mathcal{J}_M + \frac{3}{2}I$, which is known as the structure polynomial is called a complex golden structure. This name was inspired by the fact that for complex golden ratio introduced by Crasmareanu and Hreţcanu [10] this is the root of the quadratic equation $x^2 - x + \frac{3}{2} = 0$, namely $\phi_c = \frac{1+i\sqrt{5}}{2}$. It was shown in [10] that any complex golden structure induces an almost complex structure and any almost complex structure determines complex golden structure. Based on these considerations, we can say that there is a strong interrelation between complex golden structure and almost complex structure on a differentiable manifold.

Let ∇ be a linear connection on an differentiable manifold M^n. If $\mathcal{J}_M$ is parallel with respect to ∇ i.e. $\nabla\mathcal{J}_M = 0$, ∇ is called $\mathcal{J}_M$–connection. Cruceanu [11] introduced the notion of $\mathcal{J}_M$–conjugate connection associated to ∇. Several authors studied $\mathcal{J}_M$–conjugate connection on structures: almost complex structures in [4], almost product structure in [5], golden structures in [6], almost tangent structures in [7] and metallic structures in [8].

The present work deals with conjugate connections on another structures, namely complex golden conjugate connection. We study some properties of the complex golden conjugate connections. We express the complex golden conjugate connections in terms of structural and virtual tensors from the almost complex structure. In the later part of this work, we concentrate to study the existence of duality between the complex golden and complex conjugate connections.

Let M^n be an n–dim differentiable manifold of class C^∞. We denote by $F(M^n)$ the algebra of C^∞–real functions on M^n, $\Gamma(TM) = \mathfrak{X}(M^n)$ the Lie algebra of vector fields on M^n, $\mathfrak{J}_l^k(M^n)$ the $F(M^n)$–module of tensor fields of (k,l)–type on M^n, $\mathcal{C}(M^n)$ the set of all linear connections on M^n and $\mathcal{C}_{\mathcal{J}_M}(M^n)$ the set of all $\mathcal{J}_M$–connections on M^n .

Definition 1.1. [10] Let $\mathcal{J}_M$ be a $(1,1)$–tensor field on M^n. $\mathcal{J}_M$ which satisfies the equation

$$\mathcal{J}_M^2 - \mathcal{J}_M + \frac{3}{2}I = 0 \tag{1.1}$$

is called an almost complex golden structure on M^n. The pair $(M^n, \mathcal{J}_M)$ is an almost complex golden manifold.

We know that such a complex structure exists only when the manifold is even-dimensional. Therefore, we will get $n = 2k$.

Returning to $M^{2k} = \mathbb{R}^2$ we arrive at the equation

$$x^2 - x + \frac{3}{2} = 0$$

with solutions $x_1 = \frac{1}{2} + \frac{1}{2}i\sqrt{5}$, $x_2 = \overline{x}_1 = \frac{1}{2} - \frac{1}{2}i\sqrt{5}$.

The complex number

$$\phi_c = \frac{1+i\sqrt{5}}{2}$$

will be called complex golden ratio [10].

Unlike the paper [10] and in support of what was stated in [13, 3] we use the word "almost" to refer to general complex golden structures on a manifold. We say that a manifold M^{2k} is a complex golden manifold if it has an integrable almost complex golden structure.

Vanžura [29] gave two necessary and sufficient conditions for integrability of a polynomial structure. One of them, a polynomial structure $\mathcal{J}_M$ is integrable iff the Nijenhuis tensor $N_{\mathcal{J}_M}$ vanishes. Other, a polynomial structure $\mathcal{J}_M$ is integrable if and only if there exists a torsion–free linear $\mathcal{J}_M$–connection. Gezer et al. [14, 15] introduced another sufficient condition of the integrability of golden structures and metallic structures on Riemannian manifolds, respectively.

The current study involves the investigation of complex golden conjugate connection and it could be thought as the continuation of a series of previous papers [4, 5, 6, 7] and [8]. In last part of Section 2 of present study, we make use of the pair structural tensor and virtual tensor which might be considered as a very useful tool which was also defined for the almost complex geometry in [4]. In Section, we study the duality between the complex golden and complex conjugate connection.

2. Complex Golden Conjugate Connection

From now on, for the sake of simplicity, $\nabla^{(J_M)}$ will denote the complex golden conjugate connection of ∇

$$\nabla^{(\mathcal{J}_M)} := \nabla - \mathcal{J}_M \circ \nabla \mathcal{J}_M, \tag{2.1}$$

and hence if $\nabla \in \mathcal{C}_{\mathcal{J}_M}(M^{2k})$ then $\nabla^{(\mathcal{J}_M)} \in \mathcal{C}_{\mathcal{J}_M}(M^{2k})$. The mentioned connection could be given in detail as

$$\nabla^{(\mathcal{J}_M)}_X Y = -\frac{1}{2}\nabla_X Y - \mathcal{J}_M(\nabla_X \mathcal{J}_M Y - \nabla_X Y), \tag{2.2}$$

where $X, Y \in \mathfrak{X}(M^{2k})$.

Definition 2.1. Let g be a semi-Riemannian metric on M^{2k} which satisfies

$$g(\mathcal{J}_M X, Y) = g(X, \mathcal{J}_M Y) \text{ or } g\left(\mathcal{J}_M X, \mathcal{J}_M Y\right) = g\left(\mathcal{J}_M X, Y\right) - \frac{3}{2} g\left(X, Y\right)$$

namely compatible with $\mathcal{J}_M$ for every $X, Y \in \mathfrak{X}(M^{2k})$. Then the pair $(g, \mathcal{J}_M)$ is said to be an almost golden Norden structure and $(M^{2k}, g, \mathcal{J}_M)$ is said to be an almost golden Norden manifold.

Proposition 2.1. *Suppose that $\mathcal{J}_M$ is an almost complex golden structure, ∇ is a linear connection and $\nabla^{(\mathcal{J}_M)}$ is a complex golden conjugate connection of ∇. Then*

(i) $\nabla^{(\mathcal{J}_M)}\mathcal{J}_M = \mathcal{J}_M \circ \nabla \mathcal{J}_M - 2\nabla \mathcal{J}_M$*; so that if* $\nabla \in \mathcal{C}_{\mathcal{J}_M}(M^{2k})$ *then* $\nabla^{(\mathcal{J}_M)} \in \mathcal{C}_{\mathcal{J}_M}(M^{2k})$.

