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ON SOME CONVERGENCE PROPERTIES OF HYPERSPACES WITH HIT-AND-MISS TOPOLOGY

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Abstract. In this paper, the relationships between closure type properties of hyperspaces over a space X with that of covering properties of X are being investigated. Our main focus here is to study α_i-properties of X and that of hyperspaces with hit-and-miss topology. Also some weak forms of Fréchet Urysohn property have been investigated in the hyperspace $(\Lambda, \tau_{\Delta}^{+})$.
Key words: hyperspace, topology, Fréchet Urysohn property.

1. Introduction

The study of hyperspace theory started with D. Pompeiu [21], F. Hausdorff [7], L. Vietoris [27] and E. Michael [20], where for a given topological space X, the set of all nonempty closed subsets of X is denoted by $CL(X)$. The set $CL(X)$, endowed with some topology, is known as the hyperspace of X. There are many results which show that properties of hyperspaces over a space X can be described by properties of the basic space X. In recent years, several papers have been devoted to the study of classical selection principles in hyperspaces endowed with various topologies. In particular, many topological properties are defined or characterized in terms of selection principles ([15], [16], [10], [23], [17], [18]). On the other hand, selection principles started in the early 20^{th} century with the works of Borel [3],

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Menger [19], Hurewicz [8] and Rothberger [24]. Later, Scheepers [25] began a rigorous investigation on selection principles. Till now, it has become one of the most challenging area of research. Nowadays, the authors in this area are working with these concepts to get results in topological hyperspaces. Also, this theory has now applications on set theory, function spaces and hyperspaces, among other fields of mathematics. In this paper, such duality has been studied in connection with properties which are defined in terms of selection principles. We shall prove here that results concerning sets of type γ-sets and their variations can be obtained in terms of special hyperspace topologies on the subcollection Λ of $CL(X)$. Section 2 deals with some basic definitions and lemmas that are useful throughout the paper. In section 3, we discuss the covering and closure type properties $\alpha_i(\mathcal{A},\mathcal{B})$ in topological spaces, for $i = 2,3,4$. Finally, in Section 4 and Section 5, we investigate the α_i-properties and some generalizations of Fréchet Urysohn property in hyperspaces.

All spaces are assumed to be Hausdorff, non-compact. Covers always mean proper cover (i.e. a cover $\mathcal{U}$ of a space X such that $X \notin \mathcal{U}$).

Throughout the paper, Δ and Θ will denote subfamilies of $CL(X)$ which are closed under finite unions and contain all singletons. Also Λ will denote a subfamily of $CL(X)$ that is closed under finite unions. Also, we shall use $[X]^{<\omega}$ to denote all finite subsets of X.

2. Definitions and preliminaries

This section is a brief reminder of some basic concepts related to hyperspace theory as well as to the theory of selection principles.

Given a space (X,τ), we denote the family of all closed subsets of X, the family of all non-empty closed subsets of X and the family of all compact subsets of X by 2^X, $CL(X)$, and $\mathbb{K}(X)$, respectively.

For a subset $U \subseteq X$ and a family $\mathcal{U}$ of subsets of X, we write:

$$\begin{aligned} U^- &= \{A \in CL(X) : A \cap U \neq \phi\}; \\ U^+ &= \{A \in CL(X) : A \subseteq U\}; \\ U^c &= X \setminus U; \\ \mathcal{U}^c &= \{U^c : U \in \mathcal{U}\}. \end{aligned}$$

The hit-and-miss topology on $CL(X)$ with respect to Δ (first studied in the abstract in [22] and then in [14]), denoted by τ_Δ^+, has as a base, the family

$$\{(\bigcap_{i=1}^{m} V_i^-) \cap (B^c)^+ : B \in \Delta \text{ and } V_i \in \tau \text{ for } i \in \{1,2,...,m\}, m \in \mathbb{N}\}.$$

Following [28], the basic element $(\bigcap_{i=1}^{m} V_i^-) \cap (B^c)^+$ will be denoted by $(V_1,...,V_m)_B^+$.

Two important cases of the hit-and-miss topology are the Vietoris topology, τ_V, when $\Delta = CL(X)$ ([27], [20]) and the Fell topology, τ_F, when $\Delta = \mathbb{K}(X)$ ([5]).

Next, we recall two very known concepts both defined in 1996 by M. Scheepers [25]. Given an infinite set X, let $\mathcal{A}$ and $\mathcal{B}$ be collections of families of subsets of X.

- $S_1(\mathcal{A}, \mathcal{B})$ denotes the principle: For any sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$, there is a sequence $\{b_n : n \in \mathbb{N}\}$ such that for each $n \in \mathbb{N}$, $b_n \in A_n$ and $\{b_n : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$.
- $S_{fin}(\mathcal{A}, \mathcal{B})$ denotes the principle: For any sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$, there is a sequence $(B_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, B_n is a finite subset of A_n, and $\bigcup_{n \in \mathbb{N}} B_n \in \mathcal{B}$.

We denote the collection of all open covers of a topological space X by $\mathcal{O}$. When $\mathcal{A} = \mathcal{B} = \mathcal{O}$ in the definitions of $S_1(\mathcal{A}, \mathcal{B})$ and $S_{fin}(\mathcal{A}, \mathcal{B})$ above, we get the classical Rothberger property [24] and the Menger property [19], [8], whereas a space X has the Hurewicz property [8] if for each sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of open covers of X, there exists a sequence $\{\mathcal{V}_n : n \in \mathbb{N}\}$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and each $x \in X$ belongs to $\bigcup \mathcal{V}_n$ for all but finitely many n.

Arhangel'skii first introduced the α_i-properties in [1] in the following manner. Let $x \in X$ and $\{x_m \equiv (x_{m,n})_{n \in \mathbb{N}} : m \in \mathbb{N}\}$ be a countable family of sequences converging to x, i.e. $\lim_{n \to \infty} x_{m,n} = x$ for each $m \in \mathbb{N}$. There is a sequence $y \equiv (y_n)$ converging to x such that:

(α_1) $x_m \setminus y$ is finite, for each $m \in \mathbb{N}$,
(α_2) x_m and y have a joint subsequence for each $m \in \mathbb{N}$,
(α_3) x_m and y have a joint subsequence for infinitely many $m \in \mathbb{N}$,
(α_4) x_m and y have a joint element for infinitely many $m \in \mathbb{N}$.

For $\mathcal{A}$ and $\mathcal{B}$ as above, Lj. D. R. Kočinac [9] introduced the corresponding selection principles $\alpha_i(\mathcal{A}, \mathcal{B})$ in the following way:
For each sequence (A_n) of infinite elements of $\mathcal{A}$ there is an element $B \in \mathcal{B}$ such that:

$\alpha_2(\mathcal{A}, \mathcal{B})$: the set $A_n \cap B$ is infinite for each $n \in \mathbb{N}$,
$\alpha_3(\mathcal{A}, \mathcal{B})$: the set $A_n \cap B$ is infinite for infinitely many $n \in \mathbb{N}$,
$\alpha_4(\mathcal{A}, \mathcal{B})$: the set $A_n \cap B$ is non-empty for infinitely many $n \in \mathbb{N}$.

The following implications follow directly from the definitions:

$$\alpha_2(\mathcal{A}, \mathcal{B}) \Rightarrow \alpha_3(\mathcal{A}, \mathcal{B}) \Rightarrow \alpha_4(\mathcal{A}, \mathcal{B}) \text{ and } S_1(\mathcal{A}, \mathcal{B}) \Rightarrow \alpha_4(\mathcal{A}, \mathcal{B}).$$

Let us recall that an open cover $\mathcal{U}$ of a space X is called an ω-cover [6] (respectively, a k-cover [13]) if every finite (respectively, compact) subset of X is contained in a member of $\mathcal{U}$ and X is not a member of $\mathcal{U}$. An open cover $\mathcal{U}$ of X is called a γ-cover [6] if it is infinite and each $x \in X$ is contained in all but finitely many elements of $\mathcal{U}$. Notice that it is equivalent to the assertion: Each finite subset of X is contained in all but finitely many members of $\mathcal{U}$. Also, in [10], Kočinac introduced a stronger version of γ-covers as: an open cover $\mathcal{U}$ of a space X is called a γ_k-cover of X if each compact subset of X is contained in all but finitely many elements of $\mathcal{U}$ and X is not a member of the cover.

