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HORIZONTAL LIFTS OF THE GOLDEN STRUCTURES FROM A MANIFOLD TO ITS TANGENT BUNDLE

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Abstract. The present paper aims to investigate 'the horizontal lift' of J satisfying $J^2 - J - I = 0$ and demonstrate its status as a type of golden structure. The Nijenhuis tensor N^* of the horizontal lift J^H on the tangent bundle is determined. Also, a tensor field $\tilde{J}$ of type (1,1) is studied and shown to be golden structure on the tangent bundle. Furthermore, several conclusions regarding the Nijenhuis tensor and the Lie derivative of the golden structure $\tilde{J}$ on the tangent bundle are deduced. Moreover, a study is done on the golden structure $\tilde{J}$ on the tangent bundle that is equipped with projection operators $\tilde{l}$ and $\tilde{m}$. Finally, we construct an example of it.
Keywords: golden structure, tangent bundle, vertical lift, horizontal lift, almost analytic vector field, projection tensor, Nijenhuis tensor, Lie derivative.

1. Introduction

Let us consider the tangent bundle TM of a manifold M. In differential geometry, tangent bundle is a primary field of it to investigate the geometrical structures and their properties such as integrability, curvature, Lie derivative etc. Yano and Ishihara [32] introduced and studied structures like almost complex structures with some basic properties induced in tangent bundles. Das and Khan [7] have researched these lifts of an almost product structure over an almost r-contact structure along

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with TM. The work of several scholars on various geometric structures and connections has been extremely beneficial. For instance, Das and Nivas [8], Khan [19, 20, 21, 28], Omran at el [27], Tekkoyun [31] and Yano and Ishihara [32].

Consider the polynomial structure of degree n

$$Q(J) = J^n + a_n J^{n-1} + ... + a_2 J + a_1 I$$

where J is the tensor field of type (1,1) and I is an identity map on the Lie algebra of vector fields on the differentiable manifold M [12]. For instance, 'an almost complex structure' and 'an almost contact structure' are polynomial structures of degree 2 and degree 3 respectively. The polynomial structure satisfying $J^2 - J - I$ is called golden structures. The set of positive solutions of $J^2 - J - I = 0$, denoted by $\sigma = \frac{1}{2}(1+\sqrt{5})$, is named golden Mean [5, 23, 29, 30].

Hretcanu and Crasmareanu [6] introduced and studied the golden structures satisfying $J^2 - J - I = 0$, on the Riemannian manifold. Bilen et. al. [4] studied and discussed a Kähler Norden Codazzi golden structures on pseudo-Riemannian manifolds. Azami [3] introduced a tensor field of type (1,1) and showed that 'the complete' and 'horizontal' lifts of such tensor field is also metallic structure on the tangent bundle. Recently, Khan[22] studied metallic structures on the frame bundle FM and discussed diagonal lift of a Riemannian metric, derivative and co-derivative of 2-form F of metallic Riemannian structure on FM. metallic structures are extensively studied in the literature [2, 9, 10, 13, 14, 15, 16, 17, 24, 25].

The remaining structure of this paper is as follows: Section 2 presents basic definitions and results of the tangent bundle, 'the vertical' and 'horizontal lifts'. Section 3 describes 'the horizontal lift' of the golden structure in the tangent bundle and shows that it is also a golden structure. The Nijenhuis tensor N^* of the horizontal lift J^H on TM is calculated. In Section 4, a tensor field $\tilde{J}$ of type (1,1) in the tangent bundle TM is studied and found that it is also a golden structure. Furthermore, the Nijenhuis tensor and the Lie derivative of the golden structure $\tilde{J}$ in the tangent bundle TM are calculated. In Section 5, the projection operators in the tangent bundle are discussed as an application. Finally, we construct an example of it.

2. Preliminaries

Let M be a differentiable manifold of dimension n. At any point p of M, the set $TM = \bigcup_{p\in M} T_p(M)$ is the tangent bundle over M, where $T_p(M)$ denotes tangent space at p of M. The set of all tensor fields of type (r, s) in manifold M and tangent bundle TM is represented by $\wp^r_s(M)$ and $\wp^r_s(T(M))$ respectively [1, 33].

