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PROPAGATION OF POLARIZED LIGHT AND ELECTROMAGNETIC CURVES IN THE OPTICAL FIBER IN WALKER 3-MANIFOLDS

Zehra Özdemir[1] and Ameth Ndiaye[2]

[1]**Amasya University, Arts and Science Faculty**
Department of Mathematics, 05189 Amasya, Turkey
[2]**Université Cheikh Anta Diop**
Faculté des Sciences et Technologies de l'Education et de la Formation
Département de Mathématiques, BP 5036, Dakar-fann, Sénégal

Abstract. In the present paper, we define the three cases of the geometric phase equations associated with a monochromatic linearly polarized light wave traveling along an optical fiber in three-dimensional Walker manifold (M, g_f^ε). Walker manifolds have many applications in mathematics and theoretical physics. We are working in the context of a pseudo-Riemannian manifold (i.e. a manifold equipped with a non-degenerate arbitrary signature metric tensor). That is, we generalize the motion of the light wave in the optical fiber and the associated electromagnetic curves that describe the motion of a charged particle under the influence of an electromagnetic field over a Walker space defined as a pseudo-Riemannian manifold with a light-like distribution, parallel to the Levi-Civita junction. These manifolds (especially Lorentzian) are important in physics because of their applications in general relativity. Then, we obtain the Rytov curves related to the cases of geometric phase models. Moreover, we give some examples and visualize the evolution of the electric field along the optical fiber in (M, g_f^ε) via MAPLE program.
Keywords: Walker manifold, electromagnetic curves, light wave.

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Corresponding Author: Ameth Ndiaye (ameth1.ndiaye@ucad.edu.sn)
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1. Introduction

Examination of physical phenomena from the geometric point of view has been made by many researchers and is still being investigated. For this reason, the theory of curves is used in research in many fields of physics. For example, the theory of curves is used when investigating the orbits of charged particles and the geometric phase model in an electromagnetic field. Ross showed that he studied the polarization state of the polymetric fiber, which is important in optics, using space curves, [4]. Berry produced some remarkable work examining the geometric phase [5]. Berry showed that a quantum system depends on some parameters and can take on a topological phase in addition to the usual dynamic phase. Next, Kugler and Shtrikman focused on the geometric nature of the polarization rotation in the optical fiber by treating the fiber as a space curve and showed that the phase dependence of this phenomenon can be explained by parallel transport along the optical fiber, [6]. Vladimirski discussed the topological phase of quantum mechanics, like the geometric phase in [7]. Dandoloff explained the Fermi-Walker parallel transport law, which plays an important role in general relativity, with geometric phase and examined other possible parallel transport cases [8]. Kravtsov and Orlov, along with the expansion of the usage areas of geometric optics, furthered the work done in this field, [9]. Frins and Dultz geometrically interpreted the plane of return of a light wave, [10]. Then, in 3-dimensional spaces, a charged particle moving in the electromagnetic field is investigated, and also a generalization of the Landau problem is given, [11, 14, 15, 12, 16, 13]. Barros et al. discussed the motion of a charged particle in a Killing vector field. They presented a new perspective for solving many problems by examining [17, 18, 19]. Cabrerizo, who dealt with the Landau-Hall problem in the 2D unit sphere, also examined the relationship of this problem with the Killing vector field in the 3D sphere [23]. Bozkurt et al. defined a new type of magnetic curve and gave some new characterizations [21]. Körpınar and Demırkol, studied the rotation of the polarization plane in the light wave in the optical fiber [22]. Özdemir investigated the polarization state of the polarized light on the conditions that electric makes a constant angle with a tangent, normal and binormal vectors [20, 24]. Then, the evolutions of the electric field and electromagnetic curves are characterized by many authors [26, 27, 28, 25, 29, 30, 31, 32, 33].

The research is presented as follows: In the first chapter, the studies on the subject are explained and given as an introduction. In the second chapter, theoretical and basic information about the study is presented. The third chapter examines the relationship between Berry phase models and electromagnetic curves in optical fiber. Also in this chapter, the Fermi-Walker derivative is introduced and the relationship between this derivative and the motion of the polarization vector in the optical fiber is examined. In the fourth chapter, the mathematical and physical results of the new information obtained during the study are given and electromagnetic curves are investigated. In the sixth section, examples are given using the MAPLE program.

2. Preliminaries

A Walker 3-manifold is a three-dimensional Lorentzian manifold admitting a parallel degenerate line field. The metric of the Walker Manifold is investigated by Walker (see, [1]). Walker described the canonical form for a space with parallel field of null planes. The metric tensor of a three-dimensional Walker manifold (M, g_f^ε) with coordinates (x, y, z) is given by

$$g_f^\varepsilon = dx \circ dz + \varepsilon dy^2 + f(x, y, z)dz^2 \tag{2.1}$$

and its matrical representation is obtained as

$$g_f^\varepsilon = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \varepsilon & 0 \\ 1 & 0 & f \end{pmatrix} \text{ with inverse } (g_f^\varepsilon)^{-1} = \begin{pmatrix} -f & 0 & 1 \\ 0 & \varepsilon & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

for some function $f(x, y, z)$, where $\varepsilon = \pm 1$ and thus $D = \mathrm{Span}\{\partial_x\}$ as the parallel degenerate line field. Notice that when $\varepsilon = 1$ and $\varepsilon = -1$ the Walker manifold has signature $(2, 1)$ and $(1, 2)$ respectively. Thus, in both cases, it is Lorentzian. For more detail about the Walker 3-manifold, see [2].

Then, the Levi-Civita connection of any metric (2.1) is calculated as follows:

$$\begin{aligned} \nabla_{\partial_x}\partial z &= \frac{1}{2} f_x \partial_x, \quad \nabla_{\partial_y}\partial z = \frac{1}{2} f_y \partial_x, \\ \nabla_{\partial_z}\partial z &= \frac{1}{2}(f f_x + f_z)\partial_x + \frac{1}{2} f_y \partial_y - \frac{1}{2} f_x \partial_z. \end{aligned} \tag{2.2}$$

If the function f satisfies $f(x, y, z) = f(y, z)$, then (M, g_f^ε) is a strict Walker manifold and associated Levi-Civita connection is computed as

$$\nabla_{\partial_y}\partial z = \frac{1}{2} f_y \partial_x, \quad \nabla_{\partial_z}\partial z = \frac{1}{2} f_z \partial_x - \frac{\varepsilon}{2} f_y \partial_y. \tag{2.3}$$