For a space (X,τ) and a point $x \in X$ we denote

- Ω: the collection of ω-covers of X;
- $\mathcal{K}$: the collection of k-covers of X;
- Ω_x^τ (or shortly, Ω_x): the set $\{A \subset X : x \in Cl\ A \setminus A\}$;
- Γ: the collection of all γ-covers of X;
- Γ_k: the collection of all γ_k-covers of X;
- Σ_x: the set of all non-trivial sequences in X that converge to x.

It is easy to observe that

$$\Gamma_k \subset \Gamma \subset \Omega \text{ and } \Gamma_k \subset \mathcal{K} \subset \Omega.$$

G. Di Maio, Lj. D. R. Kočinac and E. Meccariello [15], [16] first investigated S_1 and S_{fin} in 2^X and $CL(X)$ under the cofinite topology $\mathbb{Z}^+$, the co-compact topology F^+, the upper Vietoris topology V^+ and the lower Vietoris topology V^-, using k-covers and ω-covers (where, the lower Vietoris topology V^- is generated by all sets A^-, for $A \subset X$ open, and the upper Vietoris topology V^+ is generated by sets B^+, for B open in X).

Motivated by [15], Z. Li in his paper [11] introduced k_F-covers and c_V-covers to characterize selection principles in $CL(X)$ under the Fell topology and the Vietoris topology. As a generalization of these concepts, in [4] is introduced the next definition.

Definition 2.1. [4] Let (X,τ) be a topological space. A family $\mathcal{U} \subseteq \Lambda^c$ is called a $c_\Delta(\Lambda)$-cover of X, if for any $D \in \Delta$ and open subsets $V_1, ..., V_m$ of X with $D^c \cap V_i \neq \phi$, for any $i \in \{1, ..., m\}$, there exist $U \in \mathcal{U}$ and $F \in [X]^{<\omega}$ such that $D \subseteq U$, $F \cap U = \phi$ and for each $i \in \{1, ..., m\}$, $F \cap V_i \neq \phi$. The family of all $c_\Delta(\Lambda)$-covers of X will be denoted by $\mathbb{C}_\Delta(\Lambda)$.

In our paper [26], the relative version of the above cover is defined as follows.

Definition 2.2. [26] Let (X,τ) be a topological space and $Y \subseteq X$, with $Y \neq X$. A family $\mathcal{U} \subseteq \Lambda^c$ is called a $c_\Delta(\Lambda)$-cover of Y, if for any $D \in \Delta$ with $D \subseteq Y$ and open subsets $V_1, ..., V_m$ of X, with $Y^c \cap V_i \neq \phi$, for any $i \in \{1, ..., m\}$, there exist $U \in \mathcal{U}$ and $F \in [X]^{<\omega}$ such that $D \subseteq U$, $F \cap U = \phi$ and for each $i \in \{1, ..., m\}$, $F \cap V_i \neq \phi$. We denote by $\mathbb{C}^*_\Delta(\Lambda)$ the family of all $c_\Delta(\Lambda)$-covers of $Y \subseteq X$ with $Y \neq X$.

Remark 2.3. (i) Consider $\Delta = \mathbb{K}(X)$ and $\Lambda = CL(X)$ in Definitions 2.1 and 2.2. We then get the notions of k_F-covers of X and k_F-covers of a subset Y of X with $Y \neq X$ as in [11] (see Definitions 2.1 and 2.2).
(ii) Consider $\Delta = \Lambda = CL(X)$ in Definitions 2.1 and 2.2. We then get the notions of c_V-covers of X and c_V-covers of a subset Y of X with $Y \neq X$ as in [11] (see Definitions 2.3 and 2.4).

In the same paper [26], the generalized version of γ-covers is defined as follows.

Definition 2.4. [26] Let (X, τ) be a topological space. A family $\mathcal{U} \subseteq \Lambda^c$ is called a $\Delta\gamma$-cover of X, if each $B \in \Delta$ belongs to all but finitely many elements of $\mathcal{U}$ and for any $B \in \Delta$ and open subsets $V_1, ..., V_m$ of X, with $B^c \cap V_i \neq \emptyset$ for any $i \in \{1, ..., m\}$, there exist $U \in \mathcal{U}$ and $F \in [X]^{<\omega}$ such that $B \subseteq U$, $F \cap U = \emptyset$ and for each $i \in \{1, ..., m\}$, $F \cap V_i \neq \emptyset$. The set of all $\Delta\gamma$-covers of X is denoted by $\Delta\Gamma$.

By letting $\Lambda = CL(X)$ and $\Delta = \mathbb{K}(X)$ (resp., $CL(X)$) we get the concepts of a γ_{k_F} (respectively, γ_{c_V})-cover of a space X. The collection of all γ_{k_F} (resp., γ_{c_V})-covers of X is denoted by $\Gamma_{\mathbb{K}_F}$ (resp., $\Gamma_{\mathbb{C}_V}$).

We next introduce the concept of the relative version of $\Delta\gamma$-covers as follows.

Definition 2.5. Let (X, τ) be a topological space and $Y \subseteq X$ with $Y \neq X$. A family $\mathcal{U} \subseteq \Lambda^c$ is called a $\Delta\gamma$-cover of Y, if each $B \in \Delta$ with $B \subseteq Y$ belongs to all but finitely many elements of $\mathcal{U}$ and for any $B \in \Delta$ with $B \subseteq Y$ and open subsets $V_1, ..., V_m$ of X, with $Y^c \cap V_i \neq \emptyset$ for any $i \in \{1, ..., m\}$, there exist $U \in \mathcal{U}$ and $F \in [X]^{<\omega}$ such that $B \subseteq U$, $F \cap U = \emptyset$ and for each $i \in \{1, ..., m\}$, $F \cap V_i \neq \emptyset$. The set of all $\Delta\gamma$-covers of Y is denoted by $\Delta\Gamma^*$.

By letting $\Lambda = CL(X)$ and $\Delta = \mathbb{K}(X)$ (resp., $CL(X)$) we get the concepts of a γ_{k_F} (respectively, γ_{c_V})-cover of a subset Y of a space X with $Y \neq X$. The collection of all γ_{k_F} (resp., γ_{c_V})-covers of a subset Y of a space X with $Y \neq X$ is denoted by $\Gamma^*_{\mathbb{K}_F}$ (resp., $\Gamma^*_{\mathbb{C}_V}$).

Definition 2.6. A topological space (X, τ) is said to be $\mathbb{C}_\Delta(\Lambda)$-Lindelöf if every $c_\Delta(\Lambda)$-cover of X has a countable subcollection which is also a $c_\Delta(\Lambda)$-cover of X.

Thus, in a $\mathbb{C}_\Delta(\Lambda)$-Lindelöf space (X, τ), without loss of generality, we may consider only countable $c_\Delta(\Lambda)$-covers.

Also considering $\Lambda = CL(X)$ and $\Delta = \mathbb{K}(X)$ (respectively, $CL(X)$), we get the notions of k_F-Lindelöf (respectively, c_V-Lindelöf) spaces.

Next we present some useful lemmas which will be used throughout the paper.

Lemma 2.7. [26] Let Y be an open subset of a space X with $Y \neq X$ and $\mathcal{U} \subseteq \Lambda^c$ be a cover of Y. Then the following statements are equivalent:

(i) $\mathcal{U}$ is a $c_\Delta(\Lambda)$-cover of Y.
(ii) $Y^c \in Cl_{\tau^+_\Delta}(\mathcal{U}^c)$.