2.1. Vertical lifts

Let f be a function in manifold M, then 'the vertical lift' of f denoted by f^V and defined as $f^V = f \circ \pi$, where $\pi : TM \to M$.

Let $X \in \wp_0^1(M)$ and 'the vertical lift' X^V of X defined in the components x^h form as

$$X^V : \begin{bmatrix} 0 \\ \partial x^h \end{bmatrix}. \tag{2.1}$$

Now, state the following proposition involving 'the horizontal lifts' for later use [33]:

Proposition 2.1. *For all $f, g \in \wp_0^0(M), X, Y \in \wp_0^1(M), \eta \in \wp_1^0(M)$. Then*

$$\begin{aligned} (f.g)^V &= f^V g^V, (f+g)^V = f^V + g^V, \\ (X+Y)^V &= X^V + Y^V, (f.X)^V = f^V X^V, \\ X^V f^V &= 0, [X^V, Y^V] = 0, \\ (f.\eta)^V &= f^V \eta^V, \eta^V(X^V) = 0 \end{aligned} \tag{2.2}$$

where η is 1-form in M.

2.2. Horizontal lifts

Let M be a differentiable manifold with an affine connection ∇. A tensor field γS in $\pi^{-1}(U)$ defined by

$$\gamma S = \left(Y^l S_{lk\ldots j}^{i\ldots h}\right) \frac{\partial}{\partial y^i} \otimes \frac{\partial}{\partial y^h} \otimes \ldots dx^k \otimes \ldots \otimes dx^j,$$

corresponding to induced coordinate (x^h, y^h), U is an arbitrary coordinate neighborhood in M [33].

Let f, X, F and η be a function, a vector field, a tensor field of type (1,1) and a 1-form on M. 'The horizontal lift' f^H of f is given by $f^H = f^C - \nabla_\gamma f$, where $\nabla_\gamma f = \gamma(\nabla f)$ and ∇f is a gradient of f. 'The horizontal lift' X^H of X is given by $X^H = X^C - \nabla_\gamma X$, where $\nabla_\gamma X = \gamma(\nabla X)$. 'The horizontal lift' F^H of F is given by $F^H = F^C - \nabla_\gamma F$, where $\nabla_\gamma F = \gamma(\nabla F)$. 'The horizontal lift' η^H of η is given by $\eta^H = \eta^C - \nabla_\gamma \eta$, where $\nabla_\gamma \eta = \gamma(\nabla \eta)$.

Now, state the following propositions involving 'the horizontal lifts' for later use [33]:

Proposition 2.2. *For all $f \in \wp_0^0(M), X, Y \in \wp_0^1(M), \eta \in \wp_1^0(M), F \in \wp_1^1(M)$*

$$\begin{aligned} X^H f^V &= (Xf)^V, F^V X^H = (FX)^V, F^H X^H = (FX)^H, \\ \eta^V(X^H) &= (\eta(X))^H, \eta^H(X^H) = 0, \end{aligned}$$

Proposition 2.3. *Let X and Y be two vector fields in M. Then*

$$\begin{aligned} [X^V, X^H] &= [X, Y]^V - (\nabla_X Y)^V = -(\hat{\nabla}_X Y)^V, \\ [X^V, X^V] &= 0, \\ [X^H, X^H] &= [X, Y]^H - \hat{R}(X, Y). \end{aligned}$$

where X^H *and* X^V *are the horizontal and vetical lifts of* X *in* TM *and* $\hat{R}$ *denotes the curvature tensor of the affine connection* $\hat{\nabla}$ *defined by*

$$\hat{\nabla}_X Y = \nabla_X Y + [X, Y]. \tag{2.3}$$

Proposition 2.4. *Let* T *and* $\tilde{T}$ *be the torsion tensors of connections* ∇ *and* ∇^H *in* M *and* TM *respectively. Then*