(ii) $T_{\nabla^{(\mathcal{J}_M)}} = T_\nabla - \mathcal{J}_M(d^\nabla \mathcal{J}_M)$, *in which* d^∇ *denotes the exterior covariant derivative that is induced by* ∇, $(d^\nabla \mathcal{J}_M)(X,Y) := (\nabla_X \mathcal{J}_M)Y - (\nabla_Y \mathcal{J}_M)X$; *it follows that for* $\nabla \in \mathcal{C}_{\mathcal{J}_M}(M^{2k})$, *the connection* ∇ *and* $\nabla^{(\mathcal{J}_M)}$ *have the same torsion.*

(iii)

$$\begin{aligned} R_{\nabla^{(\mathcal{J}_M)}}(X,Y,Z) &= (\mathcal{J}_M - \frac{1}{2}I)R_\nabla(X,Y,Z) - \mathcal{J}_M R_\nabla(X,Y,\mathcal{J}_M Z) \\ &+\frac{1}{2}(\mathcal{J}_M - I)(A^\nabla \mathcal{J}_M)(X,Y;Z) \\ &+\frac{1}{2}(A^\nabla \mathcal{J}_M)(X,Y;\mathcal{J}_M Z), \end{aligned}$$

where $(A^\nabla \mathcal{J}_M)(X,Y;Z) = (\nabla_X \mathcal{J}_M)\nabla_Y Z - (\nabla_Y \mathcal{J}_M)\nabla_X Z$; *it follows that for* $\nabla \in \mathcal{C}_{\mathcal{J}_M}(M^{2k})$, *if* ∇ *is flat then* $\nabla^{(\mathcal{J}_M)}$ *is so.*

(iv) Let $(M^{2k}, g, \mathcal{J}_M)$ *be an almost golden Norden manifold. Then*

$$\begin{aligned} (\nabla_X^{(\mathcal{J}_M)} g)(Y,Z) &= -(\nabla_X g)(\mathcal{J}_M Y, \mathcal{J}_M Z) + (\nabla_X g)(Y, \mathcal{J}_M Z) \\ &-\frac{1}{2}(\nabla_X g)(Y,Z) + g(Y, (\nabla_X \mathcal{J}_M)Z). \end{aligned}$$

We can conclude from here that, whenever ∇ *is a* g*–metric connection (i.e.* $\nabla g = 0$*) in* $\mathcal{C}_{\mathcal{J}_M}(M^{2k})$ *then* $\nabla^{(\mathcal{J}_M)}$ *is also a* g*–metric connection in* $\mathcal{C}_{\mathcal{J}_M}(M^{2k})$.

Proof.

(i) Using (1.1) and (2.2)

$$\begin{aligned} (\nabla_X^{(\mathcal{J}_M)} \mathcal{J}_M)Y &= \nabla_X^{(\mathcal{J}_M)} \mathcal{J}_M Y - \mathcal{J}_M \nabla_X^{(\mathcal{J}_M)} Y \\ &= -\frac{1}{2}\nabla_X \mathcal{J}_M Y + 2\mathcal{J}_M \nabla_X Y + \mathcal{J}_M^2(\nabla_X \mathcal{J}_M Y - \nabla_X Y) \\ &= -2\nabla_X \mathcal{J}_M Y + \mathcal{J}_M \nabla_X Y + \mathcal{J}_M \nabla_X \mathcal{J}_M Y + \frac{3}{2}\nabla_X Y \\ &= -2((\nabla_X \mathcal{J}_M)Y) + \mathcal{J}_M((\nabla_X \mathcal{J}_M)Y), \end{aligned}$$

and then

$$\nabla^{(\mathcal{J}_M)} \mathcal{J}_M = -2\nabla \mathcal{J}_M + \mathcal{J}_M \circ \nabla \mathcal{J}_M.$$

(ii) A direct computation gives

$$\begin{aligned}
T_{\nabla^{(\mathcal{J}_M)}}(X,Y) &= \nabla_X^{(\mathcal{J}_M)}Y - \nabla_Y^{(\mathcal{J}_M)}X - [X,Y] \\
&= -\frac{1}{2}\nabla_X Y - \mathcal{J}_M(\nabla_X \mathcal{J}_M Y - \nabla_X Y) + \frac{1}{2}\nabla_Y X \\
&\quad + \mathcal{J}_M(\nabla_Y \mathcal{J}_M X - \nabla_Y X) - [X,Y] \\
&= -\frac{1}{2}\nabla_X Y - \mathcal{J}_M((\nabla_X \mathcal{J}_M)Y) - \mathcal{J}_M^2 \nabla_X Y + \mathcal{J}_M \nabla_X Y \\
&\quad + \frac{1}{2}\nabla_Y X + \mathcal{J}_M((\nabla_Y \mathcal{J}_M)X) + \mathcal{J}_M^2 \nabla_Y X \\
&\quad - \mathcal{J}_M \nabla_Y X - [X,Y] \\
&= T_\nabla(X,Y) - \mathcal{J}_M((\nabla_X \mathcal{J}_M)Y - (\nabla_Y \mathcal{J}_M)X).
\end{aligned}$$