Lemma 2.8. Let X be a topological space, Y be an open subset of X with $Y \neq X$ and $\mathcal{U} = \{U_n : n \in \mathbb{N}\} \subseteq \Lambda^c$ be a cover of Y. Then the following statements are equivalent:

(i) $\mathcal{U}$ is a $\Delta\gamma$-cover of Y.
(ii) $\{U_n^c : n \in \mathbb{N}\}$ converges to Y^c in (Λ, τ^+_Δ).

Proof : (i) $\Rightarrow$ (ii): Let $W = (V_1, ..., V_m)^+_D$ be a τ^+_Δ-neighbourhood of Y^c. Since $\mathcal{U}$ is a $\Delta\gamma$-cover of Y, there exists $n_0 \in \mathbb{N}$ such that $D \subseteq U_n$, for each $n \geq n_0$. Also there exist $U_n \in \mathcal{U}$ and $F_n \in [X]^{<\omega}$ with $F_n \cap V_i \neq \emptyset$, for $i = 1, 2, ..., m$, such that $F_n \cap U_n = \emptyset$, hence $F_n \subseteq U_n^c$. Thus for each $n \geq n_0$, $U_n^c \in (V_1, ..., V_m)^+_D$, so that $\{U_n^c : n \in \mathbb{N}\}$ converges to Y^c.

(ii) $\Rightarrow$ (i): Let $B \subseteq Y$ be such that $B \in \Delta$ and $V_1, ..., V_m$ be open subsets of X such that $Y^c \cap V_i \neq \emptyset$, for $1 \leq i \leq m$, then $(V_1, ..., V_m)_B^+$ is a τ_Δ^+-neighbourhood of Y^c. By (ii), there exists $n_0 \in \mathbb{N}$ such that $U_n^c \in (V_1, ..., V_m)_B^+$, for $n \geq n_0$. Then $B \subseteq U_n$, for all $n \geq n_0$. Choose $x_i \in B^c \cap V_i$, for $1 \leq i \leq m$ and form the set $F = \{x_i : 1 \leq i \leq m\}$. Then $F \in [X]^{<\omega}$ with $F \cap V_i \neq \emptyset$, for $1 \leq i \leq m$. Also, $F \cap U_{n_0} = \emptyset$ and $B \subseteq U_{n_0}$. Thus $\mathcal{U}$ is a $\Delta\gamma$-cover of Y.

Lemma 2.9. Let $\Delta \subset CL(X) \setminus \{X\}$ and let $\mathcal{U}$ be a countable $c_\Delta(\Lambda)$-open cover of a space X. Then the following holds:

(i) Each $D \in \Delta$ is contained in infinitely many elements of $\mathcal{U}$.
(ii) If $\mathcal{V}$ is a finite subcollection of $\mathcal{U}$, then the collection $\mathcal{U} \setminus \mathcal{V}$ is a $c_\Delta(\Lambda)$-open cover of X as well.

Proof : (i) Choose $D \in \Delta$ and consider the open set $V = X$ with $(X \setminus D) \cap V \neq \phi$. Then there exists at least one $x \in X$ such that $x \notin D$. Consider $D' = D \cup \{x\}$. Then $D' \in \Delta$. Now as $\mathcal{U}$ is a $c_\Delta(\Lambda)$-cover, there exists an $U \in \mathcal{U}$ such that $D \subset D' \subseteq U$ (also the finite set can be chosen to be any $\{y\}$, where $y \notin U$, and such y exists as $X \notin \mathcal{U}$). As Δ is closed under finite unions and contains the singletons, the complement of each member of Δ contains infinitely many elements of X (as X is assumed to be infinite). Hence (i) follows.
(ii) follows from (i).

Lemma 2.10. For a space X, a sequence $\mathcal{S}$ in $CL(X)$ and $E \in CL(X)$, (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) hold:

(i) $\mathcal{S} \in \Sigma_E^{\tau_\Delta^+}$,
(ii) $\{A \cup E : A \in \mathcal{S}\} \in \Sigma_E^{\tau_\Delta^+}$,
(iii) $\{(A \cup E)^c : A \in \mathcal{S}\}$ is a $\Delta\gamma$-cover of E^c.

Proof : (i) $\Rightarrow$ (ii): Let $\mathcal{S} = \{A_n : n \in \mathbb{N}\}$ τ_Δ^+-converges to E and let $(\bigcap_{i=1}^m V_i^-) \cap (D^c)^+$ be a τ_Δ^+-neighbourhood of E. Then there exists $n_0 \in \mathbb{N}$ such that $A_n \in (\bigcap_{i=1}^m V_i^-) \cap (D^c)^+$, for all $n \geq n_0$. This implies that $A_n \cup E \subseteq D^c$, for all $n \geq n_0$ and $(A_n \cup E) \cap V_i \neq \phi$, for $1 \leq i \leq m$, for all $n \geq n_0$. Thus $\{A_n \cup E : n \in \mathbb{N}\}$ τ_Δ^+-converges to E.

(ii) $\Rightarrow$ (iii): Let $D \subseteq E^c$ be such that $D \in \Delta$ and $V_1, ..., V_m$ be open in X with $(X \setminus E^c) \cap V_i \neq \phi$, for $1 \leq i \leq m$. Then $(\bigcap_{i=1}^m V_i^-) \cap (D^c)^+$ is a τ_Δ^+-neighbourhood of E and hence by (ii), there exists $n_0 \in \mathbb{N}$ such that $A_n \cup E \in (\bigcap_{i=1}^m V_i^-) \cap (D^c)^+$, for all $n \geq n_0$. As before, we can have an $F_n \in [X]^{<\omega}$ such that $F_n \cap V_i \neq \phi$, for $1 \leq i \leq m$ with $F_n \cap (A_n \cup E)^c = \phi$ and $D \subseteq (A_n \cup E)^c$, for all $n \geq n_0$. Thus

$\{(A \cup E)^c : A \in \mathcal{S}\}$ is a $\Delta\gamma$-cover of E^c.

3. Selection principles and the $\alpha_i(\mathcal{A}, \mathcal{B})$-properties in topological spaces

Theorem 3.1. For a $\mathbb{C}_\Delta(\Lambda)$-Lindelöf space X with $\Lambda \subseteq CL(X)$, (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) holds:

(i) X satisfies $S_1(\mathbb{C}_\Delta(\Lambda), \Gamma_\Delta(\Lambda))$,
(ii) X satisfies $\alpha_2(\mathbb{C}_\Delta(\Lambda), \Gamma_\Delta(\Lambda))$,
(iii) X satisfies $\alpha_3(\mathbb{C}_\Delta(\Lambda), \Gamma_\Delta(\Lambda))$,
(iv) X satisfies $\alpha_4(\mathbb{C}_\Delta(\Lambda), \Gamma_\Delta(\Lambda))$.

Proof : As (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) always hold, we only require to prove (i) $\Rightarrow$ (ii).

(i) $\Rightarrow$ (ii): Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of $c_\Delta(\Lambda)$-covers of X and let for each $n \in \mathbb{N}$, $\mathcal{U}_n = \{U_{n,m} : m \in \mathbb{N}\}$. Set $\mathcal{V}(n,k) = \{U_{n,m} : m \geq k\}$, for each $n, k \in \mathbb{N}$. Then each $\mathcal{V}(n,k)$ is a $c_\Delta(\Lambda)$-cover of X, for each $n, k \in \mathbb{N}$. Using a bijective representation of $\mathbb{N} \times \mathbb{N}$ and applying (i) to the sequence $\mathcal{V}(n,k)$, there exists a $\Delta\gamma$-cover $\mathcal{W} = \{V_{n,k} = U_{n,m(k)} : (n,k) \in \mathbb{N} \times \mathbb{N}\}$. Thus $\mathcal{U}_n \cap \mathcal{W}$ is infinite for each $n \in \mathbb{N}$. Hence X satisfies $\alpha_2(\mathbb{C}_\Delta(\Lambda), \Gamma_\Delta(\Lambda))$.

