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ON THREE-DIMENSIONAL HOMOGENEOUS FINSLER MANIFOLDS

Akbar Tayebi[1] and Behzad Najafi[2]

[1]**Department of Mathematics, Faculty of Science, University of Qom, Qom, Iran**
[2]**Department of Mathematics and Computer Sciences, Amirkabir University, Tehran, Iran**

Abstract. In this paper, we study a long existing open problem on Landsberg metrics in Finsler geometry. For this aim, we study the Landsberg curvature of three-dimensional homogeneous Finsler manifolds. First, we express the second Matsumoto torsion of three-dimensional Finsler manifolds, explicitly. Then, we show that the mean Landsberg curvature of three-dimensional homogeneous Finsler manifolds satisfy an ODE. Finally, we prove that every homogeneous 3-dimensional L-reducible Finsler manifold has constant relatively isotropic mean Landsberg curvature if and only if it is a Landsberg metric or a Randers metric of Berwald-type.
Keywords: Homogeneous metric, Landsberg metric, Randers metric.

1. Introduction

It is a long-existing open problem in Finsler geometry to find the unicorns, that is, the Landsberg metrics which are not Berwald metrics. For the sake of simpler prose, Bao refer to the Landsberg metrics that are not of Berwald type as unicorns, by analogy with those mythical single-horned horse-like creatures for which no confirmed sighting is available [2]. In [20], Shen showed that there is not any unicorn in the class of regular (α, β)-metrics. Also, he found a family of

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Corresponding Author: Akbar Tayebi, Department of Mathematics, Faculty of Science, University of Qom, Qom, Iran | E-mail: akbar.tayebi@gmail.com
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unicorns in the class of non-regular (α,β)-metrics. Let $F=\alpha\phi(s)$, $s=\beta/\alpha$, be an (α,β)-metric defined by following

$$\phi(s)=\exp\left[\int_0^s \frac{kt+q\sqrt{b^2-t^2}}{1+kt^2+qt\sqrt{b^2-t^2}}dt\right],$$

where $q>0$ and k are real constants. Suppose that the 1-form β satisfies

$$r_{ij}=c(b^2a_{ij}-b_ib_j),\qquad s_{ij}=0,$$

where $c=c(x)$ is a scalar function on M. If $c\neq 0$, then F is a unicorn. If $c=0$, then F reduces to a Berwald metric. If $k=0$ and $c\neq 0$, then one obtains the family of unicorns constructed by Asanov in [1].

In 1975, Takano developed the theory of fields in Finsler spaces, where the fields have internal freedom. In particular, he studied the spinor fields details and found it necessary to introduce the gauge fields into the spinor field equations. Takano studied the field equations in a Finsler manifold and proposed certain interesting geometrical problems [22]. He requested mathematicians to find some proper forms of Landsberg curvature from the standpoint of physics. In 1978, Matsumoto introduced the notion of L-reducible Finsler metrics as an answer to Takano, which was a generalization of C-reducible Finsler metrics [10]. A Finsler metric F on an n-dimensional manifold M is L-reducible if its Landsberg curvature is given by

$$\mathbf{L}_y(u,v,w)=\frac{1}{n+1}\Big\{\mathbf{J}_y(u)\mathbf{h}_y(v,w)+\mathbf{J}_y(v)\mathbf{h}_y(u,w)+\mathbf{J}_y(w)\mathbf{h}_y(u,v)\Big\}, \tag{1.1}$$

where $\mathbf{J}:=trace(\mathbf{L})$ denotes the mean Landsberg curvature of F. As we mentioned, Matsumoto considered (1.1) when he studied the hv-curvature P_{ijkl} of the Cartan connection. Then, he called such Finsler metrics by the notion of *P-reducible* since it comes from the P-curvature and we call them here "L-reducible metrics" for the relation with Landsberg curvature. If $\mathbf{L}=0$, then F is called Landsberg metric [24]. We have concrete examples of non-Landsberg L-reducible Finsler metrics. For example, it is evident that every C-reducible metric is L-reducible. However, the converse of this fact may not be accurate in general. For a Finsler metric of dimension $n\geq 3$, Matsumoto found some conditions under which the Finsler metric will be L-reducible. Since the study of Landsberg curvature has become an urgent necessity for the Finsler geometry as well as for theoretical physics, Matsumoto-Shimada studied some of Riemannian and non-Riemannian curvature properties of L-reducible metrics in [16]. They proposed the following open problem:

Is there any L-reducible Finsler metric that is not C-reducible?

