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ON ALMOST PSEUDO SCHOUTEN SYMMETRIC MANIFOLDS

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Abstract. The subject of the present paper is to introduce a type of non-flat Riemannian manifold called an almost pseudo Schouten symmetric manifold $A(PSS)_n$. Some geometric properties have been studied of this manifold. Also, the existence of such a manifold is ensured by a non-trivial example. Finally, we have studied about hypersurface of an $A(PSS)_n$.
Keywords: Schouten tensor, almost pseudo symmetric manifold, almost pseudo Ricci symmetric manifold, quasi-Einstein manifold, quadratic Killing tensor, Codazzi tensor.

1. Introduction

Let (M^n, g) be a Riemannian manifold of dimension n with the Riemannian metric g and ∇ be the Levi-Civita connection with respect to the metric tensor g. Let $\mathfrak{X}(M)$ be the set of differentiable vector fields on M. That is, $X, Y, Z, U \in \mathfrak{X}(M)$. A non-flat Riemannian manifold (M^n, g), $(n \geq 3)$ is said to be an almost pseudo symmetric manifold $A(PS)_n$ [6] if its curvature tensor K satisfies the following condition:

$$
\begin{aligned}
(\nabla_U K)(X, Y, Z) =& [\alpha(U) + \beta(U)]K(X, Y, Z) + \alpha(X)K(U, Y, Z) \\
&+ \alpha(Y)K(X, U, Z) + \alpha(Z)K(X, Y, U) \\
&+ g(K(X, Y, Z), U)\rho,
\end{aligned}
\tag{1.1}
$$

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where α and β are called the associated 1-forms defined by

$$g(X,\sigma) = \alpha(X) \quad \textit{and} \quad g(X,Q) = \beta(X), \tag{1.2}$$

for all X.

A non-flat Riemannian manifold (M^n, g) is called an almost pseudo Ricci symmetric manifold $A(PRS)_n$ [3] if its Ricci tensor Ric of type $(0,2)$ satisfies the following condition:

$$(\nabla_X \mathrm{Ric})(Y,Z) = [\alpha(X)+\beta(X)]\mathrm{Ric}(Y,Z)+\alpha(Y)\mathrm{Ric}(X,Z)+\alpha(Z)\mathrm{Ric}(Y,X), \tag{1.3}$$

where α and β are two non-zero 1-forms which are defined earlier.

A non-flat Riemannian manifold is said to be a quasi-Einstein manifold $(QE)_n$ [7] if its Ricci tensor Ric of type $(0,2)$ satisfies the following condition:

$$\mathrm{Ric}(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \tag{1.4}$$

where a, b are smooth functions and η is a non-zero 1-form such that

$$g(X,\xi) = \eta(X), \tag{1.5}$$

for all vector fields X.

A quadratic Killing tensor [10] is a generalization of a Killing vector and is defined as a second order symmetric tensor A satisfying the condition

$$(\nabla_X A)(Y,Z) + (\nabla_Y A)(Z,X) + (\nabla_Z A)(X,Y) = 0. \tag{1.6}$$

A Riemannian manifold is said to be Codazzi type [8] of Ricci tensor if its Ricci tensor Ric of type $(0,2)$ satisfies the following condition:

$$(\nabla_X \mathrm{Ric})(Y,Z) = (\nabla_Y \mathrm{Ric})(X,Z). \tag{1.7}$$

On an n-dimensional Riemannian (semi-Riemannian) manifold (M^n, g), $n \geq 3$, the Schouten tensor [1] is defined by

$$\mathrm{P}(Y,Z) = \frac{1}{n-2}\Big(\mathrm{Ric}(Y,Z) - \frac{r}{2(n-1)}g(Y,Z)\Big), \tag{1.8}$$

where r is the scalar curvature. Also, the Ricci tensor L of type $(1,1)$ is defined by

$$g(L(X),Y) = \mathrm{Ric}(X,Y), \tag{1.9}$$

for any vector fields X, Y. There is a decomposition formula in which the Riemannian curvature tensor decomposes into non-conformally invariant part, the Schouten tensor ([2], [9]) and a conformally invariant part, the conformal curvature tensor [9]

$$K = \mathrm{P} \odot g + C \tag{1.10}$$

where C is the conformal curvature tensor of g and $\odot$ denotes the Kulkarni-Nomizu product. The scalar $\bar{\mathrm{P}}$ is obtained by putting $Y = Z = e_i$ in (1.8), where $\{e_i, 1 \leq i \leq n\}$ is an orthonormal basis of the tangent space at each point of the manifold

$$\bar{\mathrm{P}} = \frac{r}{2(n-1)}. \tag{1.11}$$

From (1.8), we have

$$\mathrm{P}(X,Y) = \mathrm{P}(Y,X), \tag{1.12}$$

and

$$\mathrm{P}(X,Q) = \frac{1}{n-2}\Big[\mathrm{Ric}(X,Q) - \frac{r}{2n-1}g(X,Q)\Big],$$

or

$$\mathrm{P}(X,Q) = \frac{1}{n-2}\Big[\beta(L(X)) - \frac{r}{2(n-1)}\beta(X)\Big]. \tag{1.13}$$

In the present paper, we have introduced a type of non-flat Riemannian manifold (M^n, g), $(n > 3)$ whose Schouten tensor P satisfies the condition

$$(\nabla_X \mathrm{P})(Y,Z) = [\alpha(X) + \beta(X)]\mathrm{P}(Y,Z) + \alpha(Y)\mathrm{P}(Z,X) + \alpha(Z)\mathrm{P}(X,Y), \tag{1.14}$$

where α and β are called associated 1-forms of the manifold defined by

$$g(X,\sigma) = \alpha(X) \quad \textit{and} \quad g(X,Q) = \beta(X), \tag{1.15}$$

for all X. σ and Q are called the basic vector fields of the manifold corresponding to the associated 1-forms α and β, respectively. Such an n-dimensional manifold is called an almost pseudo Schouten symmetric manifold and denoted by $A(PSS)_n$. An $A(PRS)_n$ is a particular case of an $A(PSS)_n$.