More generally,

Theorem 3.2. For a $\mathbb{C}_\Delta(\Lambda)$-Lindelöf space X with $\Lambda \subseteq CL(X)$, (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) holds:

(i) X satisfies $S_1(\mathbb{C}_\Delta(\Lambda), \Gamma_\Theta(\Lambda))$,
(ii) X satisfies $\alpha_2(\mathbb{C}_\Delta(\Lambda), \Gamma_\Theta(\Lambda))$,
(iii) X satisfies $\alpha_3(\mathbb{C}_\Delta(\Lambda), \Gamma_\Theta(\Lambda))$,
(iv) X satisfies $\alpha_4(\mathbb{C}_\Delta(\Lambda), \Gamma_\Theta(\Lambda))$.

Setting $\Delta, \Theta \in \{CL(X), \mathbb{K}(X)\}$ and $\Lambda = CL(X)$, we get

Corollary 3.3. For a c_V-Lindelöf space X, the following statements hold:

$$S_1(\mathbb{C}_V, \Gamma_{\mathbb{C}_V}) \Rightarrow \alpha_2(\mathbb{C}_V, \Gamma_{\mathbb{C}_V}) \Rightarrow \alpha_3(\mathbb{C}_V, \Gamma_{\mathbb{C}_V}) \Rightarrow \alpha_4(\mathbb{C}_V, \Gamma_{\mathbb{C}_V}).$$

Corollary 3.4. For a k_F-Lindelöf space X, the following statements hold:

$$S_1(\mathbb{K}_F, \Gamma_{\mathbb{K}_F}) \Rightarrow \alpha_2(\mathbb{K}_F, \Gamma_{\mathbb{K}_F}) \Rightarrow \alpha_3(\mathbb{K}_F, \Gamma_{\mathbb{K}_F}) \Rightarrow \alpha_4(\mathbb{K}_F, \Gamma_{\mathbb{K}_F}).$$

Similarly as above, we have,

Theorem 3.5. For a space X and $\mathcal{B} \in \{\Gamma_\Theta(\Lambda), \Gamma_\Delta(\Lambda)\}$, (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) holds:

(i) X satisfies $S_1(\Gamma_\Delta(\Lambda), \mathcal{B})$,
(ii) X satisfies $\alpha_2(\Gamma_\Delta(\Lambda), \mathcal{B})$,
(iii) X satisfies $\alpha_3(\Gamma_\Delta(\Lambda), \mathcal{B})$,
(iv) X satisfies $\alpha_4(\Gamma_\Delta(\Lambda), \mathcal{B})$.

Proof : We have to prove only (i) $\Rightarrow$ (ii).

(i) $\Rightarrow$ (ii): Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of $\Delta\gamma$-covers of X and suppose that for each $n \in \mathbb{N}$, we have $\mathcal{U}_n = \{U_{n,m} : m \in \mathbb{N}\}$. Choose an increasing sequence $k_1 < k_2 < ... < k_p < ...$ of positive integers and for each n and each k_i consider $\mathcal{V}(n, k_i) = \{U_{n,m} : m \geq k_i\}$. Then each $\mathcal{V}(n, k_i), n, i \in \mathbb{N}$, is a $\Delta\gamma$-cover of X. Now by (i) applied to the sequence $\{\mathcal{V}(n, k_i) : i \in \mathbb{N}, n \in \mathbb{N}\}$, there exists a sequence $\{V_{n,k_i} : i, n \in \mathbb{N}\}$ such that for each $(n, i) \in \mathbb{N} \times \mathbb{N}$, $V_{n,k_i} \in \mathcal{V}(n, k_i)$ and the set $\mathcal{W} = \{V_{n,k_i} : n, i \in \mathbb{N}\} \in \mathcal{B}$. Note that $\mathcal{W}$ can be chosen in such a way that for each $n \in \mathbb{N}$ the set $\mathcal{U}_n \cap \mathcal{W}$ is infinite. Therefore $\mathcal{W}$ witnesses for the sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ that X has the property $\alpha_2(\Gamma_\Delta(\Lambda), \mathcal{B})$.

Setting $\Delta, \Theta \in \{CL(X), \mathbb{K}(X)\}$ and $\Lambda = CL(X)$, we get

Corollary 3.6. For a space X, the following statements hold:

$$S_1(\Gamma_{\mathbb{C}_V}, \Gamma_{\mathbb{C}_V}) \Rightarrow \alpha_2(\Gamma_{\mathbb{C}_V}, \Gamma_{\mathbb{C}_V}) \Rightarrow \alpha_3(\Gamma_{\mathbb{C}_V}, \Gamma_{\mathbb{C}_V}) \Rightarrow \alpha_4(\Gamma_{\mathbb{C}_V}, \Gamma_{\mathbb{C}_V}).$$

Corollary 3.7. For a space X, the following statements hold:

$$S_1(\Gamma_{\mathbb{K}_F}, \Gamma_{\mathbb{K}_F}) \Rightarrow \alpha_2(\Gamma_{\mathbb{K}_F}, \Gamma_{\mathbb{K}_F}) \Rightarrow \alpha_3(\Gamma_{\mathbb{K}_F}, \Gamma_{\mathbb{K}_F}) \Rightarrow \alpha_4(\Gamma_{\mathbb{K}_F}, \Gamma_{\mathbb{K}_F}).$$

Corollary 3.8. For a space X, the following statements hold:

$$S_1(\Gamma_{\mathbb{K}_F}, \Gamma_{\mathbb{C}_V}) \Rightarrow \alpha_2(\Gamma_{\mathbb{K}_F}, \Gamma_{\mathbb{C}_V}) \Rightarrow \alpha_3(\Gamma_{\mathbb{K}_F}, \Gamma_{\mathbb{C}_V}) \Rightarrow \alpha_4(\Gamma_{\mathbb{K}_F}, \Gamma_{\mathbb{C}_V}).$$

Note 3.9. When the second coordinate in the pair $(\mathcal{A}, \mathcal{B})$ is the collection of $c_\Theta(\Lambda)$-covers of a space X, then similarly we have,

$$S_1(\mathbb{C}_\Delta(\Lambda), \mathbb{C}_\Theta(\Lambda)) \Rightarrow \alpha_2(\mathbb{C}_\Delta(\Lambda), \mathbb{C}_\Theta(\Lambda)) \Rightarrow \alpha_3(\mathbb{C}_\Delta(\Lambda), \mathbb{C}_\Theta(\Lambda)) \Rightarrow \alpha_4(\mathbb{C}_\Delta(\Lambda), \mathbb{C}_\Theta(\Lambda))$$

and

$$S_1(\Gamma_\Delta(\Lambda), \mathbb{C}_\Theta(\Lambda)) \Rightarrow \alpha_2(\Gamma_\Delta(\Lambda), \mathbb{C}_\Theta(\Lambda)) \Rightarrow \alpha_3(\Gamma_\Delta(\Lambda), \mathbb{C}_\Theta(\Lambda)) \Rightarrow \alpha_4(\Gamma_\Delta(\Lambda), \mathbb{C}_\Theta(\Lambda)).$$

4. α_i-properties in hyperspaces

In this section we consider α_1-property in hyperspaces.

Theorem 4.1. For a space X and a collection $\Lambda \subseteq CL(X)$, the following statements are equivalent:

(i) For each $E \in \Lambda$, (Λ, τ_Δ^+) satisfies $S_1(\Sigma_E, \Sigma_E)$.
(ii) Each open subset Y of X with $Y^c \in \Lambda$ satisfies $S_1(\Gamma_\Delta^*(\Lambda), \Gamma_\Delta^*(\Lambda))$.