A Finsler space (M,F) is called homogeneous Finsler space if the group of isometries of (M,F) acts transitively on M. Recently, Deng and Xu conjectured that "A homogeneous Landsberg space must be a Berwald space" [5]. In [23], the authors proved that every homogeneous Landsberg surface is Riemannian or locally

Minkowskian. This result articulates the hunters of unicorns that they do not looking forward to seeing such a creature in the jungle of homogeneous Finsler surfaces (see [7], [19] and [26]). In [6], Hu-Deng studied 3-dimensional homogeneous Finsler manifolds and obtain a complete list of invariant Finsler metrics. They considered invariant Randers metrics and presented the classification of three-dimensional homogeneous Randers spaces under isometrics.

Theorem 1.1. *Let (M, F) be a homogeneous 3-dimensional L-reducible Finsler manifold. Then F has constant relatively isotropic mean Landsberg curvature if and only if it is a Landsberg metric or a Randers metric of of Berwald-type.*

Theorem 1.1 yields a negative answer to the open problem of Matsumoto-Shimada in the class of homogeneous three-dimensional Finsler metrics with constant relatively isotropic mean Landsberg curvature.

2. Preliminary

Let M be an n-dimensional C^∞ manifold, $TM = \bigcup_{x\in M} T_xM$ the tangent bundle and $TM_0 := TM - \{0\}$ the slit tangent bundle. Let (M, F) be a Finsler manifold. The following quadratic form $\mathbf{g}_y : T_xM \times T_xM \to \mathbb{R}$ is called fundamental tensor

$$\mathbf{g}_y(u, v) = \frac{1}{2}\frac{\partial^2}{\partial s \partial t}\Big[F^2(y + su + tv)\Big]_{s=t=0}, \quad u, v \in T_xM.$$

Let $x \in M$ and $F_x := F|_{T_xM}$. To measure the non-Euclidean feature of F_x, one can define $\mathbf{C}_y : T_xM \times T_xM \times T_xM \to \mathbb{R}$ by

$$\mathbf{C}_y(u, v, w) := \frac{1}{2}\frac{d}{dt}\left[\mathbf{g}_{y+tw}(u, v)\right]_{t=0}, \quad u, v, w \in T_xM.$$

The family $\mathbf{C} := \{\mathbf{C}_y\}_{y\in TM_0}$ is called the Cartan torsion. It is well known that $\mathbf{C} = 0$ if and only if F is Riemannian. Also, for $y \in T_xM_0$, one can define $\mathbf{I}_y : T_xM \to \mathbb{R}$ by

$$\mathbf{I}_y(u) = \sum_{i=1}^{n} g^{ij}(y)\mathbf{C}_y(u, \partial_i, \partial_j),$$

where $\{\partial_i\}$ is a basis for T_xM at $x \in M$. The family $\mathbf{I} := \{\mathbf{I}_y\}_{y\in TM_0}$ is called the mean Cartan torsion. By definition, $\mathbf{I}_y(u) := I_i(y)u^i$, where $I_i := g^{jk}C_{ijk}$. Define the norm of $\mathbf{I}$ at a point $x \in M$ by

$$\|\mathbf{I}\|_x := \sup_{0\neq y\in T_xM} \sqrt{g^{ij}(x, y)I_i(x, y)I_j(x, y)}.$$

For a non-zero vector $y \in T_xM_0$, one can define the Matsumoto torsion $\mathbf{M}_y : T_xM \times T_xM \times T_xM \to \mathbb{R}$ by

$$\mathbf{M}_y(u, v, w) = \mathbf{C}_y(u, v, w) - \frac{1}{n+1}\Big\{\mathbf{I}_y(u)\mathbf{h}_y(v, w) + \mathbf{I}_y(v)\mathbf{h}_y(u, w) + \mathbf{I}_y(w)\mathbf{h}_y(u, v)\Big\}, \tag{2.1}$$

where $\mathbf{h}_y(u,v) := \mathbf{g}_y(u,v) - F^{-2}(y)\mathbf{g}_y(y,u)\mathbf{g}_y(y,v)$ is called the angular form in direction y and $\mathbf{g}_y$ is the fundamental tensor of F. Clearly, $\mathbf{M} = 0$ for all two-dimensional Finsler metrics. A Finsler metric F on a manifold M of dimension $n \geq 2$ is said to be C-reducible if $\mathbf{M}_y = 0$. In [15], Matsumoto-Hōjō proved the following.