The object of the present paper is to study $A(PSS)_n$. The paper is presented as follows:

Section 2, is devoted to the study of some properties of $A(PSS)_n$ and proved remarkable theorems on it. In section 3, we have proved that the Sufficient condition for an $A(PSS)_n$ to be quasi Einstein manifold. After that in Section 4, the existence of $A(PSS)_n$ has been shown by a non-trivial example. Last section of this paper, deals with the hypersurface of $A(PSS)_n$. It is proved that the totally geodesic hypersurface of this manifold is also $A(PSS)$. Again, it is discovered in this section that a necessary and sufficient condition for totally umbilical hypersurface of this manifold to be also $A(PSS)_n$ is that the mean curvature be constant.

2. Almost Pseudo Schouten Symmetric Manifolds

In this section, using the definitions and the concepts given in Section 1, we will prove some results on $A(PSS)_n$ satisfying certain curvature conditions.

Replacing Y and Z by X in (1.14), we get

$$(\nabla_X \mathrm{P})(X,X) = [\alpha(X)+\beta(X)]\mathrm{P}(X,X)+\alpha(X)\mathrm{P}(X,X)+\alpha(X)\mathrm{P}(X,X),$$

or

$$(\nabla_X \mathrm{P})(X,X) = [3\alpha(X)+\beta(X)]\mathrm{P}(X,X). \tag{2.1}$$

By hypothesis the Schouten tensor is non-zero, then from (2.1) it follows that

$$(\nabla_X \mathrm{P})(X,X) = 0 \quad if \quad and \quad only \quad if \quad 3\alpha(X)+\beta(X)=0.$$

Thus we can state the following:

Theorem 2.1. *In an $A(PSS)_n$, the Schouten tensor is covariantly constant in the direction of X if and only if $3\alpha+\beta=0$.*

Taking cyclic sum of (1.14) over X, Y and Z, we get

$$\begin{aligned}(\nabla_X \mathrm{P})(Y,Z)&+(\nabla_Y \mathrm{P})(Z,X)+(\nabla_Z \mathrm{P})(X,Y)\\ &= [\alpha(X)+\beta(X)]\mathrm{P}(Y,Z)+[\alpha(Y)+\beta(Y)]\mathrm{P}(Z,X)+[\alpha(Z)+\beta(Z)]\mathrm{P}(X,Y)\\ &+\alpha(Y)\mathrm{P}(Z,X)+\alpha(Z)\mathrm{P}(X,Y)+\alpha(X)\mathrm{P}(Y,Z)+\alpha(Z)\mathrm{P}(X,Y)\\ &+\alpha(X)\mathrm{P}(Y,Z)+\alpha(Y)\mathrm{P}(Z,X),\end{aligned}$$

which implies

$$\begin{aligned}(\nabla_X \mathrm{P})(Y,Z)&+(\nabla_Y \mathrm{P})(Z,X)+(\nabla_Z \mathrm{P})(X,Y)\\ &= [3\alpha(X)+\beta(X)]\mathrm{P}(Y,Z)+[3\alpha(Y)+\beta(Y)]\mathrm{P}(Z,X)+[3\alpha(Z)+\beta(Z)]\mathrm{P}(X,Y)\end{aligned}$$

or,

$$\begin{aligned}(\nabla_X \mathrm{P})(Y,Z)&+(\nabla_Y \mathrm{P})(Z,X)+(\nabla_Z \mathrm{P})(X,Y)\\ &= H(X)\mathrm{P}(Y,Z)+H(Y)\mathrm{P}(Z,X)+H(Z)\mathrm{P}(X,Y),\end{aligned} \tag{2.2}$$

where $H(X)=3\alpha(X)+\beta(X)$. If the Schouten tensor of the manifold is quadratic Killing then from (1.6), we have

$$(\nabla_X \mathrm{P})(Y,Z)+(\nabla_Y \mathrm{P})(Z,X)+(\nabla_Z \mathrm{P})(X,Y)=0. \tag{2.3}$$

By virtue of (2.3) the relation (2.2) reduces to

$$H(X)\mathrm{P}(Y,Z)+H(Y)\mathrm{P}(Z,X)+H(Z)\mathrm{P}(X,Y)=0. \tag{2.4}$$

According to Walker's Lemma [11] "If $a(X,Y)$, $b(X)$ are numbers satisfying $a(X,Y)=a(Y,X)$, and $a(X,Y)b(Z)+a(Y,Z)b(X)+a(Z,X)b(Y)=0$, then either all $a(X,Y)$ are zero or all $b(X)$ are zero", then from (2.4) we conclude that either $H(X)=0$ or $\mathrm{P}(X,Y)=0$ for all X, Y. Since $\mathrm{P}(X,Y)\neq 0$. Therefore,

$$H(X)=0 \quad for \quad all \quad X,$$

which implies that

$$3\alpha(X) + \beta(X) = 0. \tag{2.5}$$

Conversely, if $3\alpha(X) + \beta(X) = 0$, then from (2.2) we obtain

$$(\nabla_X \mathrm{P})(Y, Z) + (\nabla_Y \mathrm{P})(Z, X) + (\nabla_Z \mathrm{P})(X, Y) = 0, \tag{2.6}$$

which shows that the Schouten tensor is quadratic Killing tensor.

Thus we can state the following:

Theorem 2.2. *In an $A(PSS)_n$, $(n > 2)$ the Schouten tensor is quadratic Killing if and only if the associated 1-forms α and β satisfy the relation $3\alpha + \beta = 0$.*

Let the Schouten tensor of the manifold be quadratic Killing. Then the associated 1-forms α and β satisfy the relation (2.5) from which we get

$$\alpha(X) = -\frac{1}{3}\beta(X). \tag{2.7}$$

Taking covariant derivative of (2.7) over V, we get

$$(\nabla_V \alpha)(X) = -\frac{1}{3}(\nabla_V \beta)(X). \tag{2.8}$$

Interchanging X and V in (2.8) and then subtracting them, we get

$$(\nabla_V \alpha)(X) - (\nabla_X \alpha)(V) = -\frac{1}{3}\Big[(\nabla_V \beta)(X) - (\nabla_X \beta)(V)\Big] \tag{2.9}$$

which shows that if the 1-form α is closed, then 1-form β is also closed and vice-versa.

