FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 38, No 1 (2023), 125 - 139
https://doi.org/10.22190/FUMI221026009K
Original Scientific Paper

TANGENT BUNDLES OF LP-SASAKIAN MANIFOLD ENDOWED WITH GENERALIZED SYMMETRIC METRIC CONNECTION

Mohammad Nazrul Islam Khan[1] and Oğuzhan Bahadır[2]

[1] **Department of Computer Engineering, College of Computer, Qassim University, Buraydah, Saudia Arabia**
[2] **Department of Mathematics, Faculty of Arts and Sciences, Kahramanmaras Sutcu Imam University, Kahramanmaras, Turkey**

Abstract. The aim of the present work is to study and establish conditions for an LP-Sasakian manifold on the tangent bundle TM. An LP-Sasakian manifold with the generalized symmetric metric connection on TM is investigated. Next, the curvature tensor and the Ricci tensor of an LP-Sasakian manifold with respect to the generalized symmetric metric connection on TM are calculated. Moreover, the projective curvature tensor with respect to the generalized symmetric metric connection on TM is studied and showed that TM is not $\hat{\xi}^C$-projectively flat. In particular, if $\alpha = 0$ and $\beta = 1$ then TM is $\hat{\xi}^C$-projectively flat.
Keywords: LP-Sasakian manifold, Tangent bundle, Mathematical operators, Curvature tensor, Ricci tensor, Projective curvature tensor, Partial differential equations.

1. Introduction

H. A. Hayden introduced a metric connection with non-zero torsion on a Riemannian manifold in [8]. The properties of Riemannian manifolds with semi-symmetric (symmetric) and a non-metric connection have been studied by many authors ([1], [5], [7], [16]). The idea of quarter-symmetric linear connections in a differential

Received October 26, 2022. accepted January 10, 2023.
Communicated by Uday Chand De
Corresponding Author: Mohammad Nazrul Islam Khan, Department of Computer Engineering, College of Computer, Qassim University, Buraydah 51452, Saudia Arabia | E-mail: m.nazrul@edu.qu.sa
2010 *Mathematics Subject Classification.* Primary 53C15, 53C25; Secondary 53C40

manifold was introduced by S. Golab [7]. A linear connection is said to be a quarter-symmetric connection if its torsion tensor T is of the form

$$T(X,Y) = \eta(Y)\phi X - \eta(X)\phi Y \tag{1.1}$$

for any vector fields X, Y on a manifold, where η is a 1-form and ϕ is a tensor of type (1, 1). If $\phi = I$, then the quarter-symmetric connection is reduced to a semi-symmetric connection. Hence the quarter-symmetric connection can be viewed as a generalization of semi-symmetric connection. A linear connection ∇ is said to be a generalized symmetric connection if its torsion tensor T is of the form

$$T(X,Y) = \alpha\{\eta(Y)X - \eta(X)Y\} + \beta\{\eta(Y)\phi X - \eta(X)\phi Y\} \tag{1.2}$$

for any vector fields X, Y on a manifold, where α and β are smooth functions. ϕ is a tensor of type (1, 1) and η is a 1-form. Moreover, the connection ∇ is said to be a generalized symmetric metric connection if there is a Riemannian metric g in M such that $\nabla g = 0$, otherwise it is non-metric.

On the other hand, in 1989, K. Matsumoto [17] introduced the notion of Lorentzian para-Sasakian manifolds. I. Mihai and R. Rosca [18] studied the same manifolds independently and they obtained several results on such manifolds. Lorentzian para-Sasakian manifolds have also been studied by K. Matsumoto and I. Mihai [19], I. Mihai, A.A. Shaikh, and U. C. De [20]. S. K. Srivastava and R. P. Kushwaha [24] studied Lorentzian para-Sasakian manifolds admitting a special semi-symmetric recurrent metric connection.

In the framework of the geometry of tangent bundles, it is classical to consider geometrical structures and connections using some natural operations transforming structures and connections on the base manifold to its tangent bundle. The study of the geometry of the tangent bundle becomes then equivalent to the study of the relationship between the geometrical properties of the tangent bundle endowed with the obtained structure and the base manifold. The classical prolongations i.e. vertical, horizontal and complete lifts of the tensor fields and connections have been introduced by Ishihara, Kobayashi, and Yano. M. Tani introduced the notion of prolongations of surfaces to tangent bundle and developed the theory of the surface prolonged to the tangent bundle with respect to the metric tensor [27]. The complete, vertical and horizontal lifts of almost r-contact structures in tangent bundle were studied by Das and Khan [3]. Khan studied the lifts of hypersurface with quarter symmetric semi-metric connection and semi-symmetric non-metric connection on Kähler manifold to tangent bundle in [14] and [15], respectively.

The present paper is organized as follows: Section 2 relates to preliminaries on the subject of LP-Sasakian manifold, tangent bundle, Vertical and Complete lifts. Section 3 deals with induced metric and connection on TS, as well as theränditons for LP-Sasakian manifold in tangent bundle TM. The Section 4 is devoted to the study of LP-Sasakian manifiold with the generalized symmetric metric connection on the tangent bundle TM. The curvature tensor and the Ricci tensor of a Lorentzian para-Sasakian with respect to the generalized symmetric metric connection on TM have been calculated. In the last Section, the projective curvature tensor with respect to the generalized symmetric metric connection on TM is discussed.

2. Preliminaries

A differentiable manifold of dimension n is called Lorentzian para-Sasakian (briefly, LP-Sasakian) ([17], [18]), if it admit a $(1,1)$ tensor field ϕ, a contravariant vector field ξ, a $1-$ form η and Lorentzian metric g which satify

$$\eta(\xi) = -1, \tag{2.1}$$
$$\phi^2(X) = X + \eta(X)\xi, \tag{2.2}$$
$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), \tag{2.3}$$
$$g(X,\xi) = \eta(X), \tag{2.4}$$
$$\nabla_X \xi = \phi X, \tag{2.5}$$
$$(\nabla_X \phi)(Y) = g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi, \tag{2.6}$$

where ∇ is Levi-Civita connection with respect to the Lorentzian metric g. It can be easily seen that in an LP-Sasakian manifold the following relations hold:

$$\phi\xi = 0,\ \eta(\phi X) = 0,\ rank\phi = n-1, \tag{2.7}$$

If we write

$$\Phi(X,Y) = g(\phi X, Y), \tag{2.8}$$

for any vector field X and Y, then the tensor field $\Phi(X,Y)$ is a symmetric (0,2) tensor field. Also, since the vector η is closed in an LP-Sasakian [17] manifold, we have

$$(\nabla_X \eta)Y = \Phi(X,Y),\ \Phi(X,\xi) = 0, \tag{2.9}$$

for any vector field X and Y.

Let M be an n-dimensional LP-Sasakian manifold. Then the following relations hold ([20], [23]):

$$g(R(X,Y)Z,\xi) = \eta(R(X,Y)Z) = g(Y,Z)\eta(X) - g(X,Z)\eta(Y), \tag{2.10}$$
$$R(\xi,X)Y = g(X,Y)\xi - \eta(Y)X, \tag{2.11}$$
$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.12}$$
$$R(\xi,X)\xi = X + \eta(X)\xi, \tag{2.13}$$
$$S(X,\xi) = (n-1)\eta(X), \tag{2.14}$$
$$S(\phi X, \phi Y) = S(X,Y) + (n-1)\eta(X)\eta(Y) \tag{2.15}$$

for any vector fields X, Y and Z, where R and S are the curvature and Ricci tensors of M, respectively.

A LP-Sasakian manifold M is said to be η-Einstein if its Ricci tensor S is of the form

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \tag{2.16}$$

where a, b are scalar functions such that $b \neq 0$. If $b = 0$ then M is called η-Einstein manifold.