Lemma 2.1. (Matsumoto-Hōjō Lemma) *A positive-definite Finsler metric on a manifold of dimension $n \geq 3$ is a Randers metric if and only if the Matsumoto torsion vanishes.*

Let $c = c(t)$ be a C^∞ curve and $U(t) = U^i(t)\partial/\partial x^i|_{c(t)}$ be a vector field along c. Define the covariant derivative of $U(t)$ along c by

$$D_{\dot c}U(t) := \left\{ \frac{dU^i}{dt}(t) + U^j(t)\frac{\partial G^i}{\partial y^j}\big(c(t), \dot c(t)\big) \right\} \frac{\partial}{\partial x^i}\Big|_{c(t)}.$$

$U(t)$ is said to be *linearly parallel* if $D_{\dot c}U(t) = 0$.

For a vector $y \in T_xM$, define

$$\begin{aligned} \mathbf{L}_y(u,v,w) : &= \frac{d}{dt}\Big[\mathbf{C}_{\dot\sigma(t)}\big(U(t), V(t), W(t)\big)\Big]|_{t=0}, \\ \mathbf{J}_y(u) : &= \frac{d}{dt}\Big[\mathbf{I}_{\dot\sigma(t)}\Big(U(t)\Big)\Big]|_{t=0}, \end{aligned}$$

where $\sigma(t)$ is the geodesic with $\sigma(0) = x$, $\dot\sigma(0) = y$ and $U(t), V(t), W(t)$ are linearly parallel vector fields along σ with $U(0) = u, V(0) = v, W(0) = w$. We call $\mathbf{L}_y$ the *Landsberg curvature*. The Landsberg curvature measures the rate of change of the Cartan torsion along geodesics. Fix a local frame $\{\mathbf{b}_i\}$ for TM. Let $L_{ijk}(x,y) := \mathbf{L}_y(\mathbf{b}_i, \mathbf{b}_j, \mathbf{b}_k)$ and $J_i(x,y) := \mathbf{J}_y(\mathbf{b}_i)$. We have that $J_i(x,y) = g^{jk}(x,y)L_{ijk}(x,y)$. Thus we call $\mathbf{J}_y$ the *mean Landsberg curvature*. A Finsler metric F on a manifold M is called of relatively isotropic mean Landsberg curvature if

$$\mathbf{J} + cF\mathbf{I} = 0,$$

where $c = c(x)$ is a scalar function on M. If $c =$ *constant*, then F has constant relatively isotropic mean Landsberg curvature.

For $y \in T_xM_0$, define the second Matsumoto torsion $\widetilde{\mathbf{M}}_y : T_xM \times T_xM \times T_xM \to \mathbb{R}$ by

$$\widetilde{\mathbf{M}}_y(u,v,w) := \mathbf{L}_y(u,v,w) - \frac{1}{n+1}\Big\{\mathbf{J}_y(u)\mathbf{h}_y(v,w) + \mathbf{J}_y(v)\mathbf{h}_y(u,w) + \mathbf{J}_y(w)\mathbf{h}_y(u,v)\Big\}. \tag{2.2}$$

In local coordinates,

$$\widetilde{M}_{ijk} := L_{ijk} - \frac{1}{n+1}\Big\{J_i h_{jk} + J_j h_{ik} + J_k h_{ij}\Big\}.$$

A Finsler metric F is said to be L-reducible if $\widetilde{\mathbf{M}}_y = 0$.

Throughout this paper, we use the Berwald connection on Finsler manifolds. The pulled-back bundle π^*TM admits a unique linear connection, called the Berwald connection. Its connection forms are characterized by the structure equations as follows

- Torsion freeness:

$$d\omega^i = \omega^j \wedge \omega^i{}_j. \tag{2.3}$$

- Almost metric compatibility:

$$dg_{ij} - g_{kj}\omega^k{}_i - g_{ik}\omega^k{}_j = -2L_{ijk}\omega^k + 2C_{ijk}\omega^{n+k}, \tag{2.4}$$

  where

$$\omega^i := dx^i, \qquad \omega^{n+k} := dy^k + y^j \omega^k{}_j.$$

The horizontal and vertical covariant derivations with respect to the Berwald connection respectively are denoted by "|" and ", ". Thus

$$g_{ij|k} = -2L_{ijk}, \qquad g_{ij,k} = 2C_{ijk}.$$

Also, the structure equation of curvature tensors of Berwald connection is given by following

$$\Omega^i{}_j := d\omega^i{}_j - \omega^k{}_j \wedge \omega^i{}_k = \frac{1}{2}R^i{}_{jkl}\omega^k \wedge \omega^l - B^i{}_{jkl}\omega^k \wedge \omega^{n+l}. \tag{2.5}$$

For more details, one can see [21].