This leads to the following result:

Theorem 2.3. *In an $A(PSS)_n$, $(n > 2)$ if the Schouten tensor is quadratic Killing, then the 1-form α is closed if and only if the 1-form β is closed.*

Interchanging X and Z in (1.14) and then subtracting them, we get

$$(\nabla_X \mathrm{P})(Y, Z) - (\nabla_Z \mathrm{P})(X, Y) = \beta(X)\mathrm{P}(Y, Z) - \beta(Z)\mathrm{P}(X, Y) \tag{2.10}$$

which in view of (1.8), the relation (2.10) gives

$$\begin{aligned} &\Big((\nabla_X \mathrm{Ric})(Y, Z) - \frac{dr(X)}{2(n-1)}g(Y, Z) - (\nabla_Z \mathrm{Ric})(X, Y) + \frac{dr(Z)}{2(n-1)}g(X, Y)\Big) \\ &\qquad = (n-2)[\beta(X)\mathrm{P}(Y, Z) - \beta(Z)\mathrm{P}(X, Y)]. \end{aligned} \tag{2.11}$$

Let us suppose that the scalar curvature of $A(PSS)_n$ is constant. Taking $Y = Z = e_i$ in (2.11) and using (1.11) and (1.13), we get

$$\beta(L(X)) = \frac{r}{2}\beta(X) \tag{2.12}$$

which in view of (1.15), the relation (2.12) gives

$$\mathrm{Ric}(X, Q) = \frac{r}{2} g(X, Q). \tag{2.13}$$

This leads to the following:

Theorem 2.4. *If the scalar curvature of an $A(PSS)_n$ is constant then the vector field Q corresponding to the 1-form β is an eigenvector of the Ricci tensor Ric corresponding to the eigenvalue $\frac{r}{2}$.*

If the Schouten tensor P is of Codazzi type, then from (1.7), we find

$$(\nabla_X \mathrm{P})(Y, Z) = (\nabla_Z \mathrm{P})(X, Y). \tag{2.14}$$

Interchanging X and Z in (1.14) and then subtracting them, we get

$$(\nabla_X \mathrm{P})(Y, Z) - (\nabla_Z \mathrm{P})(X, Y) = \beta(X)\mathrm{P}(Y, Z) - \beta(Z)\mathrm{P}(X, Y), \tag{2.15}$$

which in view of (2.14), the relation (2.15) yields

$$\beta(X)\mathrm{P}(Y, Z) - \beta(Z)\mathrm{P}(X, Y) = 0. \tag{2.16}$$

Putting $X = Q$ in (2.16), we get

$$\beta(Q)\mathrm{P}(Y, Z) = \beta(Z)\mathrm{P}(Q, Y). \tag{2.17}$$

Putting $Y = Z = e_i$ in (2.16), we get

$$\frac{r}{2(n-1)}\beta(X) - \frac{1}{n-2}\Big[\beta(L(X)) - \frac{r}{2(n-1)}\beta(X)\Big] = 0. \tag{2.18}$$

By virtue of (1.13), the relation (2.18) reduces to

$$\mathrm{P}(X, Q) = \frac{r}{2(n-1)}\beta(X), \tag{2.19}$$

Using (2.19) in (2.17), we get

$$\mathrm{P}(Y, Z) = \frac{r}{2(n-1)} \frac{\beta(Y)\beta(Z)}{\beta(Q)}. \tag{2.20}$$

From (1.8), we can find

$$\mathrm{Ric}(Y, Z) = \frac{r}{2(n-1)} g(Y, Z) + (n-2)\mathrm{P}(Y, Z). \tag{2.21}$$

Now, using (2.20) in (2.21), we get

$$\mathrm{Ric}(Y, Z) = \frac{r}{2(n-1)} g(Y, Z) + \frac{(n-2)r}{2(n-1)} \frac{\beta(Y)\beta(Z)}{\beta(Q)}. \tag{2.22}$$

Equation (2.22) can be written in the following form

$$\mathrm{Ric}(Y,Z) = ag(Y,Z) + b\beta(Y)\beta(Z),$$

where $a = \frac{r}{2(n-1)}$ and $b = \frac{(n-2)r}{2(n-1)\beta(Q)}$ are non-zero scalars. Hence the manifold under consideration is a quasi-Einstein manifold.

This leads to the following theorem:

Theorem 2.5. *If the Schouten tensor of an $A(PSS)_n$ is of Codazzi type, then the manifold reduces to a quasi-Einstein manifold.*

3. Sufficient condition for an $A(PSS)_n$ to be a quasi-Einstein manifold

In an $A(PSS)_n$, the Schouten tensor satisfies the following condition

$$(\nabla_U \mathrm{P})(X,Y) = [\alpha(U) + \beta(U)]\mathrm{P}(X,Y) + \beta(X)\mathrm{P}(U,Y) + \beta(Y)\mathrm{P}(X,U). \tag{3.1}$$

In a Riemannian manifold a vector field ρ defined by $g(X,\rho) = \alpha(X)$ for all vector fields X is said to be a concircular vector field [9] if

$$(\nabla_X \alpha)(Y) = \lambda g(X,Y) + \omega(X)\alpha(Y), \tag{3.2}$$

where λ is a smooth function and ω is a closed 1-form. If ρ is a unit one then the equation (3.2) can be written as

$$(\nabla_X \alpha)(Y) = \lambda\Big(g(X,Y) - \alpha(X)\alpha(Y)\Big). \tag{3.3}$$