$$\begin{aligned} \tilde{T}(X^V, Y^V) &= \nabla^H_{X^V} Y^V - \nabla^H_{Y^V} X^V - [X^V, Y^V], \\ \tilde{T}(X^V, Y^H) &= \nabla^H_{X^H} Y^H - \nabla^H_{Y^H} X^V - [X^V, Y^H], \\ \tilde{T}(X^H, Y^H) &= \nabla^H_{X^H} Y^H - \nabla^H_{Y^H} X^H - [X^H, Y^H], \end{aligned} \tag{2.4}$$

for any $X, Y \in \wp^1_0(M)$.

2.3. Projection tensor

Let M be a manifold and a tensor field m in M. The tensor field m is said to be projection tensor in M such that $m^2 = m$ [11, 33].

3. Horizontal lifts of golden structures in the tangent bundle

In this section, a study is done on 'the horizontal lifts' of a golden structure J in the tangent bundle and show that it is also golden structure. Further, The Nijenhuis tensor N^* of the horizontal lift J^H on TM is calculated.

A golden structure is a polynomial structure of degree 2 defined by a tensor field J which satisfies

$$J^2 - J - I = 0, \tag{3.1}$$

where I is an identity tensor field and $\sigma = \frac{1}{2}(1 + \sqrt{5})$ is its positive solution [15].

Theorem 3.1. *If* $J \in \wp^1_0(M)$ *is a golden structure in* M. *Then 'the horizontal lift'* J^H *is also a golden structure in* TM *i.e.* $(J^H)^2 - J^H - I = 0$.

Proof. In the view of equation (3.1), the proof is trivial.

Let J be a tensor field of type (1,1) in M and J^H be its 'horizontal lift' in TM. The Nijenhuis tensor N of J is given by [26, 33].

$$N(X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] + J^2[X, Y], \tag{3.2}$$

for any $X, Y \in \wp^1_0(M)$.

Theorem 3.2. *Let* M *be any* C^∞ *manifold with* TM *admitting golden structure* J, *and* ∇ *be any* J *connection and* N^* *and* N *be the Nijenhuis tensors of* J^H *on*

TM and J on M respectively.The Nijenhuis tensor N^* of the horizontal lift J^H on TM is given by

$$
\begin{aligned}
N^*(X^V,Y^V) &= 0,\\
N^*(X^V,Y^H) &= (\nabla_{JX}(JY) - J(\nabla_X(JY)) - J(\nabla_{JX}Y)\\
&+ J^2(\nabla_X Y))^V,\\
N^*(X^H,Y^H) &= (N(X,Y))^H - \gamma\{R(JX,JY) - JR(JX,Y)\\
&- J\hat{R}(X,JY) + J^2\hat{R}(X,Y)\},
\end{aligned}
\tag{3.3}
$$

for any $X, Y \in \wp^1_0(M)$ and $J \in \wp^1_1(M)$.

Proof. The equation (3.3) is obtained by applying equation (2.3) and Proposition 2.1 on equation (3.2).

3.1. Golden structures in the tangent bundle

Let M be a differentiable manifold and TM its tangent bundle. In [33], Yano and Ishihara defined a tensor field $\tilde{J}$ of type (1,1) in TM by

$$\tilde{J}X^H = X^V, \quad \tilde{J}X^V = -X^H, \quad \forall X \in \wp^1_0(M). \tag{3.4}$$

It is proved that $\tilde{J}$ is 'an almost complex structure' on TM.

From these studies, the tensor field $\tilde{J}$ of type (1,1) in TM is given as [24]

$$
\begin{aligned}
\tilde{J}X^H &= \frac{1}{2}(X^H + (2\sigma - 1)X^V),\\
\tilde{J}X^V &= \frac{1}{2}(X^V + (2\sigma - 1)X^H),
\end{aligned}
\tag{3.5}
$$

for any $X \in \wp^1_0(M)$.