Let TM be the tangent bundle of n-dimensional differentiable manifold over M with the bundle projection $\pi_M : TM \to M$ define the natural bundle structure of TM over M. Let $\{U; x^i\}$ coordinate neighborhood in M where $\{x^i\}$ is a system of local coordinates in neighborhood U. Let $\{x^i, y^i\}$ be a system of local coordinates in $\pi_M^{-1}(U) \subset TM$ i.e. $\{x^i, y^i\}$ the induced coordinate in $\pi_M^{-1}(U)$.

Let $\wp_s^r(M)$ be the set of all tensor fields of type (r,s) in M, namely contravariant of degree r and covariant of degree s. If we denote by $\wp(M)$ the tensor algebra associated with M i.e. $\wp(M) = \wp_s^r(M)$. The set of tensor fields in tangent bundle represented by $\wp_s^r(TM)$ and tensor algebra on the tangent bundle by $\wp(TM)$. The set of functions, vector fields, 1-forms and tensor fields of type (1,1) are denoted by $\wp_0^0(TM), \wp_0^1(TM), \wp_1^0(TM)$ and $\wp_1^1(TM)$ respectively.

Let S be a manifold of dimension $n-1$. We denote the imbedding by $\tau : S \to M$ and by B the mapping induced by τ from $T(S)$ to TM, where $T(S)$ to TM denote tangent bundle of manifold S and M respectively. Then the mapping B induces its tangential map $dB : T(T(S)) \to T(TM)$ and denoted by $\tilde{B}$ i.e.

$$\tau : S \to M$$

$$d\tau = B : T(S) \to TM$$
$$dB = \tilde{B} : T(T(S)) \to T(TM).$$

Let the tensor algebra corresponding sets of tensor fields in $T(S,M)$ be denoted by $\wp(S,M)$. Then

$$B : \wp(S) \to \wp(S,M)$$
$$B : \wp(T(S)) \to \wp(T(S,M)).$$

Let f, X, ω and F be the elements of $\wp(S)$ and $\overline{f}, \overline{X}, \overline{\omega}$ and $\overline{F}$ be the elements of $\wp(S,M)$ [27].

2.1. Vertical and complete lifts

If $X = X^i \frac{\partial}{\partial x^i}$ is a local vector field on M, then its vertical, complete and horizontal lifts in the term of partial differential equations are

$$X^V = X^i \frac{\partial}{\partial y^i} \tag{2.17}$$

$$X^C = X^i \frac{\partial}{\partial x^i} + \frac{\partial X^i}{\partial x^j} y^j \frac{\partial}{\partial y^i} \tag{2.18}$$

(2.19)

where Γ^i_{js} represent components of the affine connection. The vertical and complete lifts of a function, a vector field and a 1-form defined as $f^V = fo\pi_M, \quad f^C = y^i\partial_i f, \quad where \quad \partial_i = \frac{\partial}{\partial x^i}$

$$(fX)^V = f^V X^V, (fX)^C = f^C X^V + f^V X^C, \tag{2.20}$$

$$X^V f^V = 0, X^V f^C = X^C f^V = (Xf)^V, X^C f^C = (Xf)^C, \tag{2.21}$$

$$\omega^V(f^V) = 0, \omega^V(X^C) = \omega^C(X^V) = \omega(X)^V, \omega^C(X^C) = \omega(X)^C, \tag{2.22}$$

$$F^V X^C = (FX)^V, F^C X^C = (FX)^C, \tag{2.23}$$

$$[X,Y]^V = [X^C, Y^V] = [X^V, Y^C], [X,Y]^C = [X^C, Y^C], \tag{2.24}$$

We extend the vertical and complete lifts to a linear isomorphism of tensor algebra $\wp(M)$ into $\wp(TM)$ with respect to constant coefficient. Let P^V and Q^V be vertical lift and P^C and Q^C be complete lift of arbitrary tensor fields P and Q of $\wp(M)$. Then by using mathematical operators [11]

$$(P \otimes Q)^V = P^V \otimes Q^V, (P \otimes Q)^C = P^C \otimes Q^V + P^V \otimes Q^C$$

$$(P + Q)^V = P^V + Q^V, (P + Q)^C = P^C + Q^C.$$

2.2. Vertical and complete lifts of $\wp^r_s(S, M)$ to TM)

If $\overline{f}$ is a function on S. The vertical lift $\overline{f}^{\overline{V}}$ of $\overline{f}$ to TM is given by $\overline{f}^{\overline{V}} = \overline{f} \circ \pi_S$. Let U be neighborhood of p in M. Then the function $\hat{f}$ fits with $\overline{f}$ in $U \cup S$ containing p. The complete lift $\hat{f}^{\hat{C}}$ of $\hat{f}$ is given as $\hat{f}^C = y^i\partial_i\hat{f}$ in $\pi_M^{-1}(U)$. If $\overline{X}$ is an element of $\wp^r_s(S, M)$. The vertical lift $\overline{X}^{\overline{V}}$ to TM is defined by $\overline{X}^{\overline{V}}\hat{f}^C = (\overline{X}\hat{f})^{\overline{V}}$ and complete lift $\overline{X}^{\overline{C}}$ to TM is defined as $\overline{X}^{\overline{C}}\hat{f}^C = (\overline{X}\hat{f})^{\overline{C}}$, for each $\hat{f} \in \wp^0_0(M)$ along S. Similarly, If $\overline{\omega}$ is an element of $\wp^0_1(S, M)$. The vertical lift $\overline{\omega}^{\overline{V}}$ and complete lift $\overline{\omega}^{\overline{C}}$ to TM are defined by $\overline{\omega}^{\overline{V}}(\overline{X}^{\overline{C}}) = (\overline{\omega}(\overline{X}))^{\overline{V}}$ and $\overline{\omega}^{\overline{C}}(\overline{X}^{\overline{C}}) = (\overline{\omega}(\overline{X}))^{\overline{C}}$ for each $\overline{X} \in \wp^0_1(M)$ respectively [10, 13].

We extend the vertical and complete lifts to linear isomorphism of $\wp(S, M)$ to $\wp(T(S, M))$ with respect to constant coefficients. Let $\overline{P}^{\overline{V}}$ and $\overline{Q}^{\overline{V}}$ be vertical and complete lifts of arbitrary tensor fields $\overline{P}$ and $\overline{Q}$ on $\wp(S, M)$. Then by definition

$$(\overline{P} \otimes \overline{Q})^{\overline{V}} = \overline{P}^{\overline{V}} \otimes \overline{Q}^{\overline{V}}$$

$$(\overline{P} \otimes \overline{Q})^{\overline{C}} = \overline{P}^{\overline{V}} \otimes \overline{Q}^{\overline{C}} + \overline{P}^{\overline{C}} \otimes \overline{Q}^{\overline{V}}.$$

The relation between lifts of $\wp^0_0(M)$ to TM and the lifts of $\wp^0_0(S)$ to $T(S)$ are given by

$$\overline{f}^{\overline{V}} = \overline{f}^V, \overline{f}^{\overline{C}} = \overline{f}^C, \forall \overline{f} \in \wp^0_0(S, M) = \wp^0_0(S).$$

Moreover, we have following properties:

$$\hat{T}^C(\overline{X}^{\overline{V}}, \overline{Y}^{\overline{C}}) = \hat{T}^V(\overline{X}^{\overline{C}}, \overline{Y}^{\overline{C}}) = \hat{T}^C(\overline{X}, \overline{Y})^V$$

$$\hat{T}^C(\overline{X}^{\overline{C}},\overline{Y}^{\overline{C}}) = \hat{T}(\overline{X},\overline{Y})^C$$
$$\hat{T}^C(\overline{X}^{\overline{V}},\overline{Y}^{\overline{V}}) = \hat{T}^V(\overline{X}^{\overline{C}},\overline{Y}^{\overline{V}}) = \hat{T}^V(\overline{X}^{\overline{V}},\overline{Y}^{\overline{V}}), \forall \overline{f} \in \wp^0_0(S), \overline{X} \in \wp^1_0(S,M).$$

3. Induced metric and connection on $T(S)$

Let g be a Riemannian metric in manifold M. The complete lift of $\hat{g}$ is $\hat{g}^C$ in TM. The induced metric from $\hat{g}^C$ on $T(S)$ is denoted by $\tilde{g}$. Then we can write

$$\tilde{g}(X^C, Y^C) = \hat{g}^C(\tilde{B}X^C, \tilde{B}Y^C), \forall X, Y \in \wp^1_0(S). \tag{3.1}$$

Let $\hat{\nabla}$ be an affine connection in M and $\hat{\nabla}^C$ be the complete lift of $\hat{\nabla}$ on TM. Then by definition

$$\hat{\nabla}^C_{X^C} Y^C = (\hat{\nabla}_{\hat{X}}\hat{Y})^C$$
$$\hat{\nabla}^C_{X^C} Y^V = (\hat{\nabla}_{\hat{X}}\hat{Y})^V$$

where $\hat{\nabla}$ is the Riemannian connection with respect to $\hat{g}$ and $\hat{\nabla}^C$ is the Riemannian connection of TM with respect to $\hat{g}^C$.