We assume that $A(PSS)_n$ admits the associated vector field ρ defined by (3.2), with a non-zero constant λ. Applying Ricci identity to (3.3), we obtain

$$\alpha(K(X,Y,Z)) = \lambda^2\Big(g(X,Z)\alpha(Y) - g(Y,Z)\alpha(X)\Big). \tag{3.4}$$

Putting $Y = Z = e_i$ in (3.4), where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and taking summation over i, $1 \le i \le n$, we get

$$\alpha(L(X)) = (n-1)\lambda^2\alpha(X), \tag{3.5}$$

where L is the Ricci tensor of type $(1,1)$ defined by $g(L(X),Y) = Ric(X,Y)$, which implies that

$$\mathrm{Ric}(X,\rho) = (n-1)\lambda^2\alpha(X). \tag{3.6}$$

Now,

$$(\nabla_Y \mathrm{Ric})(X,\rho) = \nabla_Y \mathrm{Ric}(X,\rho) - \mathrm{Ric}(\nabla_Y X,\rho) - \mathrm{Ric}(X,\nabla_Y \rho). \tag{3.7}$$

Applying (3.6) and (3.3) in (3.7), we get

$$(\nabla_Y \mathrm{Ric})(X,\rho) = (n-1)\lambda^3[g(X,Y) - \alpha(X)\alpha(Y)] - \mathrm{Ric}(X,\nabla_Y \rho). \tag{3.8}$$

Since $(\nabla_X g)(Y,\rho)=0$, we have

$$(\nabla_Y \alpha)(X) = g(X, \nabla_Y \rho). \tag{3.9}$$

Using (3.3) in (3.9) yields

$$\lambda[g(X,Y) - \alpha(X)\alpha(Y)] = g(X, \nabla_Y \rho),$$

which implies

$$\nabla_Y \rho = \lambda Y - \lambda\alpha(Y)\rho = \lambda[Y - \alpha(Y)\rho]. \tag{3.10}$$

Hence

$$\mathrm{Ric}(X, \nabla_Y \rho) = \lambda[\mathrm{Ric}(X,Y) - \alpha(Y)\mathrm{Ric}(X,\rho)]. \tag{3.11}$$

Applying (3.11) in (3.8), we get

$$\begin{aligned}(\nabla_Y \mathrm{Ric})(X,\rho) =&(n-1)\lambda^3[g(X,Y) - \alpha(X)\alpha(Y)]\\ &- \lambda\mathrm{Ric}(X,Y) + \lambda\alpha(Y)\mathrm{Ric}(X,\rho).\end{aligned} \tag{3.12}$$

Again using (3.6) in (3.12), we get

$$(\nabla_Y \mathrm{Ric})(X,\rho) = (n-1)\lambda^3 g(X,Y) - \lambda\mathrm{Ric}(X,Y). \tag{3.13}$$

By virtue of (1.8) the relation (3.1) becomes

$$\begin{aligned}&(\nabla_U \mathrm{Ric})(X,Y) - \frac{dr(U)}{2(n-1)} g(X,Y)\\ &= [\alpha(U) + \beta(U)]\Big[\mathrm{Ric}(X,Y) - \frac{r}{2(n-1)} g(X,Y)\Big]\\ &+ \beta(X)\Big[\mathrm{Ric}(U,Y) - \frac{r}{2(n-1)} g(U,Y)\Big]\\ &+ \beta(Y)\Big[\mathrm{Ric}(X,U) - \frac{r}{2(n-1)} g(X,U)\Big].\end{aligned} \tag{3.14}$$

Putting $Y=\rho$ in (3.14) and then using (3.13), we get

$$\begin{aligned}&\Big[(n-1)\lambda^3 g(X,U) - \lambda\mathrm{Ric}(X,U) - \frac{dr(U)}{2(n-1)} g(X,\rho)\Big]\\ &= [\alpha(U) + \beta(U)]\Big\{\mathrm{Ric}(X,\rho) - \frac{r}{2(n-1)} g(X,\rho)\Big\}\\ &+ \beta(X)\Big\{\mathrm{Ric}(U,\rho) - \frac{r}{2(n-1)} g(U,\rho)\Big\}\\ &+ \beta(\rho)\Big\{\mathrm{Ric}(X,U) - \frac{r}{2(n-1)} g(X,U)\Big\},\end{aligned} \tag{3.15}$$

which in view of (3.6) the relation (3.15) reduces to

$$\begin{aligned}[\lambda+\beta(\rho)]\text{Ric}(X,U)\\ &= \frac{r}{2(n-1)}\beta(\rho)g(X,U) - \frac{dr(U)}{2(n-1)}\alpha(X)\\ &- [\alpha(U)+\beta(U)]\Big\{(n-1)\lambda^2 - \frac{r}{2(n-1)}\Big\}\alpha(X)\\ &- \beta(X)\alpha(U)\Big\{(n-1)\lambda^2 - \frac{r}{2(n-1)}\Big\} + (n-1)\lambda^3 g(X,U).\end{aligned} \tag{3.16}$$

Putting $X=\rho$ in (3.16) then using (3.6), we get

$$\begin{aligned}[\lambda+\beta(\rho)](n-1)\lambda^2\alpha(U) &= \frac{r}{2(n-1)}\beta(\rho)\alpha(U) - \frac{dr(U)}{2(n-1)}\alpha(\rho)\\ &- [\alpha(U)+\beta(U)]\Big\{(n-1)\lambda^2 - \frac{r}{2(n-1)}\Big\}\alpha(\rho)\\ &- \Big\{(n-1)\lambda^2 - \frac{r}{2(n-1)}\Big\}\alpha(U)\beta(\rho) + (n-1)\lambda^3\alpha(U),\end{aligned}$$

which implies that

$$\beta(U) = -\Big\{\frac{2\beta(\rho)}{\alpha(\rho)} + 1\Big\}\alpha(U) - \frac{dr(U)}{\{2\lambda^2(n-1)^2 - r\}}. \tag{3.17}$$