Proof : (i) $\Rightarrow$ (ii): Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$, $\mathcal{U}_n = \{U_{n,m} : m \in \mathbb{N}\}$ be a sequence of countable $\Delta\gamma$-covers of an open subset Y of X with $Y^c \in \Lambda$. For each $n \in \mathbb{N}$, let $\mathcal{S}_n = \mathcal{U}_n^c$. Then $\{\mathcal{S}_n : n \in \mathbb{N}\}$ is a sequence in Λ such that each $\mathcal{S}_n$ τ_Δ^+-converges to Y^c, i.e. $\mathcal{S}_n \in \Sigma_{Y^c}$ for each $n \in \mathbb{N}$. By (i), there exists a sequence

$\mathcal{S} = \{S_n : n \in \mathbb{N}\}$ in Λ that τ_Δ^+-converges to Y^c and $S_n \in \mathcal{S}_n$, for each $n \in \mathbb{N}$. Then $\{S_n^c \equiv U_{n,m_n} : n \in \mathbb{N}\}$ is a sequence such that for each n, $U_{n,m_n} \in \mathcal{U}_n$. By Lemma 2.8, it follows that $\{U_{n,m_n} : n \in \mathbb{N}\}$ is a $\Delta\gamma$-cover of Y.

(ii) $\Rightarrow$ (i): Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence in Λ converging to $E \in \Lambda$. By Lemma 2.8, $\{\mathcal{U}_n^c : n \in \mathbb{N}\}$ is a sequence of $\Delta\gamma$-covers of E^c. Hence by (ii), there exists $U_n^c \in \mathcal{U}_n^c$ such that $\{U_n^c : n \in \mathbb{N}\}$ is a $\Delta\gamma$-cover of E^c. Using Lemma 2.8 we can say that the sequence $\{U_n : n \in \mathbb{N}\}$ τ_Δ^+-converges to E.

Corollary 4.2. For a space X the following statements are equivalent:

(i) For each $E \in CL(X)$, $(CL(X), \tau_F)$ satisfies $S_1(\Sigma_E, \Sigma_E)$.
(ii) Each open subset Y of X satisfies $S_1(\Gamma^*_{\mathbb{K}_F}, \Gamma^*_{\mathbb{K}_F})$.

Corollary 4.3. For a space X the following statements are equivalent:

(i) For each $E \in CL(X)$, $(CL(X), \tau_V)$ satisfies $S_1(\Sigma_E, \Sigma_E)$.
(ii) Each open subset Y of X satisfies $S_1(\Gamma^*_{\mathbb{C}_V}, \Gamma^*_{\mathbb{C}_V})$.

Theorem 4.4. For a space X and a collection $\Lambda \subseteq CL(X)$, the following statements are equivalent:

(i) (Λ, τ_Δ^+) is an α_1-space.
(ii) For each open set $Y \subset X$ with $Y^c \in \Lambda$ and each sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of $\Delta\gamma$-covers of Y, there exists a $\Delta\gamma$-cover $\mathcal{U}$ of Y containing all but finitely many elements of $\mathcal{U}_n$, for each $n \in \mathbb{N}$.

Proof : (i) $\Rightarrow$ (ii): Let Y be open in X with $Y^c \in \Lambda$ and $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of $\Delta\gamma$-covers of Y. Put $\mathcal{A}_n = \mathcal{U}_n^c$, $n \in \mathbb{N}$. Then $\{\mathcal{A}_n : n \in \mathbb{N}\}$ is a sequence of elements of Σ_{Y^c} in (Λ, τ_Δ^+). By (i), there exists a sequence $\mathcal{A}$ in Λ that τ_Δ^+-converges to Y^c and such that for each n, $\mathcal{A}_n \setminus \mathcal{A}$ is a finite set. Set $\mathcal{U} = \mathcal{A}^c$. Then $\mathcal{U}$ is a $\Delta\gamma$-cover of Y which shows that (ii) holds.

(ii) $\Rightarrow$ (i): Let $\{\mathcal{A}_n : n \in \mathbb{N}\}$ be a sequence of elements of Σ_E in (Λ, τ_Δ^+), for $E \in \Lambda$. By Lemma 2.8, for each $n \in \mathbb{N}$, $\mathcal{U}_n = \{A^c : A \in \mathcal{A}_n\}$ is a $\Delta\gamma$-cover of the open set $E^c \subset X$. Now by (ii), there exists a sequence $\{\mathcal{V}_n : n \in \mathbb{N}\}$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n \subset \mathcal{U}_n$ and $\mathcal{U}_n \setminus \mathcal{V}_n$ is a finite set and $\mathcal{V} = \bigcup_{n \in \mathbb{N}} \mathcal{V}_n$ is a $\Delta\gamma$-cover of E^c. Then the sequence $\mathcal{A} = \{A : A^c \in \mathcal{V}\}$ contains all but finitely many elements of $\mathcal{A}_n$, for each $n \in \mathbb{N}$.

Corollary 4.5. For a space X the following statements are equivalent:

(i) $(CL(X), \tau_F)$ is an α_1-space.
(ii) For each open set $Y \subset X$ and each sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of γ_{k_F}-covers of Y, there exists a γ_{k_F}-cover $\mathcal{U}$ of Y containing all but finitely many elements of $\mathcal{U}_n$, for each $n \in \mathbb{N}$.

Corollary 4.6. For a space X the following statements are equivalent:

(i) $(CL(X), \tau_V)$ is an α_1-space.
(ii) For each open set $Y \subset X$ and each sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of γ_{c_V}-covers of Y, there exists a γ_{c_V}-cover $\mathcal{U}$ of Y containing all but finitely many elements of $\mathcal{U}_n$, for each $n \in \mathbb{N}$.

Question 4.7. Are the properties α_2, α_3 and α_4 equivalent in (Λ, τ_Δ^+)?

5. Some generalizations of Fréchet-Urysohn property in hyperspaces

In this section we deal with some generalizations of Fréchet-Urysohn property that were introduced in [2] and [12].

Definition 5.1. A space X is said to be

(i) filter-Fréchet [2] if for each $x \in X$ and each $A \in \Omega_x$ there is a sequence $\{\mathcal{F}_n : n \in \mathbb{N}\}$ of filter-bases on A such that
(FF1) For each $n \in \mathbb{N}$, there is an $F_n \in \mathcal{F}_n$ such that $x \notin Cl\ F_n$;
(FF2) For each neighbourhood U of x there is $n_0 \in \mathbb{N}$ such that $n \geq n_0$ implies $F_n \subset U$ for some $F_n \in \mathcal{F}_n$.

(ii) strongly filter-Fréchet [2] if for each $x \in X$ and each $A \in \Omega_x$ there is a sequence $\{\mathcal{F}_n : n \in \mathbb{N}\}$ of filter-bases on A satisfying (FF1) and (FF2) above and the condition
(FF3) For each $n \in \mathbb{N}$ there is a countable $F \in \mathcal{F}_n$.

(iii) strongly set-Fréchet [2] if for each $x \in X$ and each $A \in \Omega_x$ there is a sequence $\{B_n : n \in \mathbb{N}\}$ of pairwise disjoint subsets of A such that the following conditions hold:
(SF1) $x \notin Cl\ B_n$ for each $n \in \mathbb{N}$;
(SF2) each neighbourhood U of x intersects all but finitely many sets B_n;
(SF3) each B_n is countable.

(iv) set-Fréchet [2] if only conditions (SF1) and (SF2) in the definition of strongly set-Fréchet spaces are satisfied.

(v) κ-Fréchet Urysohn [12] if for every open subset U of X and every $x \in Cl\ U$, there exists a sequence $\{x_n : n \in \mathbb{N}\} \subseteq U$ converging to x.

We next investigate the above mentioned properties in (Λ, τ_Δ^+).