The existence of a null parallel vector field (i.e $f = f(y, z)$) simplifies the non-zero components of the Christoffel symbols. Thus, we have

$$\Gamma_{23}^1 = \Gamma_{32}^1 = \frac{1}{2} f_y, \ \Gamma_{33}^1 = \frac{1}{2} f_z, \ \Gamma_{33}^2 = -\frac{\varepsilon}{2} f_y \tag{2.4}$$

Let now u and v be two vectors in M. Denoted by $(\vec{i}, \vec{j}, \vec{k})$ the canonical frame in $\mathbb{R}^3$. The vector product of u and v in (M, g_f^ε) with respect to the metric g_f^ε is the vector denoted by $u \times_f v$ in M defined by

$$g_f^\varepsilon(u \times_f v, w) = \det(u, v, w) \tag{2.5}$$

for all vector w in M, where $\det(u, v, w)$ is the determinant function associated to the canonical basis of $\mathbb{R}^3$. If $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ then by using (2.5), we have:

$$u\times = (u_1 v_2 - u_2 v_1 - (u_2 v_3 - u_3 v_2) f, -\varepsilon(u_1 v_3 - u_3 v_1), u_2 v_3 - u_3 v_2). \tag{2.6}$$

Let $\alpha : I \subset \mathbb{R} \longrightarrow (M, g_f^{\varepsilon})$ be a curve parametrized by its arc-length s.
The Frenet frame of α is the vectors $\mathbf{t}$, $\mathbf{n}$ and $\mathbf{B}$ along α where $\mathbf{t}$ is the tangent, $\mathbf{n}$ the principal normal and $\mathbf{B}$ the binormal vector. They satisfied the Frenet formulas

$$\left\{ \begin{array}{lcc} \nabla_{\mathbf{t}}\mathbf{t}(s) & = & \varepsilon_2\kappa(s)\mathbf{n}(s) \\ \nabla_{\mathbf{t}}\mathbf{n}(s) & = & -\varepsilon_1\kappa\mathbf{t}(s) - \varepsilon_3\tau\mathbf{b}(s) \\ \nabla_{\mathbf{t}}\mathbf{b}(s) & = & \varepsilon_2\tau(s)\mathbf{n}(s) \end{array} \right. \tag{2.7}$$

where κ and τ are respectively the curvature and the torsion of the curve α, with $\varepsilon_1 = g_f(\mathbf{t}, \mathbf{t})$, $\varepsilon_2 = g_f(\mathbf{n}, \mathbf{n})$ and $\varepsilon_3 = g_f(\mathbf{b}, \mathbf{b})$.

3. The evolution of the electric fields in the optical fiber

In this section, let us assume that the linearly polarized light wave with the same wavelength is combined into the optical fiber and the optical fiber injected as a space curve on a Walker manifold (M, g_f^{ε}). Thus, these give us a general viewpoint of the behavior of the polarized light along an optical fiber.

The direction of the state of a linearly polarized light wave is referred to by the direction of the electric field $\mathbf{E}$ in the optical fiber. Then, the direction along with the optical fiber can be defined as the linear combination of the Frenet frame as follows:

$$\nabla_{\mathbf{t}}\mathbf{E} = \Upsilon_1\mathbf{t}(s) + \Upsilon_2\mathbf{n}(s) + \Upsilon_3\mathbf{b}(s). \tag{3.1}$$

The related geometric phase equations is studied in three subsections. In the following subsections, We will use the following equations to shorten the operations. If $\varepsilon_E = -\varepsilon_2$, then $f_1(\phi) = \sinh\phi$ and $f_2(\phi) = \cosh\phi$. If $\varepsilon_E = \varepsilon_2 = -\varepsilon_1$=-1, then $f_1(\phi) = \cosh\phi$ and $f_2(\phi) = \sinh\phi$. If $\varepsilon_E = \varepsilon_2 = -\varepsilon_1$=1, then $f_1(\phi) = \cos\phi$ and $f_2(\phi) = \sin\phi$.

3.1. $\mathbf{E_t}$-Rytov curves

Electric field $\mathbf{E}$ lies on a plane orthogonal to the tangent vector $\mathbf{t}$. Therefore, the following equation arises:

$$g_f^{\varepsilon}(\mathbf{E}, \mathbf{t}(s)) = 0. \tag{3.2}$$

Differentiating the equation in (3.2), and using (2.7), we get

$$g_f^{\varepsilon}(\nabla_{\mathbf{t}}\mathbf{E}, \mathbf{t}(s)) = -\varepsilon_2\kappa g_f^{\varepsilon}(\mathbf{E}, \mathbf{n}). \tag{3.3}$$

Using the equations (3.1) with (3.3), we can obtain the following

$$\Upsilon_1 = -\varepsilon_1\varepsilon_2\kappa g_f^{\varepsilon}(\mathbf{E}, \mathbf{n}). \tag{3.4}$$

Due to absorption, we can suppose that there is no mechanism loss in the optical fiber, and we get

$$g_f^{\varepsilon}(\mathbf{E}, \mathbf{E}) = \varepsilon_{\mathbf{E}}c, \tag{3.5}$$

where c is constant and $\varepsilon_{\mathbf{E}} = \pm$.
Now we differentiate (3.5) and we obtain

$$g_f^{\varepsilon}(\nabla_{\mathbf{t}}\mathbf{E}, \mathbf{E}) = 0. \tag{3.6}$$

Then if we use the equation in (3.1) and the following equation

$$\mathbf{E} = \varepsilon_2 g_f^{\varepsilon}(\mathbf{E}, \mathbf{n})\mathbf{n} + \varepsilon_3 g_f^{\varepsilon}(\mathbf{E}, \mathbf{b})\mathbf{b},$$

we obtain

$$\Upsilon_2 g_f^{\varepsilon}(\mathbf{E}, \mathbf{n}(s)) = -\Upsilon_3 g_f^{\varepsilon}(\mathbf{E}, \mathbf{b}(s)). \tag{3.7}$$

If $g_f^{\varepsilon}(\mathbf{E}, \mathbf{b}(s)) \neq 0$ and $g_f^{\varepsilon}(\mathbf{E}, \mathbf{n}(s)) \neq 0$, then this case show that Υ_2 and Υ_3 are proportional and we have

$$\Upsilon_2 = \xi g_f^{\varepsilon}(\mathbf{E}, \mathbf{b}(s)) \textit{ and } \Upsilon_3 = -\xi g_f^{\varepsilon}(\mathbf{E}, \mathbf{n}(s)). \tag{3.8}$$