We suppose that $\lambda+\beta(\rho)\neq 0$ and the scalar curvature r is constant. Then from (3.16) and (3.17), we find

$$\begin{aligned}\text{Ric}(X,U) =& \frac{(n-1)\lambda^3 + \frac{r}{2(n-1)}\beta(\rho)}{\lambda+\beta(\rho)}g(X,U)\\ &+ \frac{\Big(\frac{4\beta(\rho)}{\alpha(\rho)}+1\Big)\Big\{(n-1)\lambda^2 - \frac{r}{2(n-1)}\Big\}}{\lambda+\beta(\rho)}\alpha(X)\alpha(U).\end{aligned}$$

Since λ is non-zero constant then the above relation can be written as

$$\text{Ric}(X,U) = ag(X,U) + b\alpha(X)\alpha(U),$$

where $a = \frac{1}{\lambda+\beta(\rho)}\Big[(n-1)\lambda^3 + \frac{r}{2(n-1)}\beta(\rho)\Big]$ and $b = \frac{1}{\lambda+\beta(\rho)}\Big(\frac{4\beta(\rho)}{\alpha(P)}+1\Big)\Big\{(n-1)\lambda^2 - \frac{r}{2(n-1)}\Big\}$ are two non-zero scalars. Hence the manifold under consideration is a quasi-Einstein manifold.

Thus we are in the position to state the following:

Theorem 3.1. *If the scalar curvature of an $A(PSS)_n$ is constant and the basic vector field ρ is a unit concircular vector field whose associated scalar is a non-zero constant, then the manifold reduces to a quasi-Einstein manifold provided $\lambda+\beta(\rho)\neq 0$.*

4. Existence of an $A(PSS)_n$

We define a Riemannian metric g on the 4-dimensional real number space $\mathbb{R}^4$ by the formula:

$$ds^2 = g_{ij}dx^i dx^j = \sqrt[3]{t^4}\,[(dx)^2 + (dy)^2 + (dz)^2] + (dt)^2, \tag{4.1}$$

where $0 < t < \infty$; x, y, z, t are the standard coordinates of R^4. Then the only non-vanishing components of Christoffel symbols (see [5]), and the curvature tensors are as follows:

$$\begin{aligned} \begin{Bmatrix} 1 \\ 14 \end{Bmatrix} &= \begin{Bmatrix} 2 \\ 24 \end{Bmatrix} = \begin{Bmatrix} 3 \\ 34 \end{Bmatrix} = \frac{2}{3t}, \\ \begin{Bmatrix} 4 \\ 11 \end{Bmatrix} &= \begin{Bmatrix} 4 \\ 22 \end{Bmatrix} = \begin{Bmatrix} 4 \\ 33 \end{Bmatrix} = -\frac{2\sqrt[3]{t}}{3}. \end{aligned} \tag{4.2}$$

The non-zero derivatives of equation (4.2), we get

$$\begin{aligned} \frac{\partial}{\partial t}\begin{Bmatrix} 1 \\ 14 \end{Bmatrix} &= \frac{\partial}{\partial t}\begin{Bmatrix} 2 \\ 24 \end{Bmatrix} = \frac{\partial}{\partial t}\begin{Bmatrix} 3 \\ 34 \end{Bmatrix} = -\frac{2}{3t^2}, \\ \frac{\partial}{\partial t}\begin{Bmatrix} 4 \\ 11 \end{Bmatrix} &= \frac{\partial}{\partial t}\begin{Bmatrix} 4 \\ 22 \end{Bmatrix} = \frac{\partial}{\partial t}\begin{Bmatrix} 4 \\ 33 \end{Bmatrix} = -\frac{2}{9t^{\frac{2}{3}}}. \end{aligned}$$

For the Riemannian curvature tensor

$$K^l_{ijk} = \underbrace{\begin{vmatrix} \frac{\partial}{\partial x^j} & \frac{\partial}{\partial x^k} \\ \begin{Bmatrix} l \\ ij \end{Bmatrix} & \begin{Bmatrix} l \\ ik \end{Bmatrix} \end{vmatrix}}_{=I} + \underbrace{\begin{vmatrix} \begin{Bmatrix} m \\ ik \end{Bmatrix} & \begin{Bmatrix} m \\ ij \end{Bmatrix} \\ \begin{Bmatrix} l \\ mk \end{Bmatrix} & \begin{Bmatrix} l \\ mj \end{Bmatrix} \end{vmatrix}}_{=II} \tag{4.3}$$

The non-zero components of (I) in (4.3) are:

$$K^1_{441} = \frac{\partial}{\partial t}\begin{Bmatrix} 1 \\ 14 \end{Bmatrix} = -\frac{2}{3t^2}, \quad K^2_{442} = \frac{\partial}{\partial t}\begin{Bmatrix} 2 \\ 42 \end{Bmatrix} = -\frac{2}{3t^2}, \quad K^4_{334} = -\frac{\partial}{\partial t}\begin{Bmatrix} 4 \\ 33 \end{Bmatrix} = \frac{2}{9t^{\frac{2}{3}}},$$

and the non-zero components of (II) in (4.3) are:

$$K^1_{441} = \begin{Bmatrix} m \\ 14 \end{Bmatrix}\begin{Bmatrix} 1 \\ m4 \end{Bmatrix} - \begin{Bmatrix} m \\ 44 \end{Bmatrix}\begin{Bmatrix} 1 \\ m1 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 14 \end{Bmatrix}\begin{Bmatrix} 1 \\ 14 \end{Bmatrix} - \begin{Bmatrix} 1 \\ 44 \end{Bmatrix}\begin{Bmatrix} 1 \\ 11 \end{Bmatrix} = \frac{4}{9t^2}$$