Taking complete lifts of (2.1)-(2.6), we have the following

$$\hat{\eta}^V(\hat{\xi}^V) = \hat{\eta}^C(\hat{\xi}^C) = 0, \hat{\eta}^V(\hat{\xi}^C) = \hat{\eta}^C(\hat{\xi}^V) = -1, \tag{3.2}$$

$$(\phi^C)^2 = I + \hat{\eta}^V \otimes \hat{\xi}^C + \hat{\eta}^C \otimes \hat{\xi}^V \tag{3.3}$$

$$\begin{aligned} \tilde{g}((\phi X)^C, (\phi Y)^C) &= \tilde{g}(\tilde{B}X^C, \tilde{B}Y^C) + \hat{\eta}^C(\tilde{B}X^C)\hat{\eta}^V(\tilde{B}X^C) \\ &+ \hat{\eta}^V(\tilde{B}X^C)\hat{\eta}^C(\tilde{B}X^C) \end{aligned} \tag{3.4}$$

Put $Y = \hat{\xi}$ in (2.3) and $\phi^C\hat{\xi}^C = 0$, we get

$$\tilde{g}(\tilde{B}X^C, \tilde{B}\hat{\xi}^C) = \hat{\eta}^C(\tilde{B}X^C) \tag{3.5}$$

$$\nabla^C_{X^C}\hat{\xi}^C = (\phi X)^C = \phi^C\hat{\xi}^C \tag{3.6}$$

$$\begin{aligned} (\nabla^C_{\tilde{B}X^C}\tilde{B}\phi^C)(\tilde{B}Y^C) &= \tilde{g}(\tilde{B}X^C, \tilde{B}Y^C)\hat{\xi}^V + \tilde{g}(\tilde{B}X^V, \tilde{B}Y^C)\hat{\xi}^C \\ &+ \hat{\eta}^C(\tilde{B}Y^C)(\tilde{B}X^V) + \hat{\eta}^V(\tilde{B}Y^C)(\tilde{B}X^C) \\ &+ 2\hat{\eta}^V(\tilde{B}X^C)\hat{\eta}^C(\tilde{B}Y^C)\hat{\xi}^C + 2\hat{\eta}^C(\tilde{B}X^C)\hat{\eta}^V(\tilde{B}Y^C)\hat{\xi}^C \\ &+ 2\hat{\eta}^C(\tilde{B}X^C)\hat{\eta}^C(\tilde{B}Y^C)\hat{\xi}^V \end{aligned} \tag{3.7}$$

Now in the view of (2.7) and (2.8), we have

$$\phi^C\hat{\xi}^V = \phi^C\hat{\xi}^C = \phi^V\hat{\xi}^V = 0, \tag{3.8}$$

$$\hat{\eta}^C(\phi X)^C = \hat{\eta}^C(\phi X)^V = \hat{\eta}^V(\phi X)^C = \hat{\eta}^V(\phi X)^V = 0 \tag{3.9}$$

$$\Phi^C(\tilde{B}X^C, \tilde{B}Y^C) = \hat{g}(\tilde{B}(\phi X)^C, \tilde{B}Y^C) \tag{3.10}$$

where Φ^C is a symmetric tensor field of type (0,2) in TM.
From (2.9), we have

$$(\nabla^C_{\tilde{B}X^C}\tilde{B}\hat{\eta}^C)\tilde{B}Y^C = \phi^C(\tilde{B}X^C, \tilde{B}Y^C), \hat{g}(\tilde{B}(\phi X)^C, \hat{\eta}^C) = 0 \tag{3.11}$$

Let M be n-dimensional LP- Sasakian manifold and TM its tangent bundle. Then the following conditions hold [19], [20]:

$$\begin{aligned}
\hat{g}^C(R^C(\tilde{B}X^C,\tilde{B}Y^C)\tilde{B}Z^C,\hat{\xi}^C) &= \hat{\eta}^C(R^C(\tilde{B}X^C,\tilde{B}Y^C)\tilde{B}Z^C) \\
&= \hat{g}^C(\tilde{B}Y^C,\tilde{B}Z^C)\hat{\eta}^V(\tilde{B}X^C) \\
&+ \hat{g}^C(\tilde{B}Y^V,\tilde{B}Z^C)\hat{\eta}^V(\tilde{B}X^C) \\
&- \hat{g}^C(\tilde{B}X^C,\tilde{B}Z^C)\hat{\eta}^V(\tilde{B}Y^C) \\
&- \hat{g}^C(\tilde{B}X^C,\tilde{B}Z^C)\hat{\eta}^V(\tilde{B}Y^C) \qquad (3.12)
\end{aligned}$$

$$R^C(\hat{\xi}^C,\tilde{B}X^C)\tilde{B}Y^C = \hat{g}^C(\tilde{B}X^C,\tilde{B}Y^C) \qquad (3.13)$$

$$- \hat{\eta}^C(\tilde{B}Y^C)\tilde{B}X^V - \hat{\eta}^V(\tilde{B}Y^C)\tilde{B}X^C \qquad (3.14)$$

$$\begin{aligned}
R^C(\tilde{B}X^C,\tilde{B}Y^C)\hat{\xi}^C &= \hat{\eta}^C(\tilde{B}Y^C)\tilde{B}X^V + \hat{\eta}^V(\tilde{B}Y^C)\tilde{B}X^C \\
&- \hat{\eta}^C(\tilde{B}X^C)\tilde{B}Y^V - \hat{\eta}^V(\tilde{B}X^C)\tilde{B}Y^C \qquad (3.15)
\end{aligned}$$

$$R^C(\hat{\xi}^C,\tilde{B}X^C)\hat{\xi}^C = \tilde{B}X^C + \hat{\eta}^C(\tilde{B}X^C)\hat{\xi}^V + \hat{\eta}^V(\tilde{B}X^C)\hat{\xi}^C \qquad (3.16)$$

$$S^C(\tilde{B}X^C,\hat{\xi}^C) = (n-1)\hat{\eta}^C(\tilde{B}X^C) \qquad (3.17)$$

$$\begin{aligned}
S^C(\tilde{B}(\phi X)^C,\tilde{B}(\phi Y^C)) &= S^C(\tilde{B}X^C,\tilde{B}Y^C) + (n-1)\hat{\eta}^V(\tilde{B}X^C)\hat{\hat{\eta}}^C(\tilde{B}Y^C) \\
&+ (n-1)\hat{\hat{\eta}}^C(\tilde{B}X^C)\hat{\eta}^V(\tilde{B}Y^C) \qquad (3.18)
\end{aligned}$$

for any $X^C, Y^C \in \wp_0^1(TM)$.