Theorem 3.3. *Let J be a golden structure in M with an affine connection ∇. Then a tensor field $\tilde{J}$ of type (1,1) defined in (3.5) is also a metallic structure in TM.*

Proof. By the virtue of equation (3.5), then

$$
\begin{aligned}
(\tilde{J}^2 - \tilde{J} - I)X^H &= \tilde{J}(\tilde{J}X^H) - \tilde{J}X^H - X^H,\\
&= \tilde{J}\left[\frac{1}{2}(X^H + (2\sigma - 1)X^V)\right]\\
&- \left[\frac{1}{2}(X^H + (2\sigma - 1)X^V)\right] - X^H,
\end{aligned}
$$

on solving, the obtained equation is $(\tilde{J}^2 - \tilde{J} - I)X^H = 0$.

Similarly, $(\tilde{J}^2 - \tilde{J} - I)X^V = 0$ for any $X \in \wp^1_0(M)$, which implies $\tilde{J}^2 - \tilde{J} - I = 0$.

Thus $\tilde{J}$ is also a golden structure in TM. This completes the proof.

3.2. Nijenhuis tensor of the golden structure in the tangent bundle

Let $\tilde{J}$ be a tensor field of type (1,1) in the tangent bundle TM. The Nijenhuis tensor $\tilde{N}$ of $\tilde{J}$ is given by

$$\tilde{N}(\tilde{X},\tilde{Y}) = [\tilde{J}\tilde{X},\tilde{J}\tilde{Y}] - \tilde{J}[\tilde{J}\tilde{X},\tilde{Y}] - \tilde{J}[\tilde{X},\tilde{J}\tilde{Y}] + \tilde{J}^2[\tilde{X},\tilde{Y}], \tag{3.6}$$

for any $\tilde{X},\tilde{Y} \in \wp^1_0(TM)$.

Setting $\tilde{X} = X^V$ and $\tilde{Y} = Y^V$ in equation (3.6) and using equation (3.5), the obtained equation is

$$\tilde{N}(X^V,Y^V) = [\tilde{J}X^V,\tilde{J}Y^V] - \tilde{J}[\tilde{J}X^V,Y^V] - \tilde{J}[X^V,\tilde{J}Y^V] + \tilde{J}^2[X^V,Y^V]. \tag{3.7}$$

Using equation (3.5) and Proposition 2.1 in equation (3.7), then

$$\begin{aligned}
\tilde{N}(X^V,Y^V) &= [\frac{1}{2}(X^V + (2\sigma-1)X^H), \frac{1}{2}(Y^V + (2\sigma-1)Y^H)],\\
&- \tilde{J}[\frac{1}{2}(X^V + (2\sigma-1)X^H), Y^V]\\
&- \tilde{J}[X^V, \frac{1}{2}(Y^V + (2\sigma-1)Y^H)] + \tilde{J}^2[X^V,Y^V],\\
&= \frac{1}{4}(2\sigma-1)([X,Y]^V - (\nabla_X Y)^V - [Y,X]^V + (\nabla_Y X)^V)\\
&+ \frac{(2\sigma-1)^2}{4}([X,Y]^H - \gamma\hat{R}(X,Y))\\
&+ \frac{(2\sigma-1)}{4}[Y,X]^V + \frac{(2\sigma-1)^2}{4}[Y,X]^H\\
&- \frac{(2\sigma-1)}{4}1(\nabla_Y X)^V - \frac{(2\sigma-1)^2}{4}(\nabla_Y X)^H)\\
&- \frac{(2\sigma-1)}{4}[X,Y]^V - \frac{(2\sigma-1)^2}{4}[Y,X]^H\\
&+ \frac{(2\sigma-1)}{4}(\nabla_X Y)^V + \frac{(2\sigma-1)^2}{4}(\nabla_X Y)^H)\\
&+ \tilde{J}^2[X^V,Y^V].
\end{aligned}$$

Using equation (2.2) and Proposition (2.2) in the above equation, the obtained equation is

$$\tilde{N}(X^V,Y^V) = \frac{(2\sigma-1)^2}{4}T(X,Y)^H - \frac{(2\sigma-1)^2}{4}\gamma\hat{R}(X,Y). \tag{3.8}$$