$$K^2_{442} = \begin{Bmatrix} m \\ 42 \end{Bmatrix}\begin{Bmatrix} 2 \\ m4 \end{Bmatrix} - \begin{Bmatrix} m \\ 44 \end{Bmatrix}\begin{Bmatrix} 2 \\ m2 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 42 \end{Bmatrix}\begin{Bmatrix} 2 \\ 24 \end{Bmatrix} - \begin{Bmatrix} 2 \\ 44 \end{Bmatrix}\begin{Bmatrix} 2 \\ 22 \end{Bmatrix} = \frac{4}{9t^2}$$

$$K^4_{334} = \begin{Bmatrix} m \\ 34 \end{Bmatrix}\begin{Bmatrix} 4 \\ m3 \end{Bmatrix} - \begin{Bmatrix} m \\ 33 \end{Bmatrix}\begin{Bmatrix} 4 \\ m4 \end{Bmatrix} = \begin{Bmatrix} 3 \\ 34 \end{Bmatrix}\begin{Bmatrix} 4 \\ 33 \end{Bmatrix} - \begin{Bmatrix} 3 \\ 33 \end{Bmatrix}\begin{Bmatrix} 4 \\ 34 \end{Bmatrix} = -\frac{4}{9t^{\frac{2}{3}}}$$

$$K^2_{112} = \begin{Bmatrix} m \\ 12 \end{Bmatrix}\begin{Bmatrix} 2 \\ m1 \end{Bmatrix} - \begin{Bmatrix} m \\ 11 \end{Bmatrix}\begin{Bmatrix} 2 \\ m2 \end{Bmatrix} = \begin{Bmatrix} 4 \\ 12 \end{Bmatrix}\begin{Bmatrix} 2 \\ 14 \end{Bmatrix} - \begin{Bmatrix} 4 \\ 11 \end{Bmatrix}\begin{Bmatrix} 2 \\ 42 \end{Bmatrix} = \frac{4}{9t^{\frac{2}{3}}}$$

$$K^3_{113} = \begin{Bmatrix} m \\ 13 \end{Bmatrix}\begin{Bmatrix} 3 \\ m1 \end{Bmatrix} - \begin{Bmatrix} m \\ 11 \end{Bmatrix}\begin{Bmatrix} 3 \\ m3 \end{Bmatrix} = \begin{Bmatrix} 4 \\ 13 \end{Bmatrix}\begin{Bmatrix} 3 \\ 41 \end{Bmatrix} - \begin{Bmatrix} 4 \\ 11 \end{Bmatrix}\begin{Bmatrix} 3 \\ 43 \end{Bmatrix} = \frac{4}{9t^{\frac{2}{3}}}$$

$$K^2_{332} = \begin{Bmatrix} m \\ 32 \end{Bmatrix}\begin{Bmatrix} 2 \\ m3 \end{Bmatrix} - \begin{Bmatrix} m \\ 33 \end{Bmatrix}\begin{Bmatrix} 2 \\ m2 \end{Bmatrix} = \begin{Bmatrix} 4 \\ 32 \end{Bmatrix}\begin{Bmatrix} 2 \\ 43 \end{Bmatrix} - \begin{Bmatrix} 4 \\ 33 \end{Bmatrix}\begin{Bmatrix} 2 \\ 42 \end{Bmatrix} = \frac{4}{9t^{\frac{2}{3}}}.$$

Adding components corresponding (I) and (II), we get

$$K^1_{441} = K^2_{442} = -\frac{2}{9t^2}, \quad K^4_{334} = -\frac{2}{9t^{\frac{2}{3}}}$$

$$K^2_{112} = K^2_{332} = K^3_{113} = \frac{4}{9t^{\frac{2}{3}}}.$$

Thus, the non-zero components of curvature tensor of type $(0,4)$, up to symmetry are

$$\tilde{K}_{1441} = \tilde{K}_{2442} = \tilde{K}_{4334} = -\frac{2}{9\sqrt[3]{t^2}}, \qquad \tilde{K}_{2112} = \tilde{K}_{3113} = \tilde{K}_{2332} = \frac{4\sqrt[3]{t^2}}{9}. \tag{4.4}$$

Now, we can find the non-vanishing components of the Ricci tensor are as follows:

$$\begin{aligned} Ric_{11} &= g^{jh}\tilde{K}_{1j1h} = g^{22}\tilde{K}_{2112} + g^{33}\tilde{K}_{3113} + g^{44}\tilde{K}_{1441} = \frac{2}{3\sqrt[3]{t^2}} \\ Ric_{22} &= g^{jh}\tilde{K}_{2j2h} = g^{11}\tilde{K}_{2112} + g^{33}\tilde{K}_{2323} + g^{44}\tilde{K}_{2424} = \frac{2}{3\sqrt[3]{t^2}} \\ Ric_{33} &= g^{jh}\tilde{K}_{3j3h} = g^{11}\tilde{K}_{3131} + g^{22}\tilde{K}_{3232} + g^{44}\tilde{K}_{3434} = \frac{2}{3\sqrt[3]{t^2}} \\ Ric_{44} &= g^{jh}\tilde{K}_{4j4h} = g^{11}\tilde{K}_{4141} + g^{22}\tilde{K}_{4242} + g^{33}\tilde{K}_{4343} = -\frac{2}{3t^2}, \end{aligned} \tag{4.5}$$

and the scalar curvature as follows:

$$r = g^{11}Ric_{11} + g^{22}Ric_{22} + g^{33}Ric_{33} + g^{44}Ric_{44} = \frac{4}{3t^2}.$$

. The components of the Schouten tensor and scalar $\bar{\mathrm{P}}$ are as follows:

$$\mathrm{P}_{11} = \mathrm{P}_{22} = \mathrm{P}_{33} = \frac{2}{9\sqrt[3]{t^2}}, \quad \mathrm{P}_{44} = -\frac{4}{9t^2}, \quad \bar{\mathrm{P}} = \frac{2}{9t^2}. \tag{4.6}$$

Thus from (4.5) and (4.6), we show that the relation (1.11) holds, that is $r = 2(n-1)\bar{\mathrm{P}}$. It shows that the Schouten tensor in $\mathbb{R}^4$ endowed with the metric given by (4.1) can be defined as in (1.8).