And then, from the equations (3.2)-(3.8), we obtain

$$\nabla_{\mathbf{t}}\mathbf{E} = (-\varepsilon_1\varepsilon_2\kappa g_f^{\varepsilon}(\mathbf{E}, \mathbf{n}))\mathbf{t}(s) + (\xi g_f^{\varepsilon}(\mathbf{E}, \mathbf{b}(s)))\mathbf{n}(s) + (-\xi g_f^{\varepsilon}(\mathbf{E}, \mathbf{n}(s)))\mathbf{b}(s). \tag{3.9}$$

When we use the Darboux frame fields' vector product, we obtain the following

$$\nabla_{\mathbf{t}}\mathbf{E} = (-\varepsilon_1\varepsilon_2\kappa g_f^{\varepsilon}(\mathbf{E}, \mathbf{n}))\mathbf{t}(s) + \xi\varepsilon_2(\mathbf{E} \times_f \mathbf{t}(s)). \tag{3.10}$$

The part that rotates around the tangent vector $\mathbf{t}$ is indicated by the second part of (3.10). If we suppose that $\mathbf{t}$ is parallel transported (i.e. $\xi = 0$), then it is calculated as follows

$$\nabla_{\mathbf{t}}\mathbf{E} = (-\varepsilon_1\varepsilon_2\kappa g_f^{\varepsilon}(\mathbf{E}, \mathbf{n}))\mathbf{t}(s). \tag{3.11}$$

We can also write in general

$$\mathbf{E} = \varepsilon_2 g_f^{\varepsilon}(\mathbf{E}, \mathbf{n}(s))\mathbf{n}(s) + \varepsilon_3 g_f^{\varepsilon}(\mathbf{E}, \mathbf{b}(s))\mathbf{b}(s). \tag{3.12}$$

If we compare with the derivative of the equations in (3.11) and (3.10), we find that this matrix form is

$$\begin{bmatrix} g_f^{\varepsilon}(\mathbf{E}, \mathbf{n}(s))' \\ g_f^{\varepsilon}(\mathbf{E}, \mathbf{b}(s))' \end{bmatrix} = \begin{bmatrix} 0 & \varepsilon_2\tau \\ -\varepsilon_3\tau & 0 \end{bmatrix} \begin{bmatrix} g_f^{\varepsilon}(\mathbf{E}, \mathbf{n}(s)) \\ g_f^{\varepsilon}(\mathbf{E}, \mathbf{b}(s)) \end{bmatrix}. \tag{3.13}$$

From the equality $g_f^{\varepsilon}(\mathbf{E}, \mathbf{E}) = \varepsilon_E c$, c is a constant, without losing generality, using the polar coordinates we have

$$\mathbf{E} = f_1(\phi)\mathbf{n}(s) + f_2(\phi)\mathbf{b}(s). \tag{3.14}$$

If we differentiate the equation in (3.14), we compute

$$\nabla_{\mathbf{t}}\mathbf{E} \quad = (-\varepsilon_1\kappa f_1)\mathbf{t}(s) + (\tfrac{d\phi}{ds} + \varepsilon_1\varepsilon_2\tau)\mathbf{E} \times_f \mathbf{t}. \tag{3.15}$$

We must take $\frac{d\phi}{ds} = -\varepsilon_1\varepsilon_2\tau$ along the optical fiber. Thus, the direction of the polarization state $\mathbf{E}$ is parallel to the direction of $\mathbf{t}(s)$. Then, using the Fermi-Walker transportation law, we reach

$$\nabla_{\mathbf{t}}\mathbf{E}^{FW} = \nabla_{\mathbf{t}}\mathbf{E} \pm g_f^{\varepsilon}(\mathbf{E}, \mathbf{t}(s))\nabla_{\mathbf{t}}\mathbf{t}(s) + \varepsilon_1 g_f^{\varepsilon}(\mathbf{E}, \nabla_{\mathbf{t}}\mathbf{t}(s))\mathbf{t}(s), \tag{3.16}$$

and thus we have

$$\nabla_{\mathbf{t}}\mathbf{E}^{FW} = \nabla_{\mathbf{t}}\mathbf{E} \pm g_f^{\varepsilon}(\mathbf{E}, \mathbf{t}(s))\varepsilon_2\kappa N + g_f^{\varepsilon}(\mathbf{E}, \varepsilon_2\kappa N)\mathbf{t}(s), \tag{3.17}$$

$$\nabla_{\mathbf{t}}\mathbf{E} = -\varepsilon_1\varepsilon_2\kappa g_f^{\varepsilon}(\mathbf{E}, \mathbf{n})\mathbf{t}(s). \tag{3.18}$$

The Fermi-Walker parallelism gives that the optical fiber is an $\mathbf{E_t}-$Rytov curve on the condition that $g_f^{\varepsilon}(\mathbf{E}, \mathbf{t}(s)) = 0$. Therefore, along the optical fiber, the polarization vector can be written as

$$\mathbf{E} = f_1(-\varepsilon_1\varepsilon_2 \int \tau ds)\mathbf{n}(s) + f_2(-\varepsilon_1\varepsilon_2 \int \tau ds)\mathbf{b}(s). \tag{3.19}$$

Now, we reach the following theorem.

Theorem 3.1. *During the rotation motion along the curve γ the electric field in the fiber on M trace a curve called $\mathbf{E_t}-$Rytov curve is obtained as following parametric form:*

$$\gamma_{\mathbf{E_t}} = \gamma + \mathbf{E} = f_1(-\varepsilon_1\varepsilon_2 \int \tau ds)\mathbf{n}(s) + f_2(-\varepsilon_1\varepsilon_2 \int \tau ds)\mathbf{b}(s). \tag{3.20}$$

3.2. $\mathbf{E_n}$-Rytov curves

Electric field $\mathbf{E}$ lies on a plane orthogonal to the normal vector $\mathbf{n}$. Therefore, the following equation arises:

$$g_f^{\varepsilon}(\mathbf{E}, \mathbf{n}(s)) = 0. \tag{3.21}$$

The derivative of the equation (3.21) and equation (2.7) give

$$g_f^{\varepsilon}(\nabla_{\mathbf{t}}\mathbf{E}, \mathbf{n}(s)) = \varepsilon_1\kappa g_f^{\varepsilon}(\mathbf{E}, \mathbf{t}) + \varepsilon_3\tau g_f^{\varepsilon}(\mathbf{E}, \mathbf{b}). \tag{3.22}$$

Using the equations (3.22) with (3.1), we can obtain the following

$$\Upsilon_2 = \varepsilon_1\varepsilon_2\kappa g_f^{\varepsilon}(\mathbf{E}, \mathbf{t}) + \varepsilon_2\varepsilon_3\tau g_f^{\varepsilon}(\mathbf{E}, \mathbf{b}). \tag{3.23}$$