(iii) Notice that

$$\begin{aligned}
\nabla_X^{(\mathcal{J}_M)}\nabla_Y^{(\mathcal{J}_M)}Z &= \frac{1}{4}\nabla_X\nabla_Y Z + \frac{1}{2}\nabla_X \mathcal{J}_M \nabla_Y \mathcal{J}_M Z - \frac{1}{2}\nabla_X \mathcal{J}_M \nabla_Y Z \\
&\quad + \mathcal{J}_M(\frac{1}{2}\nabla_X \mathcal{J}_M \nabla_Y Z - \frac{3}{2}\nabla_X \nabla_Y \mathcal{J}_M Z + \nabla_X \nabla_Y Z),
\end{aligned}$$

and the expression of curvature becomes

$$\begin{aligned}
R_{\nabla^{(\mathcal{J}_M)}}(X,Y,Z) &= \nabla_X^{(\mathcal{J}_M)}\nabla_Y^{(\mathcal{J}_M)}Z - \nabla_Y^{(\mathcal{J}_M)}\nabla_X^{(\mathcal{J}_M)}Z - \nabla_{[X,Y]}^{(\mathcal{J}_M)}Z \\
&= -\frac{1}{2}R_\nabla(X,Y,Z) - \mathcal{J}_M R_\nabla(X,Y,\mathcal{J}_M Z) \\
&\quad + \mathcal{J}_M R_\nabla(X,Y,Z) + \frac{1}{2}(\mathcal{J}_M - I)((\nabla_X \mathcal{J}_M)\nabla_Y Z \\
&\quad - (\nabla_Y \mathcal{J}_M)\nabla_X Z) + \frac{1}{2}((\nabla_X \mathcal{J}_M)\nabla_Y \mathcal{J}_M Z \\
&\quad - (\nabla_Y \mathcal{J}_M)\nabla_X \mathcal{J}_M Z) \\
&= (\mathcal{J}_M - \frac{1}{2}I)R_\nabla(X,Y,Z) - \mathcal{J}_M R_\nabla(X,Y,\mathcal{J}_M Z) \\
&\quad + \frac{1}{2}(\mathcal{J}_M - I)(A^\nabla \mathcal{J}_M)(X,Y;Z) \\
&\quad + \frac{1}{2}(A^\nabla \mathcal{J}_M)(X,Y;\mathcal{J}_M Z).
\end{aligned}$$

(iv) Using that $g(\mathcal{J}_M X, Y) = g(X, \mathcal{J}_M Y)$ and $g(\mathcal{J}_M X, \mathcal{J}_M Y) = g(X, \mathcal{J}_M Y) -$

$\frac{3}{2}g(X,Y)$ we obtain

$$\begin{aligned}
(\nabla_X^{(\mathcal{J}_M)} g)(Y,Z) &= X(g(Y,Z)) - g(\nabla_X^{(\mathcal{J}_M)} Y, Z) - g(Y, \nabla_X^{(\mathcal{J}_M)} Z) \\
&= X(g(Y,Z)) - g(-\frac{1}{2}\nabla_X Y - \mathcal{J}_M(\nabla_X \mathcal{J}_M Y - \nabla_X Y), Z) \\
&\quad - g(Y, -\frac{1}{2}\nabla_X Z - \mathcal{J}_M(\nabla_X \mathcal{J}_M Z - \nabla_X Z)) \\
&= -(\nabla_X g)(\mathcal{J}_M Y, \mathcal{J}_M Z) + (\nabla_X g)(Y, \mathcal{J}_M Z) \\
&\quad - \frac{1}{2}(\nabla_X g)(Y,Z) + \frac{1}{3}X(g(\mathcal{J}_M Y, \mathcal{J}_M Z)) \\
&\quad - \frac{1}{3}X(g(Y, \mathcal{J}_M Z)) + \frac{1}{2}X(g(Y,Z)) + g(Y, (\nabla_X \mathcal{J}_M)Z) \\
&= -(\nabla_X g)(\mathcal{J}_M Y, \mathcal{J}_M Z) + (\nabla_X g)(Y, \mathcal{J}_M Z) \\
&\quad - \frac{1}{2}(\nabla_X g)(Y,Z) + g(Y, (\nabla_X \mathcal{J}_M)Z).
\end{aligned}$$

□

We can give some direct consequences of Proposition 2.1:

i) If $(M^{2k}, \bar{g}, \mathcal{J}_M)$ is nearly golden Kähler manifold, which means $(M^{2k}, \bar{g}, \mathcal{J}_M)$ almost golden Hermitian manifold and $(\nabla_X \mathcal{J}_M)Y + (\nabla_Y \mathcal{J}_M)X = 0$ (see [28]), then $\mathcal{J}_M(d^\nabla \mathcal{J}_M) = 2\nabla - 2\nabla^{(\mathcal{J}_M)}$.

ii) If ∇ Levi-Civita connection of g and in addition $\nabla \in \mathcal{C}_{\mathcal{J}_M}(M^{2k})$, then $\nabla^{(\mathcal{J}_M)} = \nabla$ is the unique Levi-Civita connection.

Proposition 2.2. *Let h be a diffeomorphism of M^{2k} onto itself and be an automorphism of the G–structure defined by $\mathcal{J}_M$. Whenever h is an affine transformation for ∇ then h is an affine transformation for $\nabla^{(\mathcal{J}_M)}$ too.*

Proof. As is known that [4], there is $h_* \circ \mathcal{J}_M = \mathcal{J}_M \circ h_*$ equality for h and $\mathcal{J}_M$. If h is an affine transformation for ∇ then $h_*(\nabla_X Y) = \nabla_{h_* X} h_* Y$. The results follows from (2.2). □

A natural generalizations of the case $\nabla \in \mathcal{C}_{\mathcal{J}_M}(M^{2k})$ is given by:

Proposition 2.3. *Suppose that ∇ is a symmetric linear connection. Let $\nabla \mathcal{J}_M = \omega \otimes I$ or $\nabla \mathcal{J}_M = \omega \otimes \mathcal{J}_M^n$ where ω is a 1–form and $n \in \mathbb{N}$. Then $\nabla^{(\mathcal{J}_M)}$ is a quarter–symmetric connection.*

Proof. In [16], it is known that a linear connection is said to be a quarter-symmetric connection if its torsion is of the form: $T(X,Y) = \eta(Y)\mathcal{J}_M(X) - \eta(X)\mathcal{J}_M(Y)$, where η is a 1-form. From (2.1) we have $\nabla^{(\mathcal{J}_M)} = \nabla - \omega \otimes \mathcal{J}_M$ and from the item (ii) of Proposition 2.1 we obtain $T_{\nabla^{(\mathcal{J}_M)}} = \mathcal{J}_M \otimes \omega - \omega \otimes \mathcal{J}_M$. A similar proof can easily be given for $\nabla \mathcal{J}_M = \omega \otimes \mathcal{J}_M^n$. □

Definition 2.2. A linear connection ∇ is called special complex golden connection if it is symmetric and $d^\nabla \mathcal{J}_M = 0$.