3. Proof of Theorem 1.1

In this section, we are going to prove Theorem 1.1. For this aim, we remark that in [9], Latifi-Razavi proved that every homogeneous Finsler manifold is forward geodesically complete. In [24], Tayebi-Najafi improved their result and proved the following.

Lemma 3.1. ([24]) *Every homogeneous Finsler manifold is complete.*

By definition, every two points of a homogeneous Finsler manifold (M, F) map to each other under an isometry. This causes the norm of an invariant tensor under the isometries of a homogeneous Finsler manifold is a constant function on M, and consequently, it has a bounded norm. Then, we conclude the following.

Lemma 3.2. ([24]) *Let (M, F) be a homogeneous Finsler manifold. Then, every invariant tensor under the isometries of F has a bounded norm with respect to F.*

In order to prove Theorem 1.1, first, we consider the Landsberg curvature of homogeneous 3-dimensional Landsberg metrics.

Lemma 3.3. *The second Matsumoto torsion of a 3-dimensional Finsler metric is given by following*

$$\begin{aligned}\widetilde{\mathbf{M}}_y(u,v,w) = &-\frac{1}{2}\Big(\mathbf{g}_y(\mathbf{b},D_{\dot\sigma}\mathbf{I})+\mathbf{g}_y(D_{\dot\sigma}\mathbf{b},\mathbf{I})\Big)\Big\{\mathbf{I}_y(u)\mathbf{h}_y(v,w)+\mathbf{I}_y(v)\mathbf{h}_y(u,w)\\ &+\mathbf{I}_y(w)\mathbf{h}_y(u,v)\Big\}-\frac{1}{2}\mathbf{g}_y(\mathbf{b},\mathbf{I})\Big\{\mathbf{J}_y(u)\mathbf{h}_y(v,w)+\mathbf{J}_y(v)\mathbf{h}_y(u,w)\\ &+\mathbf{J}_y(w)\mathbf{h}_y(u,v)\Big\}-\frac{1}{4}||\boldsymbol{I}||^2\Big\{\mathbf{h}_y(v,w)D_{\dot\sigma}\mathbf{b}_y(u)+\mathbf{h}_y(u,w)D_{\dot\sigma}\mathbf{b}_y(v)\\ &+\mathbf{h}_y(u,v)D_{\dot\sigma}\mathbf{b}_y(w)\Big\}-\frac{1}{2}\mathbf{g}_y(\mathbf{I},\mathbf{J})\Big\{\mathbf{b}_y(u)\mathbf{h}_y(v,w)+\mathbf{b}_y(v)\mathbf{h}_y(u,w)\\ &+\mathbf{b}_y(w)\mathbf{h}_y(u,v)\Big\}+\mathbf{b}_y(u)\Big(\mathbf{J}_y(v)\mathbf{I}_y(w)+\mathbf{J}_y(w)\mathbf{I}_y(v)\Big)+\mathbf{b}_y(v)\Big(\mathbf{J}_y(u)\mathbf{I}_y(u)\\ &+\mathbf{J}_y(w)\mathbf{I}_y(u)\Big)+\mathbf{b}_y(w)\Big(\mathbf{J}_y(u)\mathbf{I}_y(v)+\mathbf{J}_y(v)\mathbf{I}_y(u)\Big)+\mathbf{I}_y(v)\mathbf{I}_y(w)D_{\dot\sigma}\mathbf{b}_y(u)\\ &+\mathbf{I}_y(w)\mathbf{I}_y(u)D_{\dot\sigma}\mathbf{b}_y(v)+\mathbf{I}_y(u)\mathbf{I}_y(v)D_{\dot\sigma}\mathbf{b}_y(w).\end{aligned}\tag{3.1}$$

Proof. Let $y\in T_xM$ be an arbitrary non-zero vector and let $\sigma=\sigma(t)$ be the geodesic with $\sigma(0)=x$ and $\dot\sigma(0)=y$. Since the Finsler metric is complete, one may assume that σ is defined on $(-\infty,\infty)$. $\mathbf{I}$ and $\mathbf{J}$ restricted to σ are vector fields along σ,

$$\mathbf{I}(t):=I^i\Big(\sigma(t),\dot\sigma(t)\Big)\frac{\partial}{\partial x^i}\Big|_{\sigma(t)},\qquad \mathbf{J}(t):=J^i\Big(\sigma(t),\dot\sigma(t)\Big)\frac{\partial}{\partial x^i}\Big|_{\sigma(t)}.$$