Due to absorption, we can suppose that there is no mechanism loss in the optical fiber, we get

$$g_f^{\varepsilon}(\mathbf{E}, \mathbf{E}) = \varepsilon_{\mathbf{E}} c, \;\; c = const. \tag{3.24}$$

Now we differentiate (3.24) and we obtain

$$g_f^{\varepsilon}(\nabla_{\mathbf{t}}\mathbf{E}, \mathbf{E}) = 0. \tag{3.25}$$

Then if we use the equation in (3.1) and the following equation $\mathbf{E}=\varepsilon_1 g_f^{\varepsilon}(\mathbf{E},\mathbf{t})\mathbf{t}+\varepsilon_3 g_f^{\varepsilon}(\mathbf{E},\mathbf{b})\mathbf{b}$, we obtain

$$\Upsilon_1 g_f^{\varepsilon}(\mathbf{E},\mathbf{t}(s))=-\Upsilon_3 g_f^{\varepsilon}(\mathbf{E},\mathbf{b}(s)). \tag{3.26}$$

If $g_f^{\varepsilon}(\mathbf{E},\mathbf{t}(s))\neq 0$ and $g_f^{\varepsilon}(\mathbf{E},\mathbf{b}(s))\neq 0$, then this case shows that Υ_2 and Υ_3 are proportional and we have

$$\Upsilon_1=\zeta g_f^{\varepsilon}(\mathbf{E},\mathbf{b}(s))\ \textit{and}\ \Upsilon_3=-\zeta g_f^{\varepsilon}(\mathbf{E},\mathbf{t}(s)). \tag{3.27}$$

And then, from the equations (3.2)-(3.8), we obtain

$$\nabla_{\mathbf{t}}\mathbf{E}=(\zeta g_f^{\varepsilon}(\mathbf{E},\mathbf{b}(s)))\mathbf{t}(s)+(\varepsilon_1\varepsilon_2\kappa g_f^{\varepsilon}(\mathbf{E},\mathbf{t})+\varepsilon_2\varepsilon_3\tau g_f^{\varepsilon}(\mathbf{E},\mathbf{b}))\mathbf{n}(s)+(-\zeta g_f^{\varepsilon}(\mathbf{E},\mathbf{t}(s)))\mathbf{b}(s). \tag{3.28}$$

When we use the Darboux frame fields' vector product, we obtain the following

$$\nabla_{\mathbf{t}}\mathbf{E}=(\varepsilon_2\varepsilon_3\tau g_f^{\varepsilon}(\mathbf{E},\mathbf{b}))\mathbf{n}(s)+\zeta\varepsilon_2(\mathbf{E}\times_f\mathbf{n}(s)). \tag{3.29}$$

The part that rotates around the binormal vector $\mathbf{b}$ is indicated by the second part of (3.29). If we suppose that $\mathbf{b}$ is parallel transported (i.e. $\xi=0$), then it is calculated that

$$\nabla_{\mathbf{t}}\mathbf{E}=(\varepsilon_1\varepsilon_2\kappa g_f^{\varepsilon}(\mathbf{E},\mathbf{t})+\varepsilon_2\varepsilon_3\tau g_f^{\varepsilon}(\mathbf{E},\mathbf{b}))\mathbf{n}(s). \tag{3.30}$$

We can also write in general

$$\mathbf{E}=\varepsilon_1 g_f^{\varepsilon}(\mathbf{E},\mathbf{t}(s))\mathbf{t}(s)+\varepsilon_3 g_f^{\varepsilon}(\mathbf{E},\mathbf{b}(s))\mathbf{b}(s). \tag{3.31}$$

If we compare with the derivative of the equations in (3.30) and (3.29), we find that this matrix form is

$$\begin{bmatrix} g_f^{\varepsilon}(\mathbf{E},\mathbf{t}(s))' \\ g_f^{\varepsilon}(\mathbf{E},\mathbf{b}(s))' \end{bmatrix}=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} g_f^{\varepsilon}(\mathbf{E},\mathbf{t}(s)) \\ g_f^{\varepsilon}(\mathbf{E},\mathbf{b}(s)) \end{bmatrix}. \tag{3.32}$$

From the equality $g_f^{\varepsilon}(\mathbf{E},\mathbf{E})=\varepsilon_E c$, c is a constant, without losing generality, in polar coordinates, we can write

$$\mathbf{E}=f_1(\phi)\mathbf{t}(s)+f_2(\phi)\mathbf{b}(s). \tag{3.33}$$

If we differentiate the equation in (3.33), we compute the direction of the electric field $\mathbf{E}$ as

$$\nabla_{\mathbf{t}}\mathbf{E}=(\varepsilon_1\varepsilon_2\kappa g_f^{\varepsilon}(\mathbf{E},\mathbf{t})+\varepsilon_2\varepsilon_3\tau g_f^{\varepsilon}(\mathbf{E},\mathbf{b}))\mathbf{n}(s)+\frac{d\phi}{ds}\mathbf{E}\times_f\mathbf{n}. \tag{3.34}$$

We must take $\frac{d\phi}{ds}=0$ along the optical fiber. Therefore, the direction of the polarization state $\mathbf{E}$ is parallel to the direction of $\mathbf{n}(s)$. Then, using the Fermi-Walker transportation law, we reach

$$\nabla_{\mathbf{t}}\mathbf{E}^{FW}=\nabla_{\mathbf{t}}\mathbf{E}\pm g_f^{\varepsilon}(\mathbf{E},\mathbf{n}(s))\nabla_{\mathbf{t}}\mathbf{n}(s)+\varepsilon_2 g_f^{\varepsilon}(\mathbf{E},\nabla_{\mathbf{t}}\mathbf{n}(s))\mathbf{n}(s), \tag{3.35}$$

and thus we have

$$\nabla_{\mathbf{t}}\mathbf{E}^{FW} = \nabla_{\mathbf{t}}\mathbf{E} \pm g_f^{\varepsilon}(\mathbf{E},\mathbf{n}(s))(-\varepsilon_1\kappa T - \varepsilon_3\tau B) + \varepsilon_2 g_f^{\varepsilon}(\mathbf{E}, -\varepsilon_1\kappa T - \varepsilon_3\tau B)\mathbf{n}(s), \tag{3.36}$$

$$\nabla_{\mathbf{t}}\mathbf{E}^{FW} = (\varepsilon_1\varepsilon_2\kappa g_f^{\varepsilon}(\mathbf{E},\mathbf{t}) + \varepsilon_2\varepsilon_3\tau g_f^{\varepsilon}(\mathbf{E},\mathbf{b}))\mathbf{n}(s). \tag{3.37}$$

The Fermi-Walker parallelism gives that the optical fiber is an $\mathbf{E_n}-$Rytov curve on the condition that $g_f^{\varepsilon}(\mathbf{E},\mathbf{n}(s)) = 0$. Therefore, along the optical fiber, the polarization vector is written as follows:

$$\mathbf{E} = f_1(\phi)\mathbf{t}(s) + f_2(\phi)\mathbf{b}(s), \ \phi = const. \tag{3.38}$$

Next, we obtain the following theorem.