We shall now show that $\mathbb{R}^4$ is an $A(PSS)_n$. Let us choose the associated 1-forms are as follows:

$$\alpha_i(t) = \begin{cases} -\frac{2}{3t}, & \text{if i=4} \\ 0, & \text{otherwise,} \end{cases} \qquad \beta_i(t) = \begin{cases} \frac{1}{t}, & \text{if i=1} \\ 0, & \text{otherwise} \end{cases} \tag{4.7}$$

at any point of $\mathbb{R}^4$. Now the relation (1.14) reduces to the following:

$$\mathrm{P}_{11,4} = [\alpha_4 + \beta_4]\mathrm{P}_{11} + \alpha_1 \mathrm{P}_{41} + \alpha_1 \mathrm{P}_{14}, \tag{4.8}$$

$$\mathrm{P}_{22,4} = [\alpha_4 + \beta_4]\mathrm{P}_{22} + \alpha_2 \mathrm{P}_{42} + \alpha_2 \mathrm{P}_{24}, \tag{4.9}$$

$$\mathrm{P}_{33,4} = [\alpha_4 + \beta_4]\mathrm{P}_{33} + \alpha_3 \mathrm{P}_{43} + \alpha_3 \mathrm{P}_{34}, \tag{4.10}$$

$$\mathrm{P}_{44,4} = [\alpha_4 + \beta_4]\mathrm{P}_{44} + \alpha_4 \mathrm{P}_{44} + \alpha_4 \mathrm{P}_{44} \tag{4.11}$$

since for the other cases (1.14) holds trivially. By (4.7) we get the following relation for the right hand side (R.H.S) and left hand side (L.H.S.) of (4.8)

$$\begin{aligned} R.H.S. \quad of \quad (4.8) &= [\alpha_4 + \beta_4]P_{11} + \alpha_1 P_{41} + \alpha_1 P_{14} \\ &= [\alpha_4 + \beta_4]P_{11} \\ &= \Big(-\frac{2}{3t} + 0\Big)\frac{2}{9\sqrt[3]{t^2}} \\ &= -\frac{4}{27\sqrt[3]{t^5}} \\ &= L.H.S. \quad of \quad (4.8). \end{aligned}$$

By similar argument it can be shown that (4.9), (4.10) and (4.11) are also true. Therefore, $(\mathbb{R}^4, g)$ is an $A(PSS)_n$ whose scalar curvature is non-zero and non-constant.

Thus the following theorem holds:

Theorem 4.1. *Let $(\mathbb{R}^4, g)$ be a 4-dimensional Lorentzian manifold with the Lorentzian metric g given by*

$$ds^2 = g_{ij}dx^i dx^j = \sqrt[3]{t^4}[(dx)^2 + (dy)^2 + (dz)^2] + (dt)^2, \tag{4.12}$$

where $0 < t < \infty$. Then $(\mathbb{R}^4, g)$ is an almost pseudo Schouten symmetric manifold.

We shall now show that this $(\mathbb{R}^4, g)$ is a quasi-Einstein manifold. Let us choose the scalar functions a and b (the associated scalars) and the 1-form η as follows:

$$a = \frac{4}{3t^2}, \quad b = -\frac{4}{t^2}, \qquad \eta_i(t) = \begin{cases} \dfrac{\sqrt[3]{t^2}}{\sqrt{6}}, & \text{if i=1,2,3} \\ \frac{1}{\sqrt{2}}, & \text{otherwise,} \end{cases} \tag{4.13}$$

at any point $\mathbb{R}^4$. We can easily check that $(\mathbb{R}^4, g)$ is a quasi-Einstein manifold.

5. On the hypersurface of an $A(PSS)_n$

In local coordinates the Schouten tensor P of an $A(PSS)_n$ satisfies the following condition

$$\mathrm{P}_{ij,k} = [A_k + B_k]\mathrm{P}_{ij} + B_i\mathrm{P}_{jk} + B_j\mathrm{P}_{ik} \tag{5.1}$$

where A and B denote the 1-forms of the manifold $A(PSS)_n$.

Let $(\bar{V}, \bar{g})$ be an $(n+1)-$dimensional Riemannian manifold covered by a system of coordinate neighborhoods $\{U, x^\alpha\}$. Let (V, g) be a hypersurface of $(\bar{V}, \bar{g})$ defined in a locally coordinate system by means of a system of parametric equation $x^\alpha = x^\alpha(t^i)$, where Greek indices take values $1, 2, ..., n$ and Latin indices take values $1, 2, ..., (n+1)$. Let n^α be the components of a local unit normal to (V, g). Then we have

$$g_{ij} = \bar{g}_{\alpha\beta} x_i^\alpha x_j^\beta \tag{5.2}$$

$$\bar{g}_{\alpha\beta} n^\alpha x_j^\beta = 0, \quad \bar{g}_{\alpha\beta} n^\alpha n^\beta = e = 1. \tag{5.3}$$

$$x_i^\alpha x_j^\beta g^{ij} = g^{\alpha\beta}, \quad g_{\alpha\beta} n^\alpha x_j^\beta = 0, \quad x^\alpha = \frac{\partial x^\alpha}{\partial t^i}. \tag{5.4}$$

The hypersurface (V, g) is called a totally umbilical [4] of $(\bar{V}, \bar{g})$ if its second fundamental form Ω_{ij} satisfies

$$\Omega_{ij} = H g_{ij}, \qquad x_{i,j}^\alpha = g_{ij} H n^\alpha \tag{5.5}$$

where the scalar H is called the mean curvature of (V, g) given by the equation $H = \frac{1}{n}\sum g^{ij}\Omega_{ij}$. If, in particular, $H = 0$, that is,

$$\Omega_{ij} = 0 \tag{5.6}$$

then the totally umbilical hypersurface is called a totally geodesic hypersurface of $(\bar{V}, \bar{g})$.