4. LP-Sasakian manifold with the generalized symmetric metric connection on the tangent bundle

Let $\overline{\nabla}^C$ be a complete lift of linear connection $\overline{\nabla}$ and ∇^C be complete lift of Levi-Civita connection ∇ of an LP-Sasakian manifold M such that

$$\overline{\nabla}^C_{\tilde{B}X^C}\tilde{B}Y^C = \nabla^C_{\tilde{B}X^C},\tilde{B}Y^C + \tilde{H}(X^C,Y^C). \qquad (4.1)$$

where $\tilde{H}$ is a tensor of type (0,2) in TM. and given by

$$\tilde{H}(X^C,Y^C) = \frac{1}{2}[T^C(X^C,Y^C) + T'^C(X^C,Y^C) + T'^C(X^C,Y^C)], \qquad (4.2)$$

Taking complete lift of (1.2), we get

$$\begin{aligned}
T^C(\tilde{B}X^C,\tilde{B}Y^C) &= \alpha\{\hat{\eta}^C(\tilde{B}Y^C)(\tilde{B}X^V) + \hat{\eta}^V(\tilde{B}Y^C)(\tilde{B}X^C) \\
&- \hat{\eta}^C(\tilde{B}X^C)(\tilde{B}Y^V) - \hat{\eta}^V(\tilde{B}X^C)(\tilde{B}Y^C)\} \\
&+ \beta\{\hat{\eta}^C(\tilde{B}Y^C)(\tilde{B}(\phi X)^V) + \hat{\eta}^V(\tilde{B}Y^C)(\tilde{B}(\phi X)^C) \\
&- \hat{\eta}^C(\tilde{B}X^C)(\tilde{B}(\phi Y)^V) - \hat{\eta}^V(\tilde{B}X^C)(\tilde{B}(\phi Y)^C)\} \qquad (4.3)
\end{aligned}$$

For $\overline{\nabla}^C$ to be a generalized symmetric metric connection of ∇^C, we have

$$\hat{g}^C(T'^C(X^C,Y^C),Z) = \hat{g}^C(T^C(Z^C,X^C),Y^C). \qquad (4.4)$$

where T^C is the torsion tensor of $\overline{\nabla}^C$. From (4.3) and (4.4), we get

$$\begin{aligned} T'^C(X^C, Y^C) &= \alpha\{\hat{\eta}^C(\tilde{B}X^C)(\tilde{B}Y^V) + \hat{\eta}^V(\tilde{B}X^C)(\tilde{B}Y^C) \\ &- \hat{g}^C(\tilde{B}X^C)(\tilde{B}Y^C)\hat{\xi}^V - \hat{g}^C(\tilde{B}X^V)(\tilde{B}Y^C)\hat{\xi}^C\} \\ &+ \beta\{\hat{\eta}^C(\tilde{B}X^C)(\tilde{B}(\phi Y)^V) + \hat{\eta}^V(\tilde{B}X^C)(\tilde{B}(\phi Y)^C) \\ &- \hat{g}^C(\tilde{B}(\phi X)^C), (\tilde{B}Y)^C)\hat{\xi}^V \\ &- \hat{g}^C(\tilde{B}(\phi X)^V), (\tilde{B}Y)^C)\hat{\xi}^C\} \end{aligned} \tag{4.5}$$

Using (4.3),(4.2) and (4.5), we obtain

$$\begin{aligned} \tilde{H}(X^C, Y^C) &= \alpha\{\hat{\eta}^C(\tilde{B}Y^C)(\tilde{B}X^V) + \hat{\eta}^V(\tilde{B}Y^C)(\tilde{B}X^C) \\ &- \hat{g}^C(\tilde{B}X^C)(\tilde{B}Y^C)\hat{\xi}^V - \hat{g}^C(\tilde{B}X^V)(\tilde{B}Y^C)\hat{\xi}^C\} \\ &+ \beta\{\hat{\eta}^C(\tilde{B}Y^C)(\tilde{B}(\phi X)^V) + \hat{\eta}^V(\tilde{B}Y^C)(\tilde{B}(\phi X)^C) \\ &- \hat{g}^C(\tilde{B}(\phi X)^C), (\tilde{B}Y)^C)\hat{\xi}^V \\ &- \hat{g}^C(\tilde{B}(\phi X)^V), (\tilde{B}Y)^C)\hat{\xi}^C\} \end{aligned} \tag{4.6}$$

Thus we have the following corollary:

Corollary 4.1. *For an LP-Sasakian manifold, a generalized symmetric metric connection $\overline{\nabla}^C$ of type (α, β) is given by*

$$\begin{aligned} \overline{\nabla}^C_{\tilde{B}X^C}\tilde{B}Y^C &= \nabla^C_{\tilde{B}X^C}\tilde{B}Y^C + \alpha\{\hat{\eta}^C(\tilde{B}Y^C)(\tilde{B}X^V) + \hat{\eta}^V(\tilde{B}Y^C)(\tilde{B}X^C) \\ &- \hat{g}^C(\tilde{B}X^C)(\tilde{B}Y^C)\hat{\xi}^V - \hat{g}^C(\tilde{B}X^V)(\tilde{B}Y^C)\hat{\xi}^C\} \\ &+ \beta\{\hat{\eta}^C(\tilde{B}Y^C)(\tilde{B}(\phi X)^V) + \hat{\eta}^V(\tilde{B}Y^C)(\tilde{B}(\phi X)^C) \\ &- \hat{g}^C(\tilde{B}(\phi X)^C), (\tilde{B}Y)^C)\hat{\xi}^V - \hat{g}^C(\tilde{B}(\phi X)^V), (\tilde{B}Y)^C)\hat{\xi}^C\}. \end{aligned} \tag{4.7}$$

If we choose $(\alpha, \beta) = (1, 0)$ and $(\alpha, \beta) = (0, 1)$, a generalized symmetric metric connection is reduced a semi-symetric metric connection and quarter-symetric metric connection as follows:

$$\begin{aligned} \overline{\nabla}^C_{\tilde{B}X^C}\tilde{B}Y^C &= \nabla^C_{\tilde{B}X^C}\tilde{B}Y^C + \hat{\eta}^C(\tilde{B}Y^C)(\tilde{B}X^V) + \hat{\eta}^V(\tilde{B}Y^C)(\tilde{B}X^C) \\ &- \hat{g}^C(\tilde{B}X^C)(\tilde{B}Y^C)\hat{\xi}^V - \hat{g}^C(\tilde{B}X^V)(\tilde{B}Y^C)\hat{\xi}^C \end{aligned} \tag{4.8}$$

$$\begin{aligned} \overline{\nabla}^C_{\tilde{B}X^C}\tilde{B}Y^C &= \nabla^C_{\tilde{B}X^C}\tilde{B}Y^C + \hat{\eta}^C(\tilde{B}Y^C)(\tilde{B}(\phi X)^V) + \hat{\eta}^V(\tilde{B}Y^C)(\tilde{B}(\phi X)^C) \\ &- \hat{g}^C(\tilde{B}(\phi X)^C), (\tilde{B}Y)^C)\hat{\xi}^V - \hat{g}^C(\tilde{B}(\phi X)^V), (\tilde{B}Y)^C)\hat{\xi}^C \end{aligned} \tag{4.9}$$

from (2.5), (3.7), (3.11) and (4.7), we have following proposition:

Proposition 4.1. *Let M be an LP-Sasakian manifold with the generalized symmetric metric connection and TM its tangent bundle. We have the following relations:*

$$(\overline{\nabla}^C_{\tilde{B}X^C}\tilde{B}\phi^C)(\tilde{B}Y^C) = (1-\beta)\{\hat{g}^C(\tilde{B}X^C)(\tilde{B}Y^C)\hat{\xi}^V + \hat{g}^C(\tilde{B}X^V)(\tilde{B}Y^C)\hat{\xi}^C\}$$

$$
\begin{aligned}
&+ \quad (2-2\beta)\{\hat{\eta}^V(\tilde{B}X^C)\hat{\eta}^C(\tilde{B}Y^C)\hat{\xi}^C + \hat{\eta}^C(\tilde{B}X^C)\hat{\eta}^V(\tilde{B}Y^C)\hat{\xi}^C \\
&+ \quad \hat{\eta}^C(\tilde{B}X^C)\hat{\eta}^V(\tilde{B}Y^C)\hat{\xi}^V\} \\
&- \quad \alpha\{\hat{g}^C(\tilde{B}(\phi X)^C), (\tilde{B}Y)^C)\hat{\xi}^V + \hat{g}^C(\tilde{B}(\phi X)^V), (\tilde{B}Y)^C)\hat{\xi}^C\} \\
&+ \quad (1-\beta)\{\hat{\eta}^C(\tilde{B}Y^C)(\tilde{B}X^V) + \hat{\eta}^V(\tilde{B}Y^C)(\tilde{B}X^C)\} \\
&- \quad \alpha\{\hat{\eta}^C(\tilde{B}Y^C)(\tilde{B}(\phi X)^V) + \hat{\eta}^V(\tilde{B}Y^C)(\tilde{B}(\phi X)^C)\}
\end{aligned} \tag{4.10}
$$

$$
\begin{aligned}
\overline{\nabla}^C_{\tilde{B}X^C}\hat{\xi}^C &= (1-\beta)\tilde{B}(\phi X)^C - \phi\tilde{B}X^C \\
&- \alpha\hat{\eta}^V(\tilde{B}X^C)\hat{\xi}^C - \alpha\hat{\eta}^C(\tilde{B}X^C)\hat{\xi}^V
\end{aligned} \tag{4.11}
$$

$$
\begin{aligned}
(\overline{\nabla}^C_{\tilde{B}X^C}\hat{\eta}^C)(\tilde{B}Y^C) &= (1-\beta)\{\hat{g}^C(\tilde{B}(\phi X)^C, \tilde{B}Y^C)\} \\
&- \alpha\hat{g}^C(\tilde{B}(\phi X)^C, \tilde{B}(\phi Y)^C)
\end{aligned} \tag{4.12}
$$

for any $X^C, Y^C, Z^C \in \wp^1_0(TM)$.

5. Curvature Tensor of an LP-Sasakian manifold with respect to the generalized symmetric metric connection on the tangent bundle

Let M be an $n-$ dimensional $LP-$Sasakian manifold with the generalized symmetric metric connection and TM its tangent bundle. Let $\overline{R}^C$ be the curvature tensor of $\overline{\nabla}^C$ on $T(S)$ in TM, then we have

$$
\begin{aligned}
\overline{R}^C(\tilde{B}X^C, \tilde{B}Y^C)\tilde{B}Z^C &= \overline{\nabla}^C_{\tilde{B}X^C}\overline{\nabla}^C_{\tilde{B}Y^C}\tilde{B}Z^C - \overline{\nabla}^C_{\tilde{B}Y^C}\overline{\nabla}^C_{\tilde{B}X^C}\tilde{B}Z^C \\
&- \overline{\nabla}^C_{[\tilde{B}X^C, \tilde{B}Y^C]}\tilde{B}Z^C
\end{aligned} \tag{5.1}
$$

Using (4.7), (5.1) and Proposition 4.1, we have

$$
\begin{aligned}
\overline{R}^C(\tilde{B}X^C, \tilde{B}Y^C)\tilde{B}Z^C &= R^C(\tilde{B}X^C, \tilde{B}Y^C)\tilde{B}Z^C + K^C_1(\tilde{B}Y^C, \tilde{B}Z^C)\tilde{B}X^C \\
&- K^C_1(\tilde{B}X^C, \tilde{B}Z^C)\tilde{B}Y^C + K^C_2(\tilde{B}Y^C, \tilde{B}Z^C)\tilde{B}(\phi X)^C \\
&- K^C_2(\tilde{B}X^C, \tilde{B}Z^C)\tilde{B}(\phi Y)^C + K^C_3(\tilde{B}X^C, \tilde{B}Y^C)\tilde{B}Z^C \\
&- K^C_3(\tilde{B}Y^C, \tilde{B}X^C)\tilde{B}Z^C
\end{aligned} \tag{5.2}
$$

where

$$
\begin{aligned}
K^C_1(\tilde{B}Y^C, \tilde{B}Z^C)\tilde{B}X^C &= \alpha(\beta-1)\{\hat{g}^C(\tilde{B}(\phi Y)^C, \tilde{B}Z^C)\tilde{B}X^V \\
&- \hat{g}^C(\tilde{B}(\phi Y)^V, \tilde{B}Z^C)\tilde{B}X^C\} \\
&- \alpha^2\{\hat{g}^C(\tilde{B}Y^C, \tilde{B}Z^C)\tilde{B}X^V + \hat{g}^C(\tilde{B}Y^V, \tilde{B}Z^C)\tilde{B}X^C\} \\
&+ (\alpha^2+\beta-\beta^2)\{\hat{\eta}^V(\tilde{B}Y^C)\hat{\eta}^C(\tilde{B}Z^C)\tilde{B}X^C \\
&+ \hat{\eta}^C(\tilde{B}Y^C)\hat{\eta}^V(\tilde{B}Z^C)\tilde{B}X^C \\
&+ \hat{\eta}^C(\tilde{B}Y^C)\hat{\eta}^C(\tilde{B}Z^C)\tilde{B}X^V\}
\end{aligned} \tag{5.3}
$$

$$
\begin{aligned}
K^C_2(\tilde{B}Y^C, \tilde{B}Z^C)\tilde{B}(\phi X)^C &= \beta(\beta-2)\{\hat{g}^C(\tilde{B}(\phi Y)^C, \tilde{B}Z^C)\tilde{B}(\phi X)^V \\
&+ \hat{g}^C(\tilde{B}(\phi Y)^V, \tilde{B}Z^C)\tilde{B}(\phi X)^C\}
\end{aligned}
$$

$$
\begin{aligned}
& - \quad \alpha(1-\beta)\{\hat{g}^C(\tilde{B}Y^C, \tilde{B}Z^C)(\phi X)^V \\
(5.4) \qquad & + \quad \hat{g}^C(\tilde{B}Y^V, \tilde{B}Z^C)(\phi X)^C\} \\
K_3^C(\tilde{B}X^C, \tilde{B}Y^C)\tilde{B}Z^C & = \quad (\alpha^2+\beta)\{\hat{g}^C(\tilde{B}Y^C, \tilde{B}Z^C)(\hat{\eta}(X))^V\hat{\xi}^V \\
& + \quad \hat{g}^C(\tilde{B}Y^V, \tilde{B}Z^C)(\hat{\eta}(X))^C\hat{\xi}^V \\
& + \quad \hat{g}^C(\tilde{B}Y^V, \tilde{B}Z^C)(\hat{\eta}(X))^V\hat{\xi}^C \\
& + \quad \alpha\beta\{\hat{g}^C(\tilde{B}(\phi Y)^C, \tilde{B}Z^C)(\hat{\eta}(X))^V\hat{\xi}^V) \\
& + \quad \hat{g}^C(\tilde{B}(\phi Y)^V, \tilde{B}Z^C)(\hat{\eta}(X))^C\hat{\xi}^V \\
(5.5) \qquad & + \quad \hat{g}^C(\tilde{B}(\phi Y)^V, \tilde{B}Z^C)(\hat{\eta}(X))^V\hat{\xi}^C\} \\
R^C(X^C, Y^C)Z^C & = \quad \nabla^C_{\tilde{B}X^C}\nabla^C_{\tilde{B}Y^C}\tilde{B}Z^C - \nabla^C_{\tilde{B}Y^C}\nabla^C_{\tilde{B}X^C}\tilde{B}Z^C \\
(5.6) \qquad & - \quad \nabla_{[\tilde{B}X^C, \tilde{B}Y^C]}\tilde{B}Z^C
\end{aligned}
$$