Theorem 3.2. *During the rotation motion along the curve γ the electric field in the fiber on M trace a curve called $\mathbf{E_n}-$Rytov curve is obtained as following parametric form:*

$$\gamma_{\mathbf{E_n}} = \gamma + f_1(\phi)\mathbf{t}(s) + f_2(\phi)\mathbf{b}(s), \ \phi = const. \tag{3.39}$$

3.3. $\mathbf{E_b}$-Rytov curves

Electric field $\mathbf{E}$ lies on a plane orthogonal to the binormal vector $\mathbf{b}$. Therefore, the following equation arises:

$$g_f^{\varepsilon}(\mathbf{E},\mathbf{b}(s)) = 0. \tag{3.40}$$

The derivative of the equation (3.40) and equation (2.7) imply

$$g_f^{\varepsilon}(\nabla_{\mathbf{t}}\mathbf{E},\mathbf{b}(s)) = -\varepsilon_2\tau g_f^{\varepsilon}(\mathbf{E},\mathbf{n}). \tag{3.41}$$

Using the equations (3.41) with (3.1), we can obtain the following

$$\Upsilon_3 = -\varepsilon_2\varepsilon_3\tau g_f^{\varepsilon}(\mathbf{E},\mathbf{n}). \tag{3.42}$$

Due to absorption, we can suppose that there is no mechanism loss in the optical fiber, and we get

$$g_f^{\varepsilon}(\mathbf{E},\mathbf{E}) = \varepsilon_{\mathbf{E}} c, \tag{3.43}$$

where c is constant.
Now we differentiate (3.43) and we obtain

$$g_f^{\varepsilon}(\nabla_{\mathbf{t}}\mathbf{E},\mathbf{E}) = 0. \tag{3.44}$$

Then if we use the equation in (3.1) and the following equation

$$\mathbf{E} = \varepsilon_1 g_f^{\varepsilon}(\mathbf{E},\mathbf{t})\mathbf{t} + \varepsilon_2 g_f^{\varepsilon}(\mathbf{E},\mathbf{n})\mathbf{n},$$

we obtain

$$\Upsilon_1 g_f^\varepsilon(\mathbf{E},\mathbf{t}(s)) = -\Upsilon_2 g_f^\varepsilon(\mathbf{E},\mathbf{n}(s)). \tag{3.45}$$

If $g_f^\varepsilon(\mathbf{E},\mathbf{t}(s)) \neq 0$ and $g_f^\varepsilon(\mathbf{E},\mathbf{n}(s)) \neq 0$, then this case show that Υ_1 and Υ_2 are proportional and we have

$$\Upsilon_1 = \eta g_f^\varepsilon(\mathbf{E},\mathbf{n}(s)) \textit{ and } \Upsilon_2 = -\eta g_f^\varepsilon(\mathbf{E},\mathbf{t}(s)). \tag{3.46}$$

And then, from the equations (3.40)-(3.46), we obtain

$$\nabla_{\mathbf{t}}\mathbf{E} = (\eta g_f^\varepsilon(\mathbf{E},\mathbf{n}(s)))\mathbf{t}(s) + (-\eta g_f^\varepsilon(\mathbf{E},\mathbf{t}(s)))\mathbf{n}(s) + (-\varepsilon_2\varepsilon_3\tau g_f^\varepsilon(\mathbf{E},\mathbf{n}))\mathbf{b}(s). \tag{3.47}$$

When we use the Darboux frame fields' vector product, we obtain the following

$$\nabla_{\mathbf{t}}\mathbf{E} = (-\varepsilon_2\varepsilon_3\tau g_f^\varepsilon(\mathbf{E},\mathbf{n}))\mathbf{b}(s) + \eta\varepsilon_2(\mathbf{E}\times_f \mathbf{b}(s)). \tag{3.48}$$

The part that rotates around the principal normal vector $\mathbf{t}$ is indicated by the second part of (3.48). If we suppose that $\mathbf{n}$ is parallel transported (i.e. $\xi = 0$), then it is calculated that

$$\nabla_{\mathbf{t}}\mathbf{E} = (-\varepsilon_2\varepsilon_3\tau g_f^\varepsilon(\mathbf{E},\mathbf{n}))\mathbf{b}(s). \tag{3.49}$$

We can also write in general

$$\mathbf{E} = \varepsilon_1 g_f^\varepsilon(\mathbf{E},\mathbf{t}(s))\mathbf{t}(s) + \varepsilon_2 g_f^\varepsilon(\mathbf{E},\mathbf{n}(s))\mathbf{n}(s). \tag{3.50}$$

If we compare with the derivative of the equations in (3.49) and (3.48), we find that this matrix form is

$$\begin{bmatrix} g_f^\varepsilon(\mathbf{E},\mathbf{t}(s))' \\ g_f^\varepsilon(\mathbf{E},\mathbf{n}(s))' \end{bmatrix} = \begin{bmatrix} 0 & \varepsilon_2\kappa \\ -\varepsilon_3\kappa & 0 \end{bmatrix} \begin{bmatrix} g_f^\varepsilon(\mathbf{E},\mathbf{t}(s)) \\ g_f^\varepsilon(\mathbf{E},\mathbf{n}(s)) \end{bmatrix}. \tag{3.51}$$

From the equality $g_f^\varepsilon(\mathbf{E},\mathbf{E}) = \varepsilon_E c$, c is a constant, without losing generality, in polar coordinates, we have

$$\mathbf{E} = f_1(\phi)\mathbf{t}(s) + f_2(\phi)\mathbf{n}(s). \tag{3.52}$$

If we differentiate the equation in (3.52), we compute the direction of $\mathbf{E}$ as

$$\nabla_{\mathbf{t}}\mathbf{E} = (-\varepsilon_2\varepsilon_3\tau g_f^\varepsilon(\mathbf{E},\mathbf{n}))\mathbf{b}(s) + (\frac{d\phi}{ds} - \varepsilon_1\varepsilon_3\kappa)\mathbf{E}\times_f \mathbf{b}. \tag{3.53}$$