Using similar devices, the following equations can be easily obtained:

$$\begin{aligned}
\tilde{N}(X^V,Y^H) &= -\frac{(2\sigma-1)^2}{4}T(X,Y)^V + \frac{(2\sigma-1)}{2}\tilde{J}\gamma\hat{R}(X,Y)\\
&- \frac{(2\sigma-1)}{4}\gamma\hat{R}(X,Y), \qquad (3.9)\\
\tilde{N}(X^H,Y^H) &= \frac{(2\sigma-1)^2}{4}T(X,Y)^H - \frac{(2\sigma-1)^2}{4}\gamma\hat{R}(X,Y), \qquad (3.10)
\end{aligned}$$

for any $X, Y \in \wp_0^1(M)$. Therefore, the following theorem is obtained from equations (3.8)-(3.10).

Theorem 3.4. *The golden structure $\tilde{J}$ defined in equation (3.5) is integrable i.e $\tilde{N} = 0$ if and only if $\hat{R} = 0$ and $T = 0$, where T is the torsion tensor of ∇ and $\hat{R}$ the curvature tensor of an affine connection $\hat{\nabla}$.*

3.3. The Lie derivative of the golden structure in the tangent bundle

Let $\tilde{J}$ be a tensor field of type (1,1) in the tangent bundle TM. The Lie derivative $\pounds_{\tilde{X}}\tilde{J}$ of $\tilde{J}$ is defined by [33]

$$(\pounds_{\tilde{X}}\tilde{J})\tilde{Y} = [\tilde{X}, \tilde{J}\tilde{Y}] - \tilde{J}[\tilde{X}, \tilde{Y}], \tag{3.11}$$

for any $\tilde{X}, \tilde{Y} \in \wp_0^1(TM)$.

Operating $\pounds_{Y^H}$ on both sides of equation (3.5) and taking account of Proposition 2.1, then

$$\begin{aligned}
\pounds_{Y^H}(\tilde{J}X^H) &= \frac{1}{2}\pounds_{Y^H}X^H + \frac{(2\sigma-1)}{2}\pounds_{Y^H}X^H, \\
(\pounds_{Y^H}\tilde{J})X^H + \tilde{J}(\pounds_{Y^H}X^H) &= \frac{1}{2}[Y^H, X^H] + \frac{(2\sigma-1)}{2}[Y^H, X^V], \qquad (3.12)
\end{aligned}$$

$$\begin{aligned}
&= \tilde{J}[X,Y]^H - \tilde{J}\gamma\hat{R}(X,Y) - \frac{1}{2}[X,Y]^H \\
&- \gamma\hat{R}(X,Y) - \frac{(2\sigma-1)}{2}([X,Y]^V - (\nabla_X Y)^V), \\
(\pounds_{Y^H}\tilde{J})X^H &= \frac{(2\sigma-1)}{2}(\nabla_X Y)^V - \tilde{J}\gamma\hat{R}(X,Y) \\
&- \frac{1}{2}\gamma\hat{R}(X,Y), \qquad (3.13)
\end{aligned}$$

for any $X, Y \in \wp_0^1(M)$.

Using similar devices, the following equations are obtained:

$$(\pounds_{Y^V}\tilde{J})X^H = \frac{(2\sigma-1)}{2}(\hat{\nabla}_X Y)^H \tag{3.14}$$

$$(\pounds_{Y^V}\tilde{J})X^V = -\frac{(2\sigma-1)}{2}(\hat{\nabla}_X Y)^V \tag{3.15}$$

$$(\pounds_{Y^H}\tilde{J})X^V = -\frac{(2\sigma-1)}{2}(\nabla_X Y)^H + \frac{(2\sigma-1)}{2}\gamma\hat{R}(X,Y) \tag{3.16}$$

for any $X, Y \in \wp_0^1(M)$. The following theorem is obtained from equations (3.13)-(3.16)