Theorem 5.2. For a space X, if (Λ, τ_Δ^+) is a filter-Fréchet space, then for each open subset Y of X with $Y \neq X$ and each $c_\Delta(\Lambda)$-cover $\mathcal{U}$ of Y, there is a sequence $\{\mathcal{B}_n : n \in \mathbb{N}\}$ of filter-bases on $\mathcal{U}$ such that
(i) For each $n \in \mathbb{N}$, there is $\mathcal{C}_n \in \mathcal{B}_n$ which is not a $c_\Delta(\Lambda)$-cover of Y.
(ii) For each $D \in \Delta$ and open subsets $V_1, ..., V_m$ of X with $(X \setminus Y) \cap V_i \neq \phi$, for all $1 \leq i \leq m$, there is an $n_0 \in \mathbb{N}$ such that whenever $n \geq n_0$, there exist $\mathcal{H}_n \in \mathcal{B}_n$ and $F \in [X]^{<\omega}$ satisfying $D \subset H$, $F \cap V_i \neq \phi$, for $i = 1, ...m$ and $F \cap H = \phi$, for every $H \in \mathcal{H}_n$.

Proof : Consider a $c_\Delta(\Lambda)$-cover $\mathcal{U}$ of the open subset Y of X. Then by Lemma 2.7, $\mathcal{U}^c \in \Omega_{Y^c}$. Since (Λ, τ_Δ^+) is filter-Fréchet, there is a sequence $\{\mathcal{F}_n : n \in \mathbb{N}\}$ of filter-bases on $\mathcal{U}^c$ satisfying conditions (FF1) and (FF2) of Definition 5.1. For each $n \in \mathbb{N}$, consider $\mathcal{B}_n = \{\mathcal{G}^c : \mathcal{G} \in \mathcal{F}_n\}$. Then $\{\mathcal{B}_n : n \in \mathbb{N}\}$ is a sequence of filter

bases on $\mathcal{U}$.
(i) By (FF1), for each $n \in \mathbb{N}$, there is an $\mathcal{S}_n \in \mathcal{F}_n$ such that $Y^c \notin Cl_{\tau^+_\Delta}(\mathcal{S}_n)$. Then $\mathcal{S}^c_n$ is not a $c_\Delta(\Lambda)$-cover of Y.
(ii) Next, for $D \in \Delta$ with $D \subseteq Y$ and open subsets $V_1, ..., V_m$ of X with $(X \setminus Y) \cap V_i \neq \phi$, for $1 \leq i \leq m$, $\bigcap_{i=1}^{m} V_i^- \cap (D^c)^+$ is a τ^+_Δ-neighbourhood of Y^c. Hence there exists $n_0 \in \mathbb{N}$ such that $n \geq n_0$ implies $\mathcal{S}_n \subset \bigcap_{i=1}^{m} V_i^- \cap (D^c)^+$, for some $\mathcal{S}_n \in \mathcal{F}_n$. Thus $\mathcal{S}^c_n \in \mathcal{B}_n$ such that each $S^c_n \in \mathcal{S}^c_n$ satisfies $S_n \cap V_i \neq \phi$, for all $1 \leq i \leq m$ and $S_n \subseteq D^c$. Choose $x_i \in S_n \cap V_i$, for $1 \leq i \leq m$ and consider $F = \{x_i : 1 \leq i \leq m\}$. Then $F \in [X]^{<\omega}$ with $F \cap V_i \neq \phi$, for all $1 \leq i \leq m$ and $F \cap S^c_n = \phi$, and also $D \subseteq S^c_n$.

Corollary 5.3. For a space X, if $(CL(X), \tau_F)$ is a filter-Fréchet space, then for each open subset Y of X with $Y \neq X$ and each k_F-cover $\mathcal{U}$ of Y, there is a sequence $\{\mathcal{B}_n : n \in \mathbb{N}\}$ of filter-bases on $\mathcal{U}$ such that
(i) For each $n \in \mathbb{N}$, there is $\mathcal{C}_n \in \mathcal{B}_n$ which is not a k_F-cover of Y.
(ii) For each compact set K and open subsets $V_1, ..., V_m$ of X with $(X \setminus Y) \cap V_i \neq \phi$, for all $1 \leq i \leq m$, there is an $n_0 \in \mathbb{N}$ such that whenever $n \geq n_0$, there exist $\mathcal{H}_n \in \mathcal{B}_n$ and $F \in [X]^{<\omega}$ satisfying $K \subset H$, $F \cap V_i \neq \phi$, for $i = 1, ...m$ and $F \cap H = \phi$, for every $H \in \mathcal{H}_n$.

Corollary 5.4. For a space X, if $(CL(X), \tau_V)$ is a filter Fréchet space, then for each open subset Y of X with $Y \neq X$ and each c_V-cover $\mathcal{U}$ of Y, there is a sequence $\{\mathcal{B}_n : n \in \mathbb{N}\}$ of filter-bases on $\mathcal{U}$ such that
(i) For each $n \in \mathbb{N}$, there is $\mathcal{C}_n \in \mathcal{B}_n$ which is not a c_V-cover of Y.
(ii) For each closed set D and open subsets $V_1, ..., V_m$ of X with $(X \setminus Y) \cap V_i \neq \phi$, for all $1 \leq i \leq m$, there is an $n_0 \in \mathbb{N}$ such that whenever $n \geq n_0$, there exist $\mathcal{H}_n \in \mathcal{B}_n$ and $F \in [X]^{<\omega}$ satisfying $D \subset H$, $F \cap V_i \neq \phi$, for $i = 1, ...m$ and $F \cap H = \phi$, for every $H \in \mathcal{H}_n$.

Theorem 5.5. For a space X, if (Λ, τ^+_Δ) is a strongly filter-Fréchet space, then for each open subset Y of X with $Y \neq X$ and each $c_\Delta(\Lambda)$-cover $\mathcal{U}$ of Y, there is a sequence $\{\mathcal{B}_n : n \in \mathbb{N}\}$ of filter-bases on $\mathcal{U}$ such that
(i) For each $n \in \mathbb{N}$, there is $\mathcal{C}_n \in \mathcal{B}_n$ which is not a $c_\Delta(\Lambda)$-cover of Y.
(ii) For each $D \in \Delta$ and open subsets $V_1, ..., V_m$ of X with $(X \setminus Y) \cap V_i \neq \phi$, for all $1 \leq i \leq m$, there is an $n_0 \in \mathbb{N}$ such that whenever $n \geq n_0$, there exist $\mathcal{H}_n \in \mathcal{B}_n$ and $F \in [X]^{<\omega}$ satisfying $D \subset H$, $F \cap V_i \neq \phi$, for $i = 1, ...m$ and $F \cap H = \phi$, for every $H \in \mathcal{H}_n$.
(iii) For each $n \in \mathbb{N}$, there is some countable element in $\mathcal{B}_n$.

Proof : Similar to that of the proof of Theorem 5.2.

Corollary 5.6. For a space X, if $(CL(X), \tau_F)$ is a strongly filter-Fréchet space, then for each open subset Y of X with $Y \neq X$ and each k_F-cover $\mathcal{U}$ of Y, there is a sequence $\{\mathcal{B}_n : n \in \mathbb{N}\}$ of filter-bases on $\mathcal{U}$ such that
(i) For each $n \in \mathbb{N}$, there is $\mathcal{C}_n \in \mathcal{B}_n$ which is not a k_F-cover of Y.
(ii) For each compact set K and open subsets $V_1, ..., V_m$ of X with $(X \setminus Y) \cap V_i \neq \phi$,

for all $1 \leq i \leq m$, there is an $n_0 \in \mathbb{N}$ such that whenever $n \geq n_0$, there exist $\mathcal{H}_n \in \mathcal{B}_n$ and $F \in [X]^{<\omega}$ satisfying $K \subset H$, $F \cap V_i \neq \phi$, for $i = 1, ...m$ and $F \cap H = \phi$, for every $H \in \mathcal{H}_n$.
(iii) For each $n \in \mathbb{N}$, there is some countable element in $\mathcal{B}_n$.