Thus

$$D_{\dot\sigma}\mathbf{I}(t)=I^i{}_{|m}\Big(\sigma(t),\dot\sigma(t)\Big)\dot\sigma^m(t)\frac{\partial}{\partial x^i}\Big|_{\sigma(t)}=\mathbf{J}(t).$$

In [18], Moór introduced a special orthonormal frame field $(\ell,\mathbf{m},\mathbf{n})$ in the three dimensional Finsler space. The first vector of the frame is the normalized supporting element, the second is the normalized mean Cartan torsion vector, and third is the unit vector orthogonal to them. Let (M,F) be a 3-dimensional Finsler manifold. Suppose that ℓ_i is the unit vector along the element of support, $\mathbf{m}$ is the unit vector along mean Cartan torsion $\mathbf{I}$, i.e., $\mathbf{m}:=\mathbf{I}/||\mathbf{I}||$ and $\mathbf{n}$ is a unit vector orthogonal to the vectors ℓ and $\mathbf{m}$. Then the triple $(\ell,\mathbf{m},\mathbf{n})$ is called the Moór frame. In 3-dimensional Finsler manifolds, we have

$$\mathbf{g}_y(u,v)=\ell_y(u)\ell_y(v)+\mathbf{m}_y(u)\mathbf{m}_y(v)+\mathbf{n}_y(u)\mathbf{n}_y(v).$$

Then the Cartan torsion of F is written as follows

$$\begin{aligned}F\mathbf{C}_y(u,v,w)=&\,\mathcal{H}\mathbf{m}_y(u)\mathbf{m}_y(v)\mathbf{m}_y(w)-\mathcal{J}\Big\{\mathbf{m}_y(u)\mathbf{m}_y(v)\mathbf{n}_y(w)+\mathbf{m}_y(v)\mathbf{m}_y(w)\mathbf{n}_y(u)\\ &+\mathbf{m}_y(w)\mathbf{m}_y(u)\mathbf{n}_y(v)+\mathbf{n}_y(u)\mathbf{n}_y(v)\mathbf{n}_y(w)\Big\}+\mathcal{I}\Big\{\mathbf{m}_y(u)\mathbf{n}_y(v)\mathbf{n}_y(w)\\ &+\mathbf{m}_y(v)\mathbf{n}_y(w)\mathbf{n}_y(u)+\mathbf{m}_y(w)\mathbf{n}_y(u)\mathbf{n}_y(v)\Big\},\end{aligned}\tag{3.2}$$

where $\mathcal{H} = \mathcal{H}(x, y)$, $\mathcal{I} = \mathcal{I}(x, y)$ and $\mathcal{J} = \mathcal{J}(x, y)$ are called the main scalars such that $\mathcal{H} + \mathcal{I} = F||\mathbf{I}||$. Since the angular metric is given by

$$\mathbf{h}_y(u, v) = \mathbf{m}_y(u)\mathbf{m}_y(v) + \mathbf{n}_y(u)\mathbf{n}_y(v).$$

Then (3.2) can be written as following

$$\begin{aligned} \mathbf{C}_y(u, v, w) = &\Big\{\mathbf{a}_y(u)\mathbf{h}_y(v, w) + \mathbf{a}_y(v)\mathbf{h}_y(w, u) + \mathbf{a}_y(w)\mathbf{h}_y(u, v)\Big\} \\ &+\Big\{\mathbf{b}_y(u)\mathbf{I}_y(v)\mathbf{I}_y(w) + \mathbf{b}_y(v)\mathbf{I}_y(w)\mathbf{I}_y(u) + \mathbf{b}_y(w)\mathbf{I}_y(u)\mathbf{I}_y(v)\Big\}, \end{aligned} \tag{3.3}$$

where

$$\mathbf{a}_y(u) := \frac{1}{3F}\Big[3\mathcal{I}\mathbf{m}_y(u) + \mathcal{J}\mathbf{n}_y(u)\Big], \tag{3.4}$$

$$\mathbf{b}_y(u) := \frac{1}{3F||\mathbf{I}||^2}\Big[(\mathcal{H} - 3\mathcal{I})\mathbf{m}_y(u) - 4\mathcal{J}\mathbf{n}_y(u)\Big]. \tag{3.5}$$