From (5.2) we have, the following theorem

Theorem 5.1. *Let M be an $n-$ dimensional $LP-$Sasakian manifold with the generalized symmetric metric connection and TM its tangent bundle. Then we have the following equations:*

$$
\begin{aligned}
\overline{R}^C(\tilde{B}X^C, \tilde{B}Y^C)\hat{\xi}^C & = \quad (1-\beta+\beta^2)\{\hat{\eta}^C(\tilde{B}Y^C)(\tilde{B}X^V) + \hat{\eta}^V(\tilde{B}Y^C)(\tilde{B}X^C) \\
& - \quad \hat{\eta}^C(\tilde{B}X^C)(\tilde{B}Y^V) - \hat{\eta}^V(\tilde{B}X^C)(\tilde{B}Y^C)\} \\
& + \quad \alpha(1-\beta)\{\hat{\eta}^C(\tilde{B}X^C)(\tilde{B}(\phi Y)^V) + \hat{\eta}^V(\tilde{B}X^C)(\tilde{B}(\phi Y)^C) \\
(5.7) \qquad & - \quad \hat{\eta}^C(\tilde{B}Y^C)(\tilde{B}(\phi X)^V) - \hat{\eta}^V(\tilde{B}Y^C)(\tilde{B}(\phi X)^C)) \\
\overline{R}^C(\hat{\xi}^C, \tilde{B}Y^C, \tilde{B}Z^C) & = \quad -\alpha\{\hat{g}^C(\tilde{B}(\phi Y)^C), \tilde{B}Z^C)\hat{\xi}^V + \hat{g}^C(\tilde{B}(\phi Y)^V), \tilde{B}Z^C)\hat{\xi}^C\} \\
& + \quad (1-\beta)\{\hat{g}^C(\tilde{B}Y^C, \tilde{B}Z^C)\hat{\xi}^V + \hat{g}^C(\tilde{B}Y^V), \tilde{B}Z^C)\hat{\xi}^C\} \\
& - \quad \beta^2\{\hat{\eta}^V(\tilde{B}X^C)\hat{\eta}^C(\tilde{B}Y^C)\hat{\xi}^C + \hat{\eta}^C(\tilde{B}X^C)\hat{\eta}^V(\tilde{B}Y^C)\hat{\xi}^C \\
& - \quad \hat{\eta}^C(\tilde{B}X^C)\hat{\eta}^C(\tilde{B}Y^C)\hat{\xi}^V\} + (1-\beta+\beta^2)\{\hat{\eta}^V(\tilde{B}Z^C)(\tilde{B}Y^C) \\
& + \quad \hat{\eta}^C(\tilde{B}Z^C)(\tilde{B}Y^V)\} + \alpha(1-\beta)\{\hat{\eta}^V(\tilde{B}Z^C)\tilde{B}(\phi Y^C) \\
(5.8) \qquad & + \quad \hat{\eta}^C(\tilde{B}Z^C)\tilde{B}(\phi Y^V)\} \\
\overline{R}^C(\hat{\xi}^C, \tilde{B}Y^C)\hat{\xi}^C & = \quad (1-\beta+\beta^2)\{\hat{\eta}^V(\tilde{B}Y^C)\hat{\xi}^C \\
(5.9) \qquad & + \quad \hat{\eta}^C(\tilde{B}Y^C)\hat{\xi}^V\}\alpha(\beta-1)(\phi Y)^C.
\end{aligned}
$$

where $\overline{R}^C$ be the curvature tensor of $\overline{\nabla}^C$ on $T(S)$ in TM.

The Ricci tensor $\overline{S}$ of an LP-Sasakian manifold M with respect to the generalized symmetric metric connection $\overline{\nabla}$ is given by

$$\overline{S}(Y, Z) = \sum_{i=1}^{n} \varepsilon_i g(\overline{R}(e_i, Y)Z, e_i).$$

where $\{e_1, e_2, ..., e_n\}$ is an orthonormal frame and $\varepsilon_i = g(e_i, e_i)$.

Let $\overline{S}^C$ be the complete lift of $\overline{S}^C$ in TM with respect to the generalized symmetric metric connection $\overline{\nabla}^C$ is given by

$$\overline{S}^C(\tilde{B}Y^C, \tilde{B}Z^C) = \sum_{i=1}^{n} \varepsilon_i \hat{g}^C(\overline{R}^C(e_i^C, \tilde{B}Y^C)\tilde{B}Z^C, e_i^C). \tag{5.10}$$

In the view of (5.2), we get

$$\begin{aligned}
\overline{S}^C(\tilde{B}Y^C, \tilde{B}Z^C) &= \sum_{i=1}^{n} \varepsilon_i \hat{g}^C(R^C(e_i^C, \tilde{B}Y^C)\tilde{B}Z^C, e_i^C) + K_1^C(\tilde{B}Y^C, \tilde{B}Z^C)\hat{g}^C(e_i^C, e_i^C) \\
&- K_1^C(e_i^C, \tilde{B}Z^C)\hat{g}^C(\tilde{B}Y^C, e_i^C) + K_2^C(\tilde{B}Y^C, \tilde{B}Z^C)\hat{g}^C(\tilde{B}\phi^C e_i^C, e_i^C) \\
&- K_2^C(e_i^C, \tilde{B}Z^C)\hat{g}^C(\tilde{B}(\phi Y)^C, e_i^C) + K_3^C(e_i^C, \tilde{B}Y^C)\hat{g}^C(\tilde{B}Z^C, e_i^C) \\
&- K_3^C(\tilde{B}Y^C, e_i^C,)\hat{g}^C(\tilde{B}Z^C, e_i^C).
\end{aligned} \tag{5.11}$$

since the Ricci tensor S^C of an LP-Sasakian manifold TM with respect to ∇^C is given by

$$S^C(\tilde{B}Y^C, \tilde{B}Z^C) = \sum_{i=1}^{n} \varepsilon_i \hat{g}^C(R^C(e_i^C, \tilde{B}Y^C)\tilde{B}Z^C, e_i^C)$$

Then using (5.4),(5.5),(5.6) and (5.2), we get

$$\begin{aligned}
\overline{S}^C(\tilde{B}Y^C, \tilde{B}Z^C) &= S^C(\tilde{B}Y^C, \tilde{B}Z^C) + \{-\alpha\beta + (n-2)(\alpha\beta - \alpha) \\
&+ (\beta^2 - 2\beta)Trace\Phi^C\}\Phi^C(\tilde{B}(\phi Y)^C, \tilde{B}Z^C) \\
&- \{2\alpha^2 + \beta + \beta^2 + n\alpha^2 + (\alpha\beta - \alpha)Trace\Phi^C\}\hat{g}^C(\tilde{B}Y^C, \tilde{B}Z^C) \\
&+ \{-2\alpha^2 + n(\alpha^2 + \beta - \beta^2)\}\{\hat{\eta}^V(\tilde{B}Y^C)\hat{\eta}^C(\tilde{B}Z^C) \\
&+ \hat{\eta}^C(\tilde{B}Y^C)\hat{\eta}^V(\tilde{B}Z^C)\}
\end{aligned} \tag{5.12}$$

Φ^C and Ricci tensor S^C of Levi-Civita connection are symmetric. Then from (5.12), we have the following corollary

Corollary 5.1. *Let M be an n-dimensional LP-Sasakain manifold and TM its tangent bundle then the Ricci tensor $\overline{S}^C$ of the generalized symmetric metric connection $\overline{\nabla}^C$ is symmetric.*

Theorem 5.2. *Let M be an n-dimensional LP-Sasakain manifold and TM its tangent bundle then we have*

$$\begin{aligned}
\overline{S}^C(\tilde{B}Y^C, \hat{\xi}^C) &= (n-1)(1 - \beta + \beta^2)\hat{\eta}^C(\tilde{B}Y^C) \\
&+ (\alpha\beta - 1)\{\hat{g}^C(\tilde{B}\phi^C e_i^C, e_i^C)\hat{\eta}^V(\tilde{B}Y^C) \\
&+ \hat{g}^C(\tilde{B}\phi^V e_i^C, e_i^C)\hat{\eta}^C(\tilde{B}Y^C)\}
\end{aligned} \tag{5.13}$$

$$
\begin{aligned}
\overline{S}^C(\tilde{B}(\phi Y)^C, \tilde{B}(\phi Z^C)) &= \overline{S}^C(\tilde{B}Y^C, \tilde{B}Z^C) \\
&+ (n-1)(1-\beta+\beta^2)\{\hat{\eta}^V(\tilde{B}Y^C)\hat{\eta}^C(\tilde{B}Z^C) + \hat{\eta}^C(\tilde{B}Y^C)\hat{\eta}^V(\tilde{B}Z^C)\} \\
&+ \alpha(\beta-1)\{\hat{g}^C(\tilde{B}\phi^C e_i^C, e_i^C)\hat{\eta}^V(\tilde{B}Y^C)\hat{\eta}^C(\tilde{B}Z^C) \\
&+ \hat{g}^C(\tilde{B}\phi^C e_i^C, e_i^C)\hat{\eta}^C(\tilde{B}Y^C)\hat{\eta}^V(\tilde{B}Z^C) \\
(5.14) \qquad &+ \hat{g}^C(\tilde{B}\phi^V e_i^C, e_i^C)\hat{\eta}^C(\tilde{B}Y^C)\hat{\eta}^C(\tilde{B}Z^C)\}
\end{aligned}
$$

Proof: Using (3.2), (3.9) and (3.17) in (5.12), we get (5.13).
By using (3.4), (3.9) and (3.18) in (5.12), we get (5.14).