Corollary 5.7. For a space X, if $(CL(X), \tau_V)$ is a strongly filter-Fréchet space, then for each open subset Y of X with $Y \neq X$ and each c_V-cover $\mathcal{U}$ of Y, there is a sequence $\{\mathcal{B}_n : n \in \mathbb{N}\}$ of filter-bases on $\mathcal{U}$ such that
(i) For each $n \in \mathbb{N}$, there is $\mathcal{C}_n \in \mathcal{B}_n$ which is not a c_V-cover of Y.
(ii) For each closed set D and open subsets $V_1, ..., V_m$ of X with $(X \setminus Y) \cap V_i \neq \phi$, for all $1 \leq i \leq m$, there is an $n_0 \in \mathbb{N}$ such that whenever $n \geq n_0$, there exist $\mathcal{H}_n \in \mathcal{B}_n$ and $F \in [X]^{<\omega}$ satisfying $D \subset H$, $F \cap V_i \neq \phi$, for $i = 1, ...m$ and $F \cap H = \phi$, for every $H \in \mathcal{H}_n$.
(iii) For each $n \in \mathbb{N}$, there is some countable element in $\mathcal{B}_n$.

Theorem 5.8. For a space X if (Λ, τ_Δ^+) has the strong set-Fréchet property, then for each open subset Y of X with $Y \neq X$ and each $c_\Delta(\Lambda)$-cover $\mathcal{U}$ of Y, there is a sequence $\{\mathcal{V}_n : n \in \mathbb{N}\}$ of countable, pairwise disjoint subsets of $\mathcal{U}$ such that
(i) No $\mathcal{V}_n$ is a $c_\Delta(\Lambda)$-cover of Y.
(ii) For each $D \in \Delta$ with $D \subseteq Y$ and open subsets $V_1, ..., V_m$ of X with $(X \setminus Y) \cap V_i \neq \phi$, for all $1 \leq i \leq m$, there is an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, there exist $B_n^c \in \mathcal{V}_n$ and $F \in [X]^{<\omega}$ satisfying $D \subseteq B_n^c$, $F \cap V_i \neq \phi$, for $i = 1, ...m$ and $F \cap B_n^c = \phi$.

Proof : Let $\mathcal{U}$ be a $c_\Delta(\Lambda)$-cover of the open subset Y of X. By Lemma 2.7, $\mathcal{U}^c \in \Omega_{Y^c}$.
(i) As (Λ, τ_Δ^+) has the strong set-Fréchet property, choose countable, pairwise disjoint sets $\mathcal{B}_n \subset \mathcal{U}^c$, $n \in \mathbb{N}$ such that $Y^c \notin Cl_{\tau_\Delta^+}(\mathcal{B}_n)$, for each $n \in \mathbb{N}$, but each τ_Δ^+-neighbourhood U of Y^c intersects all but finitely many $\mathcal{B}_n$, i.e. there exists $n_0 \in \mathbb{N}$ such that $U \cap \mathcal{B}_n \neq \phi$, for all $n \geq n_0$. Then the sets $\mathcal{V}_n = \mathcal{B}_n^c$, $n \in \mathbb{N}$ are countable, pairwise disjoint subsets of $\mathcal{U}$. No $\mathcal{V}_n$ is a $c_\Delta(\Lambda)$-cover of Y.
(ii) Now, for $D \in \Delta$ with $D \subseteq Y$ and open subsets $V_1, ..., V_m$ of X with $(X \setminus Y) \cap V_i \neq \phi$, for all $1 \leq i \leq m$, $\bigcap_{i=1}^{m} V_i^- \cap (D^c)^+$ is a τ_Δ^+-neighbourhood of Y^c, so that $\bigcap_{i=1}^{m} V_i^- \cap (D^c)^+ \cap \mathcal{B}_n \neq \phi$, for all $n \geq n_0$. Pick any $B_n \in \bigcap_{i=1}^{m} V_i^- \cap (D^c)^+ \cap \mathcal{B}_n$, for each n. Then $B_n \cap V_i \neq \phi$, $B_n \subseteq D^c$. Consider $F = \{x_i : 1 \leq i \leq m\}$, where $x_i \in B_n \cap V_i$, for all $1 \leq i \leq m$. Then $F \cap V_i \neq \phi$, for all $1 \leq i \leq m$, $D \subseteq B_n^c$ with $F \cap B_n^c = \phi$, for all $n \geq n_0$.

Corollary 5.9. For a space X, if $(CL(X), \tau_F)$ has the strong set-Fréchet property, then for each open subset Y of X with $Y \neq X$ and each k_F-cover $\mathcal{U}$ of Y, there is a sequence $\{\mathcal{V}_n : n \in \mathbb{N}\}$ of countable, pairwise disjoint subsets of $\mathcal{U}$ such that
(i) No $\mathcal{V}_n$ is a k_F-cover of Y.
(ii) For each compact set $K \subseteq Y$ and open subsets $V_1, ..., V_m$ of X with $(X \setminus Y) \cap V_i \neq \phi$, for all $1 \leq i \leq m$, there is an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, there exist $B_n^c \in \mathcal{V}_n$ and $F \in [X]^{<\omega}$ satisfying $K \subset B_n^c$, $F \cap V_i \neq \phi$, for $i = 1, ...m$ and $F \cap B_n^c = \phi$.

Corollary 5.10. For a space X, if $(CL(X), \tau_V)$ has the strong set-Fréchet property,

then for each open subset Y of X with $Y \neq X$ and each $c_{_V}$-cover $\mathcal{U}$ of Y, there is a sequence $\{\mathcal{V}_n : n \in \mathbb{N}\}$ of countable, pairwise disjoint subsets of $\mathcal{U}$ such that
(i) No $\mathcal{V}_n$ is a $c_{_V}$-cover of Y.
(ii) For each closed set $D \subseteq Y$ and open subsets $V_1, ..., V_m$ of X with $(X\backslash Y)\cap V_i \neq \phi$, for all $1 \leq i \leq m$, there is an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, there exist $B_n^c \in \mathcal{V}_n$ and $F \in [X]^{<\omega}$ satisfying $D \subset B_n^c$, $F \cap V_i \neq \phi$, for $i = 1, ...m$ and $F \cap B_n^c = \phi$.

Theorem 5.11. For a space X, if (Λ, τ_Δ^+) has the set-Fréchet property, then for each open subset Y of X with $Y \neq X$ and each $c_\Delta(\Lambda)$-cover $\mathcal{U}$ of Y, there is a sequence $\{\mathcal{V}_n : n \in \mathbb{N}\}$ of pairwise disjoint subsets of $\mathcal{U}$ such that
(i) No $\mathcal{V}_n$ is a $c_\Delta(\Lambda)$-cover of Y.
(ii) For each $D \in \Delta$ with $D \subseteq Y$ and open subsets $V_1, ..., V_m$ of X with $(X\backslash Y)\cap V_i \neq \phi$, for all $1 \leq i \leq m$, there is an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, there exist $B_n^c \in \mathcal{V}_n$ and $F \in [X]^{<\omega}$ satisfying $D \subset B_n^c$, $F \cap V_i \neq \phi$, for $i = 1, ...m$ and $F \cap B_n^c = \phi$.

Proof : Similar to that of the proof of Theorem 5.8.

Corollary 5.12. For a space X, if $(CL(X), \tau_{_F})$ has the set-Fréchet property, then for each open subset Y of X with $Y \neq X$ and each $k_{_F}$-cover $\mathcal{U}$ of Y, there is a sequence $\{\mathcal{V}_n : n \in \mathbb{N}\}$ of pairwise disjoint subsets of $\mathcal{U}$ such that
(i) No $\mathcal{V}_n$ is a $k_{_F}$-cover of Y.
(ii) For each compact set $K \subseteq Y$ and open subsets $V_1, ..., V_m$ of X with $(X\backslash Y)\cap V_i \neq \phi$, for all $1 \leq i \leq m$, there is an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, there exist $B_n^c \in \mathcal{V}_n$ and $F \in [X]^{<\omega}$ satisfying $K \subset B_n^c$, $F \cap V_i \neq \phi$, for $i = 1, ...m$ and $F \cap B_n^c = \phi$.