After that, the set of all special complex golden connections with $\mathcal{C}_{CG}(M^{2k})$ will be shown.

Proposition 2.4. *Let $\bar{\mathcal{C}}_{\mathcal{J}_M}(M^{2k})$ be the set of all symmetric $\mathcal{J}_M$–connection on M^{2k}. Then $\bar{\mathcal{C}}_{\mathcal{J}_M}(M^{2k}) \subset \mathcal{C}_{CG}(M^{2k})$.*

Proof. The statement is a direct consequence of Proposition 2.1 and Definition 2.2. □

Proposition 2.5. *If $\nabla \in \mathcal{C}_{CG}(M^{2k})$ then complex golden conjugate connection $\nabla^{(\mathcal{J}_M)}$ is symmetric.*

Proof. The statement immediately follows from item (ii) of Proposition 2.1 and Definition 2.2. □

Proposition 2.6. *Let ∇ be a special complex golden connection, then*

(i) $\nabla^{(\mathcal{J}_M)}$ is also special complex golden connection.

(ii) $\mathcal{J}_M$ is integrable.

Proof.

(i) The result follows from Proposition 2.1.

(ii) Applying a standard computation,

$$N_{\mathcal{J}_M}(X,Y) = \left(d^{\nabla}\mathcal{J}_M\right)(\mathcal{J}_M X, Y) - \left(d^{\nabla}\mathcal{J}_M\right)(\mathcal{J}_M Y, X) - \left(d^{\nabla}\mathcal{J}_M\right)(X,Y),$$

and by $d^{\nabla}\mathcal{J}_M = 0$, the desired result is obtained.

□

Definition 2.3. (i) $(g, \mathcal{J}_M, \nabla)$ is said to be a special almost golden Norden structure whenever $(g, \mathcal{J}_M)$ is an almost golden Norden structure and additionally ∇ is a special complex golden connection.

(ii) $(\bar{g}, \mathcal{J}_M, \nabla)$ is said to be a special almost golden Hermitian structure whenever $(\bar{g}, \mathcal{J}_M)$ is an almost golden Hermitian structure and additionally ∇ is a special complex golden connection.

The following results are a direct consequence of Proposition 2.6 and Definition 2.3:

Proposition 2.7. *(i) $(g, \mathcal{J}_M, \nabla^{(\mathcal{J}_M)})$ is a special almost golden Norden structure if $(g, \mathcal{J}_M, \nabla)$ is a special almost golden Norden structure.*

(ii) $(\bar{g}, \mathcal{J}_M, \bar{\nabla}^{(\mathcal{J}_M)})$ is a special nearly golden Kähler structure if $(\bar{g}, \mathcal{J}_M, \bar{\nabla})$ is a special nearly golden Kähler structure, where $\bar{\nabla}$ is a Levi-Civita connection with respect to $\bar{g}$.

Example 2.1. The following examples of special almost golden Norden structures on the 4–dim Euclidean space can be generalized to special almost golden Norden structures defined on Euclidean spaces of any even dimension.

Let $\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$ be the Euclidean space with the metric $g : \mathfrak{X}(\mathbb{R}^4) \times \mathfrak{X}(\mathbb{R}^4) \to F\left(\mathbb{R}^4\right)$ given by

$$g\left(Z, \bar{Z}\right) = -X^1\overline{X}^1 + X^2\overline{X}^2 + Y^1\overline{Y}^1 + Y^2\overline{Y}^2,$$

for any $Z = \left(X^1, X^2, Y^1, Y^2\right), \overline{Z} = \left(\overline{X}^1, \overline{X}^2, \overline{Y}^1, \overline{Y}^2\right) \in \mathfrak{X}(\mathbb{R}^4)$ and denotes (x^1, x^2, y^1, y^2) the local coordinates in $\mathbb{R}^4$.

Define $(1,1)$–tensor field $\mathcal{J}_{M_1} : \mathfrak{X}(\mathbb{R}^4) \longrightarrow \mathfrak{X}(\mathbb{R}^4)$ by

$$\mathcal{J}_{M_1}\left(\partial_1, \partial_2, \bar{\partial}_1, \bar{\partial}_2\right) = \left(\bar{\phi}_c\partial_1, \phi_c\partial_2, \bar{\phi}_c\bar{\partial}_1, \phi_c\bar{\partial}_2\right),$$

where $\partial_i = \frac{\partial}{\partial x^i}$, $\bar{\partial}_i = \frac{\partial}{\partial y^i}$, $1 \leq i \leq 2$, $\phi_c = \frac{1+i\sqrt{5}}{2}$ is the complex golden ratio and $\bar{\phi}_c = \frac{1-i\sqrt{5}}{2} = 1 - \phi_c$. We can check that $\mathcal{J}_{M_1}$ is an almost complex golden structure on $\mathbb{R}^4$:

$$\begin{aligned}
\mathcal{J}^2_{M_1}(X^1, X^2, Y^1, Y^2) &= \left(\bar{\phi}_c^2 X^1, \phi_c^2 X^2, \bar{\phi}_c^2 Y^1, \phi_c^2 Y^2\right) \\
&= \mathcal{J}_{M_1}(X^1, X^2, Y^1, Y^2) - \frac{3}{2}I(X^1, X^2, Y^1, Y^2),
\end{aligned}$$

for any $(X^1, X^2, Y^1, Y^2) \in \mathfrak{X}(\mathbb{R}^4)$. Thus, $\mathcal{J}^2_{M_1} = \mathcal{J}_{M_1} - \frac{3}{2}I$.