Theorem 5.3. *Let M be an n-dimensional LP-Sasakain manifold and TM its tangent bundle. If Ricci tensor $\overline{S}^C$ is a semi-symmetric with respect to the generalized symmetric metric connection on TM, then we have*

$$
\begin{aligned}
& \{(1-\beta+\beta^2)^2 - (\alpha\beta-\alpha)^2\}\overline{S}^C(\tilde{B}Y^C, \tilde{B}U^C) \\
= \; & \{(n-1)(1-\beta+\beta^2) + \alpha(\beta-1)\hat{g}^C(\tilde{B}\phi^C e_i^C, e_i^C)\} \\
& \{\alpha\beta\hat{g}^C(\tilde{B}(\phi Y)^C, \tilde{B}U^C) \\
- \; & (1-\beta)(1-\beta+\beta^2-\alpha^2)\hat{g}^C(\tilde{B}Y^C, \tilde{B}U^C) \\
+ \; & (-\beta^4+\beta^3-\beta^2+\alpha^2\beta^2-\beta\alpha^2) \\
& (\hat{\eta}^V(\tilde{B}Y^C)\hat{\eta}^C(\tilde{B}U^C) \\
(5.15) \qquad + \; & \hat{\eta}^C(\tilde{B}Y^C)\hat{\eta}^V(\tilde{B}U^C))\}
\end{aligned}
$$

Proof: Let $\overline{R}^C(\tilde{B}X^C, \tilde{B}Y^C)\overline{S}^C = 0$ be on TM for any $X^C, Y^C, Z^C, U^C \in TM$, then we have

$$
\overline{S}^C(\overline{R}^C(\tilde{B}X^C, \tilde{B}Y^C)\tilde{B}Z^C, \tilde{B}U^C) \quad + \quad \overline{S}^C(\tilde{B}Z^C, \overline{R}^C(\tilde{B}X^C, \tilde{B}Y^C), \tilde{B}U^C) = 0
$$
(5.16)

if we choose $Z^C = \hat{\xi}$ and $X^C = \hat{\xi}$ in (5.16)

$$
\overline{S}^C(\overline{R}^C(\tilde{B}\hat{\xi}^C, \tilde{B}Y^C)\tilde{B}\hat{\xi}^C, \tilde{B}U^C) + \overline{S}^C(\tilde{B}\hat{\xi}^C, \overline{R}^C(\tilde{B}\hat{\xi}^C, \tilde{B}Y^C), \tilde{B}U^C) \quad = \quad 0
$$
(5.17)

Using Theorem 5.1 and Theorem 5.2 in (5.17), we obtain

$$
\begin{aligned}
(1-\beta+\beta^2)\overline{S}^C(\tilde{B}Y^C, \tilde{B}U^C) \quad + \quad & \alpha(\beta-1)\overline{S}^C(\tilde{B}(\Phi Y)^C, \tilde{B}U^C) \\
= \; & \{(n-1)(1-\beta+\beta^2) + \alpha(\beta-1)\hat{g}^C(\tilde{B}\phi^C e_i^C, e_i^C)\} \\
& \{-\alpha\hat{g}^C(\tilde{B}(\phi Y)^C, \tilde{B}U^C) + (1-\beta)\hat{g}^C(\tilde{B}Y^C, \tilde{B}U^C) \\
- \; & \beta^2(\hat{\eta}^V(\tilde{B}Y^C)\hat{\eta}^C(\tilde{B}U^C) \\
(5.18) \qquad + \; & \hat{\eta}^C(\tilde{B}Y^C)\hat{\eta}^V(\tilde{B}U^C))\}
\end{aligned}
$$

If $Y = \phi Y$ in (5.18) and using (5.13) we obtain

$$
(1-\beta+\beta^2)\overline{S}^C(\tilde{B}(\phi Y)^C, \tilde{B}U^C) \quad + \quad \alpha(\beta-1)\overline{S}^C(\tilde{B}Y^C, \tilde{B}U^C)
$$

$$
\begin{aligned}
&= \{(n-1)(1-\beta+\beta^2)+\alpha(\beta-1)\hat{g}^C(\tilde{B}\phi^C e_i^C, e_i^C)\} \\
&\quad \{(1-\beta)\hat{g}^C(\tilde{B}(\phi Y)^C, \tilde{B}U^C) - \alpha\hat{g}^C(\tilde{B}Y^C, \tilde{B}U^C) \\
&- \alpha\beta\{(\hat{\eta}^V(\tilde{B}Y^C)\hat{\eta}^C(\tilde{B}U^C) \\
&+ \hat{\eta}^C(\tilde{B}Y^C)\hat{\eta}^V(\tilde{B}U^C))\}
\end{aligned} \tag{5.19}
$$

from (5.18) and (5.19), we obtain (5.15). Hence the proof is completed.

Corollary 5.2. *Let M be an n-dimensional LP-Sasakain manifold and TM its tangent bundle. If $\overline{S}^C$ is Ricci tensor with respect to the generalized symmetric metric connection on TM and if $\alpha = 0$ and $\beta = 0$ in (5.15), we get the following equations:*

$$
\begin{aligned}
\overline{S}^C(\tilde{B}Y^C, \tilde{B}U^C) &= (n-1)(1-\beta)\hat{g}^C(\tilde{B}Y^C, \tilde{B}U^C) \\
&- (n-1)\beta^2\{(\hat{\eta}^V(\tilde{B}Y^C)\hat{\eta}^C(\tilde{B}U^C)+\hat{\eta}^C(\tilde{B}Y^C)\hat{\eta}^V(\tilde{B}U^C)\} \\
\overline{S}^C(\tilde{B}Y^C, \tilde{B}U^C) &= (n-1-\alpha\hat{g}^C(e_i^C, e_i^C))\hat{g}^C(\tilde{B}Y^C, \tilde{B}U^C)\}.
\end{aligned} \tag{5.20}
$$

6. Projective curvature tensor of an LP-Sasakian manifold with respect to the generalized symmetric metric connection on the tangent bundle

Let M be an $n-$ dimensional $LP-$Sasakian manifold. The projective curvature tensor $\overline{P}$ of type $(1,3)$ of M with respect to the generalized metric connection $\overline{\nabla}$ is defined by

$$
\overline{P}(X,Y)Z = \overline{R}(X,Y)Z - \frac{1}{n-1}\{\overline{S}(Y,Z)X - \overline{S}(X,Z)Y\}. \tag{6.1}
$$

Let $\overline{P}^C$ be complete lift of $\overline{P}$ in TM with respect to the generalized symmetric metric connection $\overline{\nabla}^C$ is given by

$$
\begin{aligned}
\overline{P}^C(\tilde{B}X^C, \tilde{B}Y^)\tilde{B}Z^C &= \overline{R}^C(\tilde{B}X^C, \tilde{B}Y^C)\tilde{B}Z^C - \frac{1}{n-1}\{\overline{S}^C(\tilde{B}Y^C, \tilde{B}Z^C)\tilde{B}X^C \\
&- \overline{S}^C(\tilde{B}X^C, \tilde{B}Z^C)\tilde{B}Y^C\}.
\end{aligned} \tag{6.2}
$$

Definition Let M be an $n-$ dimensional $LP-$Sasakian manifold and TM its tangent bundle. Then TM is said to be $\hat{\xi}^C-$ projectively flat with respect to the generalized metric connection $\overline{\nabla}^C$ if $\overline{P}^C(\tilde{B}X^C, \tilde{B}Y^C)\hat{\xi}^C = 0$ on TM.