Corollary 5.13. For a space X, if $(CL(X), \tau_{_V})$ has the set-Fréchet property, then for each open subset Y of X with $Y \neq X$ and each $c_{_V}$-cover $\mathcal{U}$ of Y, there is a sequence $\{\mathcal{V}_n : n \in \mathbb{N}\}$ of pairwise disjoint subsets of $\mathcal{U}$ such that
(i) No $\mathcal{V}_n$ is a $c_{_V}$-cover of Y.
(ii) For each closed set $D \subseteq Y$ and open subsets $V_1, ..., V_m$ of X with $(X\backslash Y)\cap V_i \neq \phi$, for all $1 \leq i \leq m$, there is an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, there exist $B_n^c \in \mathcal{V}_n$ and $F \in [X]^{<\omega}$ satisfying $D \subset B_n^c$, $F \cap V_i \neq \phi$, for $i = 1, ...m$ and $F \cap B_n^c = \phi$.

Theorem 5.14. For a space X, (Λ, τ_Δ^+) is κ-Fréchet Urysohn if and only if for each open subset $\mathcal{A}$ of Λ and every open subset A of X with $\mathcal{A}^c$ a $c_\Delta(\Lambda)$-cover of A, there exists a sequence $\{A_1, A_2, ...\} \subseteq \mathcal{A}$ such that $\{A_1^c, A_2^c, ...\}$ is a $\Delta\gamma$-cover of A^c.

Proof : First let $\mathcal{A}$ be an open subset of Λ and $A \subseteq X$ be such that $\mathcal{A}^c$ is a $c_\Delta(\Lambda)$-cover of A. Then by Lemma 2.7, $\mathcal{A} \in \Omega_{A^c}$ and then by hypothesis there exists a sequence $\{A_1, A_2, ...\} \subseteq \mathcal{A}$ such that $\{A_1, A_2, ...\}$ converges to A. Then by Lemma 2.8, $\{A_1^c, A_2^c, ...\}$ becomes a $\Delta\gamma$-cover of A^c.

Conversely, let $\mathcal{A}$ be an open subset of Λ and $\mathcal{A} \in \Omega_{A^c}$. Then by Lemma 2.7, $\mathcal{A}^c$ is a $c_\Delta(\Lambda)$-cover of A. Hence by the given condition, there exists a sequence $\{A_1, A_2, ...\} \subseteq \mathcal{A}$ such that $\{A_1^c, A_2^c, ...\}$ is a $\Delta\gamma$-cover of A^c. Hence by Lemma 2.8 $\{A_1, A_2, ...\}$ converges to A.

Corollary 5.15. For a space X, $(CL(X), \tau_{_F})$ is κ-Fréchet Urysohn if and only if for each open subset $\mathcal{A}$ of $CL(X)$ and every open subset A of X with $\mathcal{A}^c$ a $k_{_F}$-cover of A, there exists a sequence $\{A_1, A_2, ...\} \subseteq \mathcal{A}$ such that $\{A_1^c, A_2^c, ...\}$ is a $\gamma_{_{k_F}}$-cover of A^c.

Corollary 5.16. For a space X, $(CL(X), \tau_V)$ is κ-Fréchet Urysohn if and only if for each open subset $\mathcal{A}$ of $CL(X)$ and every open subset A of X with $\mathcal{A}^c$ a c_V-cover of A, there exists a sequence $\{A_1, A_2, ...\} \subseteq \mathcal{A}$ such that $\{A_1^c, A_2^c, ...\}$ is a γ_{c_V}-cover of A^c.

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REFERENCES

1. A. V. Arhangel'skii: *Frequency spectrum of a topological space and the classification of spaces.* Dokl. Akad. Nauk SSSR 206, 265-268, 1972 Sov. Math. Dokl. **13**, 1185-1189, 1972.
2. A. V. Arhangel'skii and T. Nogura: *Relative sequentiality.* Topol. Appl. **82**, no. 1-3, 49-58, 1998.
3. E. Borel: *Sur la classification des ensembles de mesure nulle.* Bull. Soc. Math. de France. **47**, 97-125, 1919.
4. R. Cruz-Castillo, A. Ramírez-Páramo and J. F. Tenorio: *Menger and Menger-type star selection principles for hit-and-miss topology.* Topol. Appl. **290**, 107574, 2021.
5. J. Fell: *Hausdorff topology for the closed subsets of a locally compact non-Hausdorff spaces.* Proc. Amer. Math. Soc. **13**, 472-476, 1962.
6. J. Gerlits and Zs. Nagy: *Some properties of C(X), I.* Topol. Appl. **14**, 151-161, 1982.
7. F. Hausdorff: *Grundzuge der Mengenlehre.* Leipzig, 1914.
8. W. Hurewicz: *Über eine Verallgemeinerung des Borelschen Theorems.* Math. Z. **24**, 401-425, 1925.
9. Lj. D. R. Kočinac: *Selection principles related to α_i-properties.* Taiwanese J. Math. **12**, 561-571, 2008.
10. Lj. D. R. Kočinac: *Selected results on selection principles.* in Proceedings of the 3rd Seminar on Geometry and Topology (Sh. Rezapour, ed.)July 15-17 Tabriz, Iran, 71-104, 2004.
11. Z. Li: *Selection principles of the Fell topology and the Vietoris topology.* Topol. Appl. **212**, 90-104, 2016.
12. C. Liu and L. D. Ludwig: *κ-Fréchet Urysohn spaces.* Houston J. Math. 31, 391-401, 2005.
13. R. A. McCoy: *Function spaces which are k-spaces.* Topology Proc. **5**, 139-146, 1980.
14. G. Di Maio and Ĺ. Holá: *On hit-and-miss topologies.* Rend. Accad. Sci. Fis. Mat. Napoli **62**, 103-124, 1995.
15. G. Di Maio, Lj. D. R. Kočinac and E. Meccariello: *Selection principles and hyperspace topologies.* Topol. Appl. **153**, 912-923, 2005.

16. G. Di Maio and Lj. D. R. Kočinac: *Some covering properties of hyperspaces.* Topol. Appl. **155**, 1959-1969, 2008.

17. G. Di Maio, Lj. D. R. Kočinac and E. Meccariello: *Applications of k-covers.* Acta Math. Sin. Engl. Ser. **22** (4), 1151-1160, 2006.

18. G. Di Maio, Lj. D. R. Kočinac and T. Nogura: *Convergence properties of hyperspaces.* J. Korean Math. Soc. **44**, 845-854, 2007.

19. K. Menger: *Einige Überdeckungssätze der Punltmengenlehre, in: Sitzungsberichte Abt. 2a.* Mathematik, Astronomie, Physik, Meteorologie und Mechanik (Wiener Akademie, Wien) **133**, 421-444, 1924.

20. E. Michael: *Topologies on spaces of subsets.* Trans. Am. Math. Soc. **71**, 152-182, 1951.

21. D. Pompeiu: *Sur la continuité des fonctions de variables complexes.* Ann. Fac. Sci. de Toulouse : Mathématiques, Sér. 2, Tome 7, no. **3**, 265-315, 1905.

22. H. Poppe: *Eine Bemerkung über Trennungsaxiome in Raumen von abgeschlossenen Teilmengen topologisher Raume.* Arch. Math. **16**, 197-198, 1965.

23. A. V. Osipov: *Selectors for sequences of subsets of hyperspaces.* Topol. Appl. **275**, 107007, 2020.

24. F. Rothberger: *Eine Verschärfung der Eigenschaft C.* Fundam. Math. **30**, 50-55, 1938.

25. M. Scheepers: *Combinatorics of open covers I: Ramsey theory.* Topol. Appl. **69**, 31-62, 1996.

26. R. Sen and A. Ramírez-Páramo: *On $c_\Delta(\Lambda)$-covers and $\Delta\gamma$-sets.* Topol. Appl. **307**, 107940, 2022.

27. L. Vietoris: *Bereiche Zweiter Ordnung.* Monatshefte Math. Phys. **33**, 49-62, 1923.

28. L. Zsilinszky: *Baire spaces and hyperspace topologies.* Proc. Am. Math. Soc. **124**, 3175-3184, 1996.