Moreover, the metric is $\mathcal{J}_{M_1}$ compatible:

$$g(\mathcal{J}_{M_1}Z, \overline{Z}) = \bar{\phi}_c\left(-X^1\overline{X}^1 + Y^1\overline{Y}^1\right) + \phi_c\left(X^2\overline{X}^2 + Y^2\overline{Y}^2\right) = g(Z, \mathcal{J}_{M_1}\overline{Z}),$$

for any $Z = \left(X^1, X^2, Y^1, Y^2\right), \overline{Z} = \left(\overline{X}^1, \overline{X}^2, \overline{Y}^1, \overline{Y}^2\right) \in \mathfrak{X}(\mathbb{R}^4)$. Thus, $(\mathbb{R}^4, g, \mathcal{J}_{M_1})$ is an almost golden Norden manifold.

Similarly we can show that the $(1,1)$-tensor fields $\mathcal{J}_{M_2}$, $\mathcal{J}_{M_3}$, $\mathcal{J}_{M_4} : \mathfrak{X}(\mathbb{R}^4) \longrightarrow \mathfrak{X}(\mathbb{R}^4)$ defined by

$$\begin{aligned}
\mathcal{J}_{M_2}\left(\partial_1, \partial_2, \bar{\partial}_1, \bar{\partial}_2\right) &= \left(\phi_c\partial_1, \phi_c\partial_2, \phi_c\bar{\partial}_1, \bar{\phi}_c\bar{\partial}_2\right), \\
\mathcal{J}_{M_3}\left(\partial_1, \partial_2, \bar{\partial}_1, \bar{\partial}_2\right) &= \left(\phi_c\partial_1, \phi_c\partial_2, \bar{\phi}_c\bar{\partial}_1, \bar{\phi}_c\bar{\partial}_2\right), \\
\mathcal{J}_{M_4}\left(\partial_1, \partial_2, \bar{\partial}_1, \bar{\partial}_2\right) &= \left(\bar{\phi}_c\partial_1, \bar{\phi}_c\partial_2, \phi_c\bar{\partial}_1, \phi_c\bar{\partial}_2\right)
\end{aligned}$$

are almost complex golden structures on $\mathbb{R}^4$ and $(\mathbb{R}^4, g, \mathcal{J}_{M_i})$, $i \in \{2,3,4\}$ are almost golden Norden manifolds. Since the Levi-Civita ∇^g is trivial $(g, \mathcal{J}_{M_i}, \nabla^g)$, $i \in \{1, ..., 4\}$ are special almost golden Norden structures on $\mathbb{R}^4$.

In the last part of this section, we consider two different tensor fields related with an almost complex structure [22]. In [22], the authors described structural tensor fields and virtual tensor fields of an almost product structure. Going back to our context, suppose that $(\nabla, \mathcal{J}_M)$ is a pair of tensor field of the type $(1,2)$.

The structural tensor field is given by

$$C^{\mathcal{J}_M}_{\nabla}(X, Y) := \frac{1}{2}[(\nabla_{\mathcal{J}_M X}\mathcal{J}_M)Y + (\nabla_X\mathcal{J}_M)\mathcal{J}_M Y].$$

The virtual tensor field is given by

$$B_{\nabla}^{\mathcal{J}_M}(X,Y) := \frac{1}{2}[(\nabla_{\mathcal{J}_M X}\mathcal{J}_M)Y - (\nabla_X \mathcal{J}_M)\mathcal{J}_M Y].$$

It result that

$$C_{\nabla^{(\mathcal{J}_M)}}^{\mathcal{J}_M} = -\frac{1}{2}C_{\nabla}^{\mathcal{J}_M} + \mathcal{J}_M^2 \circ C_{\nabla}^{\mathcal{J}_M}, \quad B_{\nabla^{(\mathcal{J}_M)}}^{\mathcal{J}_M} = -\frac{1}{2}B_{\nabla}^{\mathcal{J}_M} + \mathcal{J}_M^2 \circ B_{\nabla}^{\mathcal{J}_M}.$$

Also

$$C_{\nabla}^{\mathcal{J}_M}(\mathcal{J}_M X, \mathcal{J}_M Y) = (\nabla_{\mathcal{J}_M X}\mathcal{J}_M)\mathcal{J}_M Y - \frac{3}{2}C_{\nabla}^{\mathcal{J}_M}(X,Y),$$

$$B_{\nabla}^{\mathcal{J}_M}(\mathcal{J}_M X, \mathcal{J}_M Y) = \frac{3}{2}B_{\nabla}^{\mathcal{J}_M}(X,Y).$$

The relation given below underlines the importance of the tensor fields mentioned above,

$$\nabla^{(\mathcal{J}_M)} = \nabla - \nabla\mathcal{J}_M + C_{\nabla}^{\mathcal{J}_M} - B_{\nabla}^{\mathcal{J}_M}. \tag{2.3}$$

Example 2.2. Let η be a 1-form. Suppose that the linear connection ∇ satisfies $\nabla\mathcal{J}_M = \eta \otimes \mathcal{J}_M^n$ then we have the following relations on structural and virtual tensor fields.

$$C_{\nabla}^{\mathcal{J}_M} = \frac{1}{2}[(\eta \circ \mathcal{J}_M) \otimes \mathcal{J}_M^n + (\eta \otimes \mathcal{J}_M^{n+1})],$$

$$B_{\nabla}^{\mathcal{J}_M} = \frac{1}{2}[(\eta \circ \mathcal{J}_M) \otimes \mathcal{J}_M^n - (\eta \otimes \mathcal{J}_M^{n+1})],$$

$$C_{\nabla^{(\mathcal{J}_M)}}^{\mathcal{J}_M} = -\frac{1}{4}[(\eta \circ \mathcal{J}_M) \otimes \mathcal{J}_M^n + (\eta \otimes \mathcal{J}_M^{n+1})] + \frac{1}{2}[(\eta \circ \mathcal{J}_M) \otimes \mathcal{J}_M^{n+2} + (\eta \otimes \mathcal{J}_M^{n+3})],$$

$$B_{\nabla^{(\mathcal{J}_M)}}^{\mathcal{J}_M} = -\frac{1}{4}[(\eta \circ \mathcal{J}_M) \otimes \mathcal{J}_M^n - (\eta \otimes \mathcal{J}_M^{n+1})] + \frac{1}{2}[(\eta \circ \mathcal{J}_M) \otimes \mathcal{J}_M^{n+2} - (\eta \otimes \mathcal{J}_M^{n+3})].$$

Let us consider $\nabla^{(\cdot)}$ and its behavior for almost complex golden structures families. We can give following proposition.