We must take $\frac{d\phi}{ds} = \varepsilon_1\varepsilon_3\kappa$ along the optical fiber. Thus, the direction of the polarization vector $\mathbf{E}$ is parallel to the direction of $\mathbf{b}(s)$. Then, using the Fermi-Walker transportation law, we reach

$$\nabla_{\mathbf{t}}\mathbf{E}^{FW} = \nabla_{\mathbf{t}}\mathbf{E} \pm g_f^\varepsilon(\mathbf{E},\mathbf{b}(s))\nabla_{\mathbf{t}}\mathbf{n}(s) + \varepsilon_3 g_f^\varepsilon(\mathbf{E},\nabla_{\mathbf{t}}\mathbf{b}(s))\mathbf{b}(s), \tag{3.54}$$

and thus we have

$$\nabla_{\mathbf{t}}\mathbf{E}^{FW} = \nabla_{\mathbf{t}}\mathbf{E} \pm g_f^\varepsilon(\mathbf{E},\mathbf{b}(s))(\varepsilon_2\tau N) + \varepsilon_2 g_f^\varepsilon(\mathbf{E},\varepsilon_2\tau N)\mathbf{b}(s), \tag{3.55}$$

$$\nabla_{\mathbf{t}}\mathbf{E}^{FW} = (-\varepsilon_2\varepsilon_3\tau g_f^{\varepsilon}(\mathbf{E},\mathbf{n}))\mathbf{b}(s). \tag{3.56}$$

The Fermi-Walker parallelism gives that the optical fiber is an $\mathbf{E_b}-$Rytov curve on the condition that $g_f^{\varepsilon}(\mathbf{E},\mathbf{b}(s)) = 0$. Therefore, along the optical fiber, the polarization vector can be written as

$$\mathbf{E} = f_1(-\varepsilon_1\varepsilon_3\int \kappa ds)\mathbf{n}(s) + f_2(-\varepsilon_1\varepsilon_3\int \kappa ds)\mathbf{b}(s). \tag{3.57}$$

Now, we get the following theorem.

Theorem 3.3. *During the rotation motion along the curve γ the electric field in the fiber on M trace a curve called $\mathbf{E_b}-$Rytov curve is obtained as following parametric form:*

$$\gamma_{\mathbf{E_b}} = \gamma + \mathbf{E} = f_1(-\varepsilon_1\varepsilon_3\int \kappa ds)\mathbf{n}(s) + f_2(-\varepsilon_1\varepsilon_3\int \kappa ds)\mathbf{b}(s). \tag{3.58}$$

4. Electromagnetic curves related to the evolution of the electric field in the optical fiber

When a charged particle enters the electromagnetic field, it may be exposed to the Lorentz force during the rotation of the electric field. Then it follows a new trajectory called an electromagnetic trajectory. Using the polarization state of $\mathbf{E}$ and the Killing magnetic vector field $\mathbf{B}$ in the electromagnetic wave along with the optical fiber, we can define the electromagnetic trajectories in the Walker manifold (M, g_f^{ε}) as follows;

$$\Phi(\mathbf{E}) = \mathbf{B}\times_f \mathbf{E} = \nabla_{\mathbf{t}}\mathbf{E}. \tag{4.1}$$

Then, considering the three evolutions of the electric fields, we obtain three types of electromagnetic trajectories First, let us suppose that the electric field be orthogonal to the tangent vector field $\mathbf{t}$ in the Walker manifold (M, g_f^{ε}). Then, we have;

$$\begin{aligned}\nabla_{\mathbf{t}}\mathbf{E} &= (\varepsilon_3\kappa g_f^{\varepsilon}(\mathbf{E},\mathbf{n}))\mathbf{t} + (\tfrac{d\phi}{ds} + \varepsilon_2\tau)g_f^{\varepsilon}(\mathbf{E},\mathbf{b}))\mathbf{n} \\ &\quad + (\varepsilon_3\tfrac{d\phi}{ds} - \varepsilon_2\varepsilon_3\tau)g_f^{\varepsilon}(\mathbf{E},\mathbf{n}))\mathbf{b} \\ &= (\varepsilon_3\kappa g_f^{\varepsilon}(\mathbf{E},\mathbf{n}))\mathbf{t} + \varepsilon_3(\tfrac{d\phi}{ds} + \varepsilon_2\tau)\mathbf{E}\times T.\end{aligned} \tag{4.2}$$

The equations (4.1) and (4.2) gives

$$g_f^{\varepsilon}(\Phi(\mathbf{E}),\mathbf{t}) = -\varepsilon_2\kappa g_f^{\varepsilon}(\mathbf{E},\mathbf{n}) = -g_f^{\varepsilon}(\Phi(\mathbf{t}),\mathbf{E}),$$

$$g_f^{\varepsilon}(\Phi(\mathbf{E}),\mathbf{n}) = \varepsilon_1\kappa g_f^{\varepsilon}(\mathbf{E},\mathbf{t}) + \varepsilon_3(\varepsilon_2\frac{d\phi}{ds} + \tau)g_f^{\varepsilon}(\mathbf{E},\mathbf{b}) = -g_f^{\varepsilon}(\Phi(\mathbf{n}),\mathbf{E}), \tag{4.3}$$

$$g_f^{\varepsilon}(\Phi(\mathbf{E}),\mathbf{b}) = -\varepsilon_3(\varepsilon_2\frac{d\phi}{ds} + \tau)g_f^{\varepsilon}(\mathbf{E},\mathbf{n})) = -g_f^{\varepsilon}(\Phi(\mathbf{b}),\mathbf{E}). \tag{4.4}$$

Thus, we can reach

$$\begin{bmatrix}\Phi(\mathbf{t})\\ \Phi(\mathbf{n})\\ \Phi(\mathbf{b})\end{bmatrix} = \begin{bmatrix}0 & \varepsilon_2\kappa & 0\\ -\varepsilon_1\kappa & 0 & -\varepsilon_3(\varepsilon_2\frac{d\phi}{ds}+\tau)\\ 0 & \varepsilon_3(\varepsilon_2\frac{d\phi}{ds}+\tau) & 0\end{bmatrix}\begin{bmatrix}T\\ N\\ B\end{bmatrix}. \tag{4.5}$$