Using (6.2) we have

$$
\begin{aligned}
\overline{P}^C(\tilde{B}X^C, \tilde{B}Y^)\tilde{B}\hat{\xi}^C &= \overline{R}^C(\tilde{B}X^C, \tilde{B}Y^C)\tilde{B}\hat{\xi}^C - \frac{1}{n-1}\{\overline{S}^C(\tilde{B}Y^C, \tilde{B}\hat{\xi}^C)\tilde{B}X^C \\
&- \overline{S}^C(\tilde{B}X^C, \tilde{B}\hat{\xi}^C)\tilde{B}Y^C\}.
\end{aligned} \tag{6.3}
$$

From the last equation, using (5.13) we obtain

$$
\begin{aligned}
\overline{P}^C(\tilde{B}X^C,\tilde{B}Y^C)\tilde{B}\hat{\xi}^C &= \alpha(1-\beta)\{\frac{1}{n-1}(\hat{\eta}^C(\tilde{B}Y^C)(\tilde{B}X^V)+\hat{\eta}^V(\tilde{B}Y^C)(\tilde{B}X^C)\\
&- \hat{\eta}^C(\tilde{B}X^C)(\tilde{B}Y^V)+\hat{\eta}^V(\tilde{B}X^C)(\tilde{B}Y^C)+\hat{\eta}^C(\tilde{B}X^C)(\tilde{B}(\phi Y)^V)\\
&+ \hat{\eta}^V(\tilde{B}X^C)(\tilde{B}(\phi Y)^C)-\hat{\eta}^C(\tilde{B}Y^C)(\tilde{B}(\phi X)^V)\\
&+ \hat{\eta}^V(\tilde{B}Y^C)(\tilde{B}(\phi X)^C)) \qquad (6.4)
\end{aligned}
$$

Then we have the following theorem

Theorem 6.1. *Let M be an $n-$ dimensional $LP-$Sasakian manifold with the generalized metric connection and TM its tangent bundle.*
(i) *TM is not $\hat{\xi}^C-$ projectively flat with respect to the generalized metric connection.*
(ii) *TM is $\hat{\xi}^C-$ projectively flat with respect to the generalized metric connection of type $(0,\beta)$.*
(iii) *TM is $\hat{\xi}^C-$ projectively flat with respect to the generalized metric connection of type $(\alpha,1)$.*

Acknowledgement

We would like to thank the Reviewers and the Editor for reviewing the paper carefully and their valuable comments that helped to improve the overall quality of the paper.

REFERENCES

1. N. S. Agashe and M. R. Chafle: *A semi symetric non-metric connection in a Riemannian manifold*. Indian J. Pure Appl. Math.**23** (1992), 399–409.
2. O. Bahadir: *Lorentzian para-Sasakian manifold with quartersymmetric non-metric connection*. Journal of Dynamical Systems and Geometric Theories **14**(1) (2016), 17–33.
3. L. S. Das and M. N. I. Khan: *Almost r-contact structure in the tangent bundle*. Differential Geometry-Dynamical System **7** (2005), 34–41.
4. L. S. Das and M. N. I. Khan: *Symmetric and Ricci LP-Sasakian manifold*. Mathematical Sciences Research Journal **17**(10) (2013), 263–268.
5. U. C. De and D. Kamilya: *Hypersurfaces of Rieamnnian manifold with semi-symmetric non-metric connection*. J. Indian Inst. Sci. **75** (1995), 707–710.
6. A. Friedmann and J. A. Schouten: *ÄUber diegeometrie der halbsymmetrischen Äubertragung*. Math. Zeitschr. **21** (1924), 211–223.
7. S. Golab: *On semi-symmetric and quarter-symmetric linear connections*. Tensor N. S. **29** (1975), 249–254.
8. H. A. Hayden: *Subspaces of a space with torsion*. Proc. London Math. Soc. **34** (1932), 27–50.

9. S. K. Hui, S. Uddin, C. Ozel and A. A. Mustafa: *Warped product submanifolds of LP-Sasakian manifolds*: Hindawi Publishing Corporation, Discrete Dynamics in Nature and Society, **2012**, Article ID 868549, 11 pages.
10. M. N. I. Khan and J. B. Jun: *Lorentzian almost r-para-contact structure in tangent bundle.* Journal of the Chungcheong Mathematical Society **27**(1) (2014), 29–34.
11. M. N. I. Khan: *Quarter-symmetric semi-metric connection on a Sasakian manifold.* Tensor N.S., **68**(2) (2007), 154–157.
12. M. N. I. Khan: *Novel theorems for the frame bundle endowed with metallic structures on an almost contact metric manifold.* Chaos, Solitons & Fractals. **146**, May 2021, 110872.
13. M. N. I. Khan: *Tangent bundle endowed with quarter-symmetric non-metric connection on an almost Hermitian manifold.* Facta Universitis (NIS) Ser. Math. Inform. **35**(1) (2020), 167–178.
14. M. N. I. Khan: *Lifts of hypersurfaces with quarter-symmetric semi-metric connection to tangent bundles.* Afrika Matematika **25** (2014), 475–482.
15. M. N. I. Khan: *Lifts of semi-symmetric non-metric connection on a Kahler manifold.* Afrika Matematika **27**(3) (2016), 345–352.
16. Y. Liang: *On semi-symmetric recurrent-metric connection.* Tensor **55** (1994), 107–112.
17. K. Matsumoto: *On Lorentzian Paracontact manifolds.* Bull. Yamagata Univ. Natur. Sci. **12**(2) (1989), 151–156.
18. I. Mihai and R. Rosca: *On Lorentzian P-Sasakian manifolds, Clssical Analysis.* World Scientific Publ., Signapore, (1992), 155–169.
19. K. Matsumoto and I. Mihai: *On a certain transformation in a Lorentzian para Sasakian manifold.* Tensor N. S. **47** (1988), 189–197.
20. I. Mihai, A. A. Shaikh and U. C. De: *On Lorentzian para-Sasakian manifolds.* Rendiconti del Seminario Mat. di Messina, Serie II 1999.
21. A. K. Mondal and U. C. De: *Some properties of a quarter-symmetric metric connection on a sasakian manifold.* Bulletin of Mathematical analysis and applications, **3**(1) (2009), 99–108.
22. A. K. Mondal and U. C. De: *Quarter-symmetric Nonmetric Connection on P-sasakian manifolds.* ISRN Geometry, **2012**, Article ID 659430, 14 pages.
23. S. Y. Perktas, E. Kilic and S. Keles: *On a semi-symmetric non-metric connection in an LP-Sasakian manifold.* Int. Electron. J. Geom. **3**(2), (2010), 15–25.
24. S. K. Srivastava and R. P. Kushwaha: *Lorentzian para Sasakian manifolds admitting special semi-symmetric recurrent metric connection.* Global Journal of Science Frontier Research Mathematics and Decision Sciences **13**(7) Version 1.0 (2013), 1–7.
25. S. Sharfuddin and S. I. Husain:*Semi-symmetric metric connexions in almost contact manifolds.* Tensor N. S. **30** (1976), 133–139.
26. B. G. Schmidt: *Conditions on a connection to be a metric connection.* Commun. Math. Phys. **29** (1973), 55–59.
27. M. Tani: *Prolongations of hypersurfaces of tangent bundles.* Kodai Math. Semp. Rep. **21** (1969), 85–96.