Proposition 2.8. *Assuming $\mathcal{J}_{M_1}$ and $\mathcal{J}_{M_2}$ are two almost complex golden structures will yield $\mathcal{J}_{M_1}+\mathcal{J}_{M_2}$ to be an almost complex golden structure too iff $\mathcal{J}_{M_1}\mathcal{J}_{M_2} + \mathcal{J}_{M_2}\mathcal{J}_{M_1} = \frac{3}{2}I$, in this case*

$$\nabla_X^{(\mathcal{J}_{M_1}+\mathcal{J}_{M_2})}Y = \nabla_X^{(\mathcal{J}_{M_1})}Y + \nabla_X^{(\mathcal{J}_{M_2})}Y + \frac{1}{2}\nabla_X Y - \mathcal{J}_{M_1}(\nabla_X \mathcal{J}_{M_2} Y) - \mathcal{J}_{M_2}(\nabla_X \mathcal{J}_{M_1} Y),$$

$$\begin{aligned} C_{\nabla}^{(\mathcal{J}_{M_1}+\mathcal{J}_{M_2})} - B_{\nabla}^{(\mathcal{J}_{M_1}+\mathcal{J}_{M_2})} &= \left(C_{\nabla}^{\mathcal{J}_{M_1}} + C_{\nabla}^{\mathcal{J}_{M_2}}\right) - \left(B_{\nabla}^{\mathcal{J}_{M_1}} + B_{\nabla}^{\mathcal{J}_{M_2}}\right) \\ &\quad - \mathcal{J}_{M_1} \circ \nabla\mathcal{J}_{M_2} - \mathcal{J}_{M_2} \circ \nabla\mathcal{J}_{M_1}. \end{aligned}$$

3. On the Duality of Complex Golden and Complex Conjugate Connections

Crasmareanu and Hreţcanu have shown [10] that any almost complex structure $\mathcal{J}$ determines two almost complex golden structures:

$$\mathcal{J}_M^{\pm} = \frac{1}{2}(I \pm \sqrt{5}\mathcal{J}) \tag{3.1}$$

and, any almost complex golden structure $\mathcal{J}_M$ induces two almost complex structures:

$$\mathcal{J}^{\pm} = \pm\frac{1}{\sqrt{5}}(2\mathcal{J}_M - I). \tag{3.2}$$

Then $\nabla\mathcal{J}_M^{\pm} = \pm\frac{\sqrt{5}}{2}\nabla\mathcal{J}$ and $\nabla\mathcal{J}^{\pm} = \pm\frac{2}{\sqrt{5}}\nabla\mathcal{J}_M$. Hence, ∇ is a $\mathcal{J}_M$–connection iff ∇ is a $\mathcal{J}$–connection.

Our goal here is to find the connection between the conjugate connection associated to $\mathcal{J}$ and $\mathcal{J}_M$.

Proposition 3.1. *(i) Let $\mathcal{J}_M$ be an almost complex golden structure on M^{2k} and $\mathcal{J}^{\pm}$ be the almost complex structure which is given in (3.2), then*

$$5\nabla^{(\mathcal{J}^{\pm})} - 4\nabla^{(\mathcal{J}_M)} = \nabla + 2\nabla\mathcal{J}_M.$$

(ii) Let $\mathcal{J}$ be an almost complex structure on M^{2k} and $\mathcal{J}_M^{\pm}$ be the almost complex golden structure which is given in (3.1), then

$$-4\nabla^{(\mathcal{J}_M^{\pm})} + 5\nabla^{(\mathcal{J})} = \nabla \pm \sqrt{5}\nabla\mathcal{J}.$$

(iii) For all cases defined earlier $(\nabla^{(\mathcal{J}_M)})^{(\mathcal{J})} = (\nabla^{(\mathcal{J})})^{(\mathcal{J}_M)}$.

Proof.

(i) We know the expression of $\nabla^{(\mathcal{J}^{\pm})}$ from [4]:

$$\nabla_X^{(\mathcal{J}^{\pm})}Y = -\mathcal{J}^{\pm}(\nabla_X\mathcal{J}^{\pm}Y). \tag{3.3}$$

Using (3.2), we have

$$\nabla_X^{(\mathcal{J}^{\pm})}Y = -\frac{4}{5}\mathcal{J}_M(\nabla_X\mathcal{J}_MY) + \frac{2}{5}\mathcal{J}_M(\nabla_XY) + \frac{2}{5}\nabla_X\mathcal{J}_MY - \frac{1}{5}\nabla_XY,$$

and from (2.2) replacing

$$\mathcal{J}_M(\nabla_X\mathcal{J}_MY) = -\nabla_X^{(\mathcal{J}_M)}Y - \frac{1}{2}\nabla_XY + \mathcal{J}_M(\nabla_XY),$$

we have

$$\nabla_X^{(\mathcal{J}^{\pm})}Y = \frac{1}{5}\nabla_XY + \frac{4}{5}\nabla^{(\mathcal{J}_M)} + \frac{2}{5}(\nabla_X\mathcal{J}_M)Y.$$

(ii) From (2.2)

$$\nabla_X^{(\mathcal{J}_M^{\pm})}Y = -\frac{1}{2}\nabla_X Y - \mathcal{J}_M^{\pm}(\nabla_X \mathcal{J}_M^{\pm} Y - \nabla_X Y),$$

and replacing $\mathcal{J}_M^{\pm}$ from (3.1) we get

$$\nabla_X^{(\mathcal{J}_M^{\pm})}Y = -\frac{1}{4}\nabla_X Y - \frac{5}{4}\mathcal{J}(\nabla_X \mathcal{J} Y) \mp \frac{\sqrt{5}}{4}(\nabla_X \mathcal{J})Y. \tag{3.4}$$