The equation of Weingarten for (V, g) can be written as $n_{,j}^\alpha = -\frac{H}{n} x_j^\alpha$. The structure equations of Gauss and Codazzi [4] for (V, g) and $(\bar{V}, \bar{g})$ are respectively given by

$$K_{ijkl} = \bar{K}_{\alpha\beta\gamma\delta} A_{ijkl}^{\alpha\beta\gamma\delta} + H^2 G_{ijkl} \tag{5.7}$$

$$\bar{K}_{\alpha\beta\gamma\delta} A_{ijk}^{\alpha\beta\gamma} n^\delta = H_{,i g_{jk}} - H_{,j g_{ik}} \tag{5.8}$$

where K_{ijkl} and $\bar{K}\alpha\beta\gamma\delta$ are the curvature tensors of (V, g) and $(\bar{V}, \bar{g})$, respectively, and

$$A_{ijkl}^{\alpha\beta\gamma\delta} = A_i^\alpha A_j^\beta A_k^\gamma A_l^\delta, \quad A_i^\alpha = x_i^\alpha, \quad G_{ijkl} = g_{il}g_{jk} - g_{ik}g_{jl}. \tag{5.9}$$

Also we have [4]

$$\overline{\mathrm{Ric}}_{\alpha\beta} A_i^\alpha A_j^\beta = \mathrm{Ric}_{ij} - H^2 g_{ij} \tag{5.10}$$

$$\overline{\mathrm{Ric}}_{\alpha\beta} n^\alpha A_i^\beta = (n-1) H_i \tag{5.11}$$

$$\bar{r} = r - n(n-1)H^2 \tag{5.12}$$

where Ric_{ij} and $\bar{\mathrm{Ric}}_{\alpha\beta}$ are the Ricci tensors and r and $\bar{r}$ are the scalar curvatures of (V, g) and $(\bar{V}, \bar{g})$, respectively.

Now we prove the following theorem:

Theorem 5.1. *The totally geodesic hypersurface of an $A(PSS)_n$ is also a $A(PSS)_n$.*

Proof. Let us consider the totally geodesic hypersurface of an $A(PSS)_n$. Then from (5.6) and (5.10), we have

$$\mathrm{Ric}_{ij} = \overline{\mathrm{Ric}} \alpha\beta A_i^\alpha A_j^\beta. \tag{5.13}$$

By virtue of (5.2) and (5.13) in (5.13), we get

$$P_{ij} = \bar{P}_{\alpha\beta} A_i^\alpha A_j^\beta. \tag{5.14}$$

Since $(\bar{V}, \bar{g})$ be an $A(PSS)_n$, then from (5.1) we find

$$\bar{\mathrm{P}}_{\alpha\beta,\gamma} = [A_\gamma + B_\gamma]\bar{\mathrm{P}}_{\alpha\beta} + B_\alpha \bar{\mathrm{P}}_{\gamma\beta} + B_\beta \bar{\mathrm{P}}_{\gamma\alpha}. \tag{5.15}$$

Multiplying both sides of (5.15) by $A_{ijk}^{\alpha\beta\gamma}$ and using (5.14), we finally get

$$\mathrm{P}_{ij,k} = [A_k + B_k]\mathrm{P}_{ij} + B_i \mathrm{P}_{jk} + B_j \mathrm{P}_{ik}. \tag{5.16}$$

Hence the froof is completed. □

Now we assume that our manifold is $A(PSS)_n$. Multiplying (3.13) by $A_{ijk}^{\alpha\beta\gamma}$, we obtain

$$A_{ijk}^{\alpha\beta\gamma} \bar{\mathrm{P}}_{\alpha\beta,\gamma} = [A_k + B_k]\mathrm{P}_{ij} + B_i \mathrm{P}_{jk} + B_j \mathrm{P}_{ik}. \tag{5.17}$$

Let the scalar curvature r be constant. Then, from (1.8), for $A(PSS)_n$, we find

$$\bar{P}_{\alpha\beta,\gamma} = \frac{1}{n-2} \overline{\mathrm{Ric}}_{\alpha\beta,\gamma}. \tag{5.18}$$

Combining the equations (5.17) and (5.18), we have

$$\frac{1}{n-2} \overline{\mathrm{Ric}}_{\alpha\beta,\gamma} A_{ijk}^{\alpha\beta\gamma} = [A_k + B_k]\mathrm{P}_{ij} + B_i \mathrm{P}_{jk} + B_j \mathrm{P}_{ik}. \tag{5.19}$$

We consider that the hypersurface is totally umbilical then by taking the covariant derivative of (5.10), it can be seen that

$$\overline{\mathrm{Ric}}_{\alpha\beta,\gamma} A^{\alpha\beta\gamma}_{ijk} = \mathrm{Ric}_{ij,k} - 2(n-1) H\, H_{,k g_{ij}}. \tag{5.20}$$

From (5.19) and (5.20), we conclude that

$$\frac{1}{n-2}\mathrm{Ric}_{ij,k} - (n-1) H\, H_{,k g_{ij}} = [A_k + B_k] P_{ij} + B_i P_{jk} + A_j P_{ik}. \tag{5.21}$$

If this hypersurface of $A(PSS)_n$ is also $A(PSS)_n$ then by virtue of (5.1) and (1.8) the equation (5.21) reduces to

$$H\, H_{,k g_{ij}} = 0. \tag{5.22}$$

This means that either $H = 0$ or $H_{,k} = 0$, that is, H is constant. Conversely, if $H = 0$ or H is constant from (1.8) and (5.21), we get (5.1). In this case, the totally umbilical hypersurface of this manifold is also $A(PSS)_n$.

Hence, we conclude the following:

Theorem 5.2. *If the scalar curvature of an $A(PSS)_n$ is constant then the totally umbilical hypersurface of an $A(PSS)_n$ be also $A(PSS)_n$, provided that the mean curvature be constant.*

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