Then, $\mathbf{B}$ can be written as follows;

$$\mathbf{B} = \Upsilon_1 \mathbf{t} + \Upsilon_2 \mathbf{n} + \Upsilon_3 \mathbf{b}, \tag{4.6}$$

where Υ_i, $i \in \{1,2,3\}$ are differentiable functions. Therefore, we found

$$\Phi(\mathbf{B}) = \Upsilon_1 \Phi(\mathbf{t}) + \Upsilon_2 \Phi(\mathbf{n}) + \Upsilon_3 \Phi(\mathbf{b}). \tag{4.7}$$

By using the equation, $\Phi(\mathbf{B}) = 0$, and equation (4.5). Thus, we obtain that γ is an electromagnetic trajectory of $\mathbf{B}$ if and only if $\mathbf{B}$ can be written as follows:

$$\mathbf{B} = -\varepsilon_3(\varepsilon_2 \frac{d\phi}{ds} + \tau)\mathbf{t} + \varepsilon_2 \kappa \mathbf{b}. \tag{4.8}$$

On the condition that $\phi = -\varepsilon_2 \int \tau ds$, the Lorentz force and Killing magnetic vector field are given as follows;

$$\begin{bmatrix} \Phi(\mathbf{t}) \\ \Phi(\mathbf{n}) \\ \Phi(\mathbf{b}) \end{bmatrix} = \begin{bmatrix} 0 & \varepsilon_2 \kappa & 0 \\ -\varepsilon_1 \kappa & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{4.9}$$

$$B = \varepsilon_2 \kappa \mathbf{b}. \tag{4.10}$$

Thus, we can give the following theorem.

Theorem 4.1. *Let γ be an optical fiber in the Walker manifold (M, g_f^ε) such that electric field E is orthogonal to the tangent vector $\mathbf{t}$ of γ. Then, γ is an electromagnetic trajectory associated with the magnetic field $\mathbf{B}$ if and only if along the optical fiber $\mathbf{B}$ can be written as follows;*

$$\mathbf{B} = \varepsilon_2 \kappa \mathbf{b}. \tag{4.11}$$

Corollary 4.1. *Let γ be an optical fiber in the Walker manifold (M, g_f^ε) such that electric field E is orthogonal to the tangent vector $\mathbf{t}$ of γ. Then γ is a curve with the torsion $\tau = 0$.*

Proof. If we use the equation (3.19) and the equation (4.11) with the condition $g_f^\varepsilon(\mathbf{E}, \mathbf{B}) = 0$, we obtain that $\tau = 0$. □

Similar calculations above the characterizations of the electromagnetic curves for the conditions that $\mathbf{n}$ and $\mathbf{b}$ orthogonal to the electric field $\mathbf{E}$ can be given in the following theorems and corollaries.

Theorem 4.2. *Assume that γ be an optical fiber in the Walker manifold (M, g_f^ε) such that electric field E is orthogonal to the normal vector $\mathbf{n}$ of γ. Then, γ is an electromagnetic trajectory associated with the magnetic field $\mathbf{B}$ if and only if along the optical fiber $\mathbf{B}$ is the Darboux vector of the curve γ. Namely,*$\mathbf{b}$ *can be written as follows;*

$$\mathbf{B} = \varepsilon_3 \tau \mathbf{t} + \varepsilon_2 \kappa \mathbf{b}. \tag{4.12}$$

Corollary 4.2. *Let γ be an optical fiber in the Walker manifold (M, g_f^ε) such that electric field E is orthogonal to the tangent vector $\mathbf{n}$ of γ. Then γ is a curve with the curvature $\kappa(s) = \cos\phi$ and torsion $\tau = -\sin\phi$, where $\phi = const.$*

Theorem 4.3. *Suppose that γ is an optical fiber in the Walker manifold (M, g_f^ε) such that electric field E is orthogonal to the binormal vector $\mathbf{b}$ of γ. Then, γ is an electromagnetic trajectory associated with the magnetic field $\mathbf{B}$ if and only if along the optical fiber $\mathbf{B}$ can be written as follows;*

$$\mathbf{B} = \varepsilon_3 \tau \mathbf{t}. \tag{4.13}$$

Corollary 4.3. *Let γ be an optical fiber in the Walker manifold (M, g_f^ε) such that electric field E is orthogonal to the tangent vector $\mathbf{b}$ of γ. Then γ is a curve with the curvature $\kappa(s) = 0$.*

5. Example of propagation of the light in the optical fiber in the Walker 3-manifolds

In this section, we will consider a unit speed spacelike curve in the strict Walker manifold (M, g_f^ε), for the constant function, $f(x, y, z) = 1$, is defined by

$$\gamma(s) = [\frac{1}{8}\sin(2s) - \frac{s}{4}, \sin(s), s]. \tag{5.1}$$

Then, the tangent, normal and binormal vectors of γ in the strict Walker manifold (M, g_f^ε), are computed as follows:

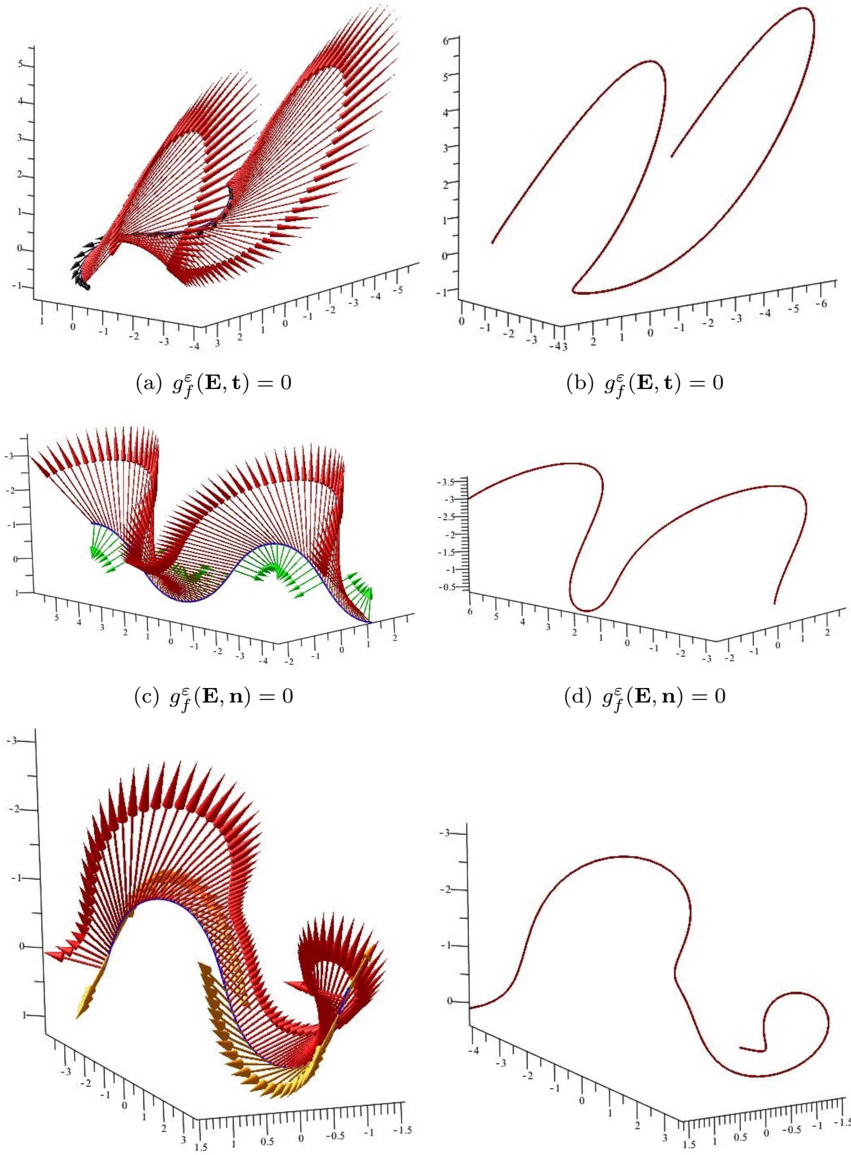
$$\begin{aligned}
\mathbf{t}(s): \quad &= [\tfrac{\cos s^2}{2}, \cos s, 1],\\
\mathbf{n}(s): \quad &= [-\cos s, -1, 0],\\
\mathbf{b}(s): \quad &= [\tfrac{\cos s^2}{2} - 1, \cos s, 1],
\end{aligned}$$