Equation (3.4), thanks to $\nabla_X^{(\mathcal{J})}Y = -\mathcal{J}(\nabla_X \mathcal{J} Y)$ gives

$$\nabla_X^{(\mathcal{J}_M^{\pm})}Y = -\frac{1}{4}\nabla_X Y + \frac{5}{4}\nabla_X^{(\mathcal{J})}Y \mp \frac{\sqrt{5}}{4}(\nabla_X \mathcal{J})Y.$$

(iii) By direct computation using (2.2) and (3.3), we get

$$\begin{aligned}(\nabla^{(\mathcal{J}_M)})_X^{(\mathcal{J}^{\pm})}Y &= \frac{1}{2}\mathcal{J}^{\pm}(\nabla_X \mathcal{J}^{\pm} Y) - (\mathcal{J}^{\pm} \circ \mathcal{J}_M)(\nabla_X \mathcal{J}^{\pm} Y)\\ &\quad +(\mathcal{J}^{\pm} \circ \mathcal{J}_M)(\nabla_X(\mathcal{J}_M \circ \mathcal{J}^{\pm})Y),\end{aligned}$$

and

$$\begin{aligned}(\nabla^{(\mathcal{J}^{\pm})})_X^{(\mathcal{J}_M)}Y &= \frac{1}{2}\mathcal{J}^{\pm}(\nabla_X \mathcal{J}^{\pm} Y) - (\mathcal{J}_M \circ \mathcal{J}^{\pm})(\nabla_X \mathcal{J}^{\pm} Y)\\ &\quad +(\mathcal{J}_M \circ \mathcal{J}^{\pm})(\nabla_X(\mathcal{J}^{\pm} \circ \mathcal{J}_M)Y),\end{aligned}$$

and taking into account that $\mathcal{J}_M \circ \mathcal{J}^{\pm} = \mathcal{J}^{\pm} \circ \mathcal{J}_M$, we obtain $(\nabla^{(\mathcal{J}_M)})^{(\mathcal{J}^{\pm})} = (\nabla^{(\mathcal{J}^{\pm})})^{(\mathcal{J}_M)}$.

Moreover,

$$(\nabla^{(\mathcal{J})})_X^{(\mathcal{J}_M^{\pm})}Y = \frac{1}{2}\mathcal{J}(\nabla_X \mathcal{J} Y) - (\mathcal{J}_M^{\pm} \circ \mathcal{J})(\nabla_X \mathcal{J} Y) + (\mathcal{J}_M^{\pm} \circ \mathcal{J})(\nabla_X(\mathcal{J} \circ \mathcal{J}_M^{\pm})Y),$$

and

$$(\nabla^{(\mathcal{J}_M^{\pm})})_X^{(\mathcal{J})}Y = \frac{1}{2}\mathcal{J}(\nabla_X \mathcal{J} Y) - (\mathcal{J} \circ \mathcal{J}_M^{\pm})(\nabla_X \mathcal{J} Y) + (\mathcal{J} \circ \mathcal{J}_M^{\pm})(\nabla_X(\mathcal{J}_M^{\pm} \circ \mathcal{J})Y),$$

and taking into account that $\mathcal{J} \circ \mathcal{J}_M^{\pm} = \mathcal{J}_M^{\pm} \circ \mathcal{J}$, we obtain $(\nabla^{(\mathcal{J})})^{(\mathcal{J}_M^{\pm})} = (\nabla^{(\mathcal{J}_M^{\pm})})^{(\mathcal{J})}$.

□

We know from [4] that for an almost complex structure $\mathcal{J}$, the structural and virtual tensor fields relation is given with

$$\nabla^{(\mathcal{J})} = \nabla + C_{\nabla}^{\mathcal{J}} - B_{\nabla}^{\mathcal{J}}. \tag{3.5}$$

From (2.3) and (3.5) we can give following corollary.

Corollary 3.1. *(i) Let $\mathcal{J}_M$ be an almost complex golden structure on M^{2k} and $\mathcal{J}^{\pm}$ be the almost complex structure which is given in (3.2), then*

$$5(C_{\nabla}^{\mathcal{J}^{\pm}} - B_{\nabla}^{\mathcal{J}^{\pm}}) = 4(C_{\nabla}^{\mathcal{J}_M} - B_{\nabla}^{\mathcal{J}_M}) - 2\nabla\mathcal{J}_M.$$

(ii) Let $\mathcal{J}$ be an almost complex structure on M^{2k} and $\mathcal{J}_M^{\pm}$ be the almost complex golden structure which is given in (3.1), then

$$4(C_{\nabla}^{\mathcal{J}_M^{\pm}} - B_{\nabla}^{\mathcal{J}_M^{\pm}}) = 5(C_{\nabla}^{\mathcal{J}} - B_{\nabla}^{\mathcal{J}}) \pm \sqrt{5}\nabla\mathcal{J}.$$

4. Invariant Distribution

Suppose that $D \subset TM$ is a fixed distribution which is considered as a vector subbundle of TM.

Definition 4.1. (i) If $X \in \Gamma(D) \Rightarrow \mathcal{J}_M X \in \Gamma(D)$, then D is said to be the $\mathcal{J}_M$ invariant.

(ii) ([9]) If $Y \in \Gamma(D)$, $\nabla_X Y \in \Gamma(D)$ for any $X \in \Gamma(TM)$, then the linear connection ∇ is restricted to D.

∇ might be seen as a connection in D whenever ∇ is restricted to D. Considering this truth from [1], a connection is said to be adapted to D if it is restricted to D.