Theorem 3.5. *Let Y be a vector field in M and Y^V and Y^H its 'vertical' and 'horizontal lifts' in TM. Let $\tilde{J}$ be a golden structure defined by the equation (3.5), then $\pounds_{Y^V}\tilde{J} = 0$ if and only if $\hat{\nabla}Y = 0$ and $\pounds_{Y^H}\tilde{J} = 0$ if and only if $\hat{\nabla}Y = 0$ and $\gamma\hat{R}(X,Y), X$ being an arbitrary element of $\wp_0^1(M)$, where $\hat{R}$ is the curvature tensor of $\hat{\nabla}$.*

4. Application

The goal of this section is to study a golden structure $\tilde{J}$ endowed with projection operators $\tilde{l}$ and $\tilde{m}$ in the tangent bundle TM.

Let $\tilde{l}$ and $\tilde{m}$ be the projection operators in the tangent bundle TM and defined by

$$\tilde{l} = \tilde{J}^2 - \tilde{J}, \tag{4.1}$$

$$\tilde{m} = I - \tilde{J}^2 + \tilde{J}, \tag{4.2}$$

and let $\tilde{P}$ and $\tilde{Q}$ be the complementary distributions corresponding to $\tilde{l}$ and $\tilde{m}$ respectively.

Theorem 4.1. *Let $\tilde{J}$ be the golden structure and $\tilde{l}$ and $\tilde{m}$ be projection operators in TM. Then*

$$\tilde{l} + \tilde{m} = 0, \quad \tilde{l}^{\,2} = \tilde{l}, \quad \tilde{m}^2 = \tilde{m}, \quad \tilde{l}\tilde{m} = \tilde{m}\tilde{l} = 0, \tag{4.3}$$

$$\tilde{J}\tilde{l} = \tilde{l}\tilde{J} = \tilde{J}, \quad \tilde{J}\tilde{m} = \tilde{m}\tilde{J} = 0 \tag{4.4}$$

Proof. The proof follows easily by virtue of equations (4.1) and (4.2).

Theorem 4.2. *Let X^H and X^V be 'the horizontal' and 'vertical lifts' of a vector field X in the tangent bundle TM. Then*

$$\begin{aligned} \tilde{l}(X^H) &= X^H, \quad \tilde{l}(X^V) = X^V, \\ \tilde{m}(X^H) &= 0, \quad \tilde{m}(X^V) = 0. \end{aligned}$$

where $\tilde{l}$ and $\tilde{m}$ are projection operators in TM.

Proof. Using equations (3.5) and (4.1), then

$$\begin{aligned} \tilde{l}(X^H) &= \tilde{J}^2(X^H) - \tilde{J}(X^H), \\ \tilde{l}(X^H) &= \tilde{J}[\frac{1}{2}(X^H + (2\sigma - 1)X^V)] \\ &- [\frac{1}{2}(X^H + (2\sigma - 1)X^V)], \\ 2\tilde{l}(X^H) &= [(X^H + (2\sigma - 1)X^V)] \\ &- [\frac{(2\sigma - 1)}{2}[(X^V + (2\sigma - 1)X^H)] \\ &- X^H - (2\sigma - 1)X^V, \\ \tilde{l}(X^H) &= X^H. \end{aligned}$$

Using similar devices, other identities can be easily obtained.

5. Example

Let us consider M be a manifold and TM its tangent bundle. There exists a tensor field $\tilde{F}(\neq 0)$ of the type (1, 1) on TM such that

$$\tilde{F}^2 + I = 0 \tag{5.1}$$

such structure on TM is called an almost complex structure.

The projection operators are given by

$$l = -\tilde{F}^2, \quad m = \tilde{F}^2 + I \tag{5.2}$$

where I denotes the identity operator on M. It can be easily obtain

$$l + m = I, \quad l^2 = l, \quad and \quad m^2 = m. \tag{5.3}$$

$$\tilde{F}l = l\tilde{F} = \tilde{F}, \quad \tilde{F}m = m\tilde{F} = 0. \tag{5.4}$$

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