and the curvatures are calculated as $\kappa(s) = \sin s$ and $\tau(s) = -\sin s$. In Figure 5.1, we showed that the electric field (red vector) parallel transported through the Fermi Walker transportation rule during the propagation of linearly polarized light and the tip point of the electric field traced the Rytov curve (red curve) during the propagation in the optical fiber.

Next, let us take an electromagnetic trajectory γ with the parametric equation;

$$\gamma(s) = [\frac{s^3}{3\sqrt{2}}, \frac{s^2}{2}, \frac{s}{\sqrt{2}}],$$

with the curvatures $\kappa(s) = \frac{1}{\sqrt{2}}$ and $\tau(s) = -\frac{1}{\sqrt{2}}$. Then we compute that $f(x, y, z) = 2$. In Figure 5.2, we show the electromagnetic trajectory (blue) of a charged point-particle (red point) in the magnetic vector field $\mathbf{B} = \tau\mathbf{t} + \kappa\mathbf{b}$ (green) during the propagation of light on the condition that $g_f^\varepsilon(\mathbf{E}, \mathbf{n}) = 0$.



(a) $g_f^{\varepsilon}(\mathbf{E}, \mathbf{t}) = 0$ (b) $g_f^{\varepsilon}(\mathbf{E}, \mathbf{t}) = 0$

(c) $g_f^{\varepsilon}(\mathbf{E}, \mathbf{n}) = 0$ (d) $g_f^{\varepsilon}(\mathbf{E}, \mathbf{n}) = 0$

(e) $g_f^{\varepsilon}(\mathbf{E}, \mathbf{b}) = 0$ (f) $g_f^{\varepsilon}(\mathbf{E}, \mathbf{b}) = 0$

FIG. 5.1: Rotation of the polarization plane and associated Rytov curves.

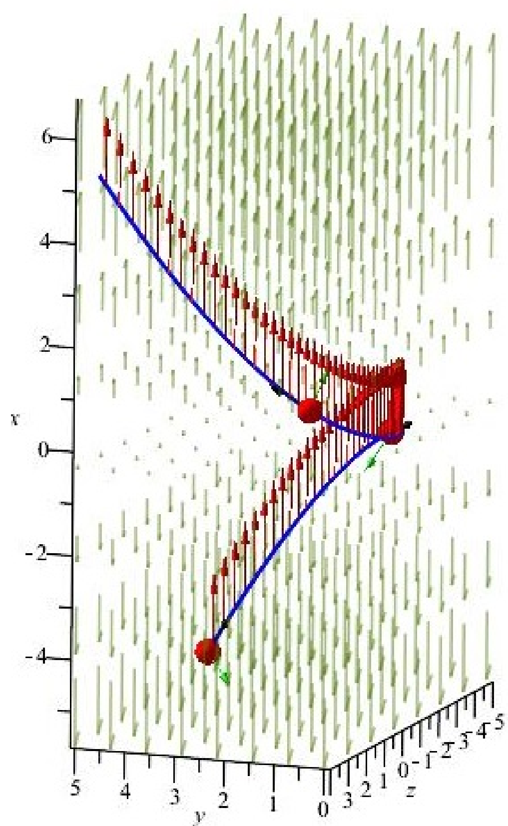


FIG. 5.2: Electromagnetic curve in the electromagnetic field

6. Conclusions and some discussions for physical results

Pseudo-Riemannian manifolds (especially Lorentzian) have important applications in physics because of their application to general relativity. Walker manifolds that work in the context of a pseudo-Riemann manifold have various applications in mathematics and theoretical physics, as seen in [36, 37]. Lorentzian Walker manifolds offer many special properties from the physical and geometric perspectives, [43, 44, 45, 46]. Moreover, Lorentzian Walker manifolds have been extensively studied in the physics literature as they form the background metric of pp-wave models, [39, 40, 41, 42]. It has a light-like distribution that is parallel to the Levi-Civita junction, [38]. An electromagnetic curve or electromagnetic trajectory in physics is described as the motion of a charged particle under magnetic influence. In this paper, we investigated the electromagnetic curves on Walker 3-manifolds. We found that when a charged-point particle enters the electromagnetic field, it follows one of the following trajectories: If the tangent vector is orthogonal to the electric field then the particle follows a trajectory that has zero torsion. If the normal vector is orthogonal to the electric field then the particle follows a trajectory that has constant curvature and constant torsion. If the tangent vector is orthogonal to the electric field then the particle follows a trajectory that has zero curvature. The difference of this work from others is the use of different metrics, which is a non-degenerate arbitrary signature metric tensor, in calculations. Thus, this gives an application in general relativity. This topic has not been dealt with from this perspective before, so this study is an original. We think that it is a very useful study for those working in these fields, as it includes differential geometry and physics together. Therefore, we can say that the results and characterizations will contribute to the optical theory.

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Conflict of interest Authors ZEHRA ÖZDEMİR and AMETH NDIAYE declare that they have no conflict of interest.

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