Proposition 4.1. *Consider a $\mathcal{J}_M$–invariant distribution D and a linear connection ∇ is restricted to D. Then the conjugate connection $\nabla^{(\mathcal{J}_M)}$ is restricted to D too.*

Proof. Fix $Y \in \Gamma(D)$. Then $\mathcal{J}_M Y \in \Gamma(D)$ and for any $X \in \Gamma(TM)$. We have $\nabla_X Y \in \Gamma(D)$. Therefore $\nabla_X^{(\mathcal{J}_M)} Y = -\frac{1}{2}\nabla_X Y - \mathcal{J}_M(\nabla_X \mathcal{J}_M Y - \nabla_X Y) \in \Gamma(D)$. □

REFERENCES

1. A. Bejancu and H. R. Farran: *Foliations and Geometric Structures.* Springer, Dordrecht, Netherlands, 2006.
2. G. Beldjilali: *s-Golden manifolds.* Mediterranean Journal of Mathematics **16** (2019) Article ID 56. https://doi.org/10.1007/s00009-019-1343-9
3. G. Beldjilali: *A new class of golden Riemannian manifold.* International Electronic Journal of Geometry **13(1)** (2020), 1–8. https://doi.org/10.36890/iejg.690479
4. A. M. Blaga and M. Crasmareanu: *The geometry of complex conjugate connections.* Hacettepe Journal of Mathematics and Statistics **41(1)** (2012), 119–126.
5. A. M. Blaga and M. Crasmareanu: *The geometry of product conjugate connections.* Analele Ştiinţifice Ale Universităţii "Alexandru Ioan Cuza" Din Iaşi (S N) Matematică **59(1)** (2013), 73–84.

6. A. M. Blaga: *The geometry of golden conjugate connections.* Sarajevo Journals of Mathematics **10(23)** (2014), 237–245. https://doi.org/10.5644/SJM.10.2.09
7. A. M. Blaga and M. Crasmareanu: *The geometry of tangent conjugate connections.* Hacettepe Journal of Mathematics and Statistics 44(4) (2015), 767–774.
8. A. M. Blaga and C. E. Hreţcanu: *Metallic conjugate connections.* Revista de la Unión Matemática Argentina **59(1)** (2018), 179–192. https://doi.org/10.33044/revuma.v59n1a09
9. F. Bullo and A. D. Lewis: *Geometric Control of Mechanical Systems: Modeling, Analysis, and Design for Simple Mechanical Control Systems.* Springer, New York, 2005.
10. M. Crasmareanu and C. E. Hreţcanu: *Golden differential geometry.* Chaos, Solitons and Fractals **38(5)** (2008), 1229–1238. https://doi.org/10.1016/j.chaos.2008.04.007
11. V. Cruceanu: *Connexions compatibles avec certaines structures sur un fibré vectoriel banachique.* Czechoslovak Mathematical Journal **24(99)** (1974), 126–142 (in French). https://doi.org/10.21136/CMJ.1974.101223
12. F. E. Erdoğan, S. Y. Perktaş, B. E. Acet and A. M. Blaga: *Screen transversal lightlike submanifolds of metallic semi-Riemannian manifolds.* Journal of Geometry and Physics **142** (2019), 111–120. https://doi.org/10.1016/j.geomphys.2019.04.010
13. F. Etayo, R. Santamaría and A. Upadhyay: *On the geometry of almost golden Riemannian manifolds.* Mediterranean Journal of Mathematics **14** (2017), Article ID 187. https://doi.org/10.1007/s00009-017-0991-x
14. A. Gezer, N. Cengiz and A. Salimov: *On integrability of Golden Riemannian structures.* Turkish Journal of Mathematics **37(4)** (2013), 693–703. https://doi.org/10.3906/mat-1108-35
15. A. Gezer and Ç. Karaman : *On metallic Riemannian structures.* Turkish Journal of Mathematics **39(6)** (2015), 954–962. https://doi.org/10.3906/mat-1504-50
16. S. Golab: *On semi-symmetric and quarter-symmetric linear connections.* Tensor New Series **29(3)** (1975), 249–254.
17. S. I. Goldberg and K. Yano: *Polynomial structures on manifolds.* Kodai Mathematical Seminar Reports **22** (1970), 199–218.
18. S. I. Goldberg and N.C. Petridis: *Differentiable solutions of algebraic equations on manifolds.* Kodai Mathematical Seminar Reports **25** (1973), 111–128.
19. C. E. Hreţcanu and M. Crasmareanu: *Metallic structure on Riemannian manifolds.* Revista de la Unión Mathemática Argentina **54(2)** (2013), 15–27.
20. B. Kacimi, M. Özkan and F. Hathout: *Bifoliated homotopy invariant and metallic submersion.* Mediterranean Journal of Mathematics **18(4)** (2021), Article ID 163. https://doi.org/10.1007/s00009-021-01801-w
21. M. N. Khan: *Novel theorems for the frame bundle endowed with metallic structures on an almost contact metric manifold.* Chaos, Solitons and Fractals **146** (2021) Article ID 110872. https://doi.org/10.1016/j.chaos.2021.110872
22. V. F. Kirichenko: *Method of generalized Hermitian geometry in the theory of almost contact manifold.* Journal of Soviet Mathematics **42(5)** (1988), 1885–1919. https://doi.org/10.1007/BF01094419

23. N. Ö. Poyraz: *Screen semi-invariant lightlike submanifolds of golden semi-Riemannian manifolds*. International Electronic Journal of Geometry **14(1)** (2021), 207–216. https://doi.org/10.36890/IEJG.893398

24. M. Özkan and B. Peltek: *A new structure on manifolds: Silver structure*. International Electronic Journal of Geometry **9(2)** (2016), 59–69. https://doi.org/10.36890/iejg.584592

25. M. Özkan and F. Yilmaz: *Metallic structures on differential manifolds*. Journal of Science and Arts **3(44)** (2018), 645–660.

26. M. Özkan and S. Doğan: *Almost bronze structures on differentiable manifolds*. Journal of Mathematics **2022** (2022) Article ID 6940387. https://doi.org/10.1155/2022/6940387

27. B. Şahin: *Almost poly-Norden manifolds*. International Journal of Maps in Mathematics **1(1)** (2018), 68–79.

28. S. Turanli, A. Gezer and H. Cakicioglu: Metallic *Kähler and nearly metallic Kähler manifolds*. International Journal of Geometric Methods in Modern Physics **18(9)** (2021) Article ID 2150146. https://doi.org/10.1142/S0219887821501462

29. J. Vanžura: *Integrability conditions for polynomial structures*. Kodai Mathematical Seminar Reports **27(1-2)** (1976), 42–50.