It is easy to see that $\mathbf{g}_y(\mathbf{a}, \mathbf{y}) = 0$ and $\mathbf{g}_y(\mathbf{b}, \mathbf{y}) = 0$. It follows from (3.3) that

$$\mathbf{a}_y(u) = \frac{1}{4}\Bigg[\big(1 - 2\mathbf{g}_y(\mathbf{b}, \mathbf{I})\big)\mathbf{I}_y(u) - ||\mathbf{I}||^2\mathbf{b}_y(u)\Bigg]. \tag{3.6}$$

Substituting (3.6) into (3.3), we get

$$\begin{aligned} \mathbf{C}_y(u, v, w) = &\frac{1}{4}\Big\{\mathbf{I}_y(u)\mathbf{h}_y(v, w) + \mathbf{I}_y(v)\mathbf{h}_y(u, w) + \mathbf{I}_y(w)\mathbf{h}_y(u, v)\Big\} \\ &-\frac{1}{2}\mathbf{g}_y(\mathbf{b}, \mathbf{I})\Big\{\mathbf{I}_y(u)\mathbf{h}_y(v, w) + \mathbf{I}_y(v)\mathbf{h}_y(u, w) + \mathbf{I}_y(w)\mathbf{h}_y(u, v)\Big\} \\ &-\frac{1}{4}||\mathbf{I}||^2\Big\{\mathbf{b}_y(u)\mathbf{h}_y(v, w) + \mathbf{b}_y(v)\mathbf{h}_y(u, w) + \mathbf{b}_y(w)\mathbf{h}_y(u, v)\Big\} \\ &+\Big\{\mathbf{b}_y(u)\mathbf{I}_y(v)\mathbf{I}_y(w) + \mathbf{b}_y(v)\mathbf{I}_y(w)\mathbf{I}_y(u) + \mathbf{b}_y(w)\mathbf{I}_y(u)\mathbf{I}_y(v)\Big\}. \end{aligned} \tag{3.7}$$

The relation (3.7) can be written as follows

$$\begin{aligned} \mathbf{M}_y(u, v, w) = &-\frac{1}{2}\mathbf{g}_y(\mathbf{b}, \mathbf{I})\Big\{\mathbf{I}_y(u)\mathbf{h}_y(v, w) + \mathbf{I}_y(v)\mathbf{h}_y(u, w) + \mathbf{I}_y(w)\mathbf{h}_y(u, v)\Big\} \\ &-\frac{1}{4}||\mathbf{I}||^2\Big\{\mathbf{b}_y(u)\mathbf{h}_y(v, w) + \mathbf{b}_y(v)\mathbf{h}_y(u, w) + \mathbf{b}_y(w)\mathbf{h}_y(u, v)\Big\} \\ &+\Big\{\mathbf{b}_y(u)\mathbf{I}_y(v)\mathbf{I}_y(w) + \mathbf{b}_y(v)\mathbf{I}_y(w)\mathbf{I}_y(u) + \mathbf{b}_y(w)\mathbf{I}_y(u)\mathbf{I}_y(v)\Big\}. \end{aligned} \tag{3.8}$$

By taking a horizontal derivation of (3.8), we get (3.1). □

Let us define the norm of the second Matsumoto torsion at $x \in M$ by

$$\|\widetilde{\mathbf{M}}\|_x := \sup_{y,u,v,w \in T_xM\setminus\{0\}} \frac{F(y)|\widetilde{\mathbf{M}}_y(u, v, w)|}{\sqrt{\mathbf{g}_y(u, u)\mathbf{g}_y(v, v)\mathbf{g}_y(w, w)}}.$$

Then, the following holds.

Lemma 3.4. *The second Matsumoto torsion of an n-dimensional homogeneous Finsler manifold is bounded.*

Proof. By (2.2), we get

$$||\widetilde{\mathbf{M}}||^2 = ||\mathbf{L}||^2 - \frac{3}{n+1}\ \|\mathbf{J}\|^2. \tag{3.9}$$

According to definition, we have $\mathbf{J}_y(u) = \sum_{i=1}^n g^{ij}(y)\mathbf{L}_y(u, \partial_i, \partial_j)$, where $\{\partial_i\}$ is a basis for T_xM at $x \in M$. Then

$$\|\mathbf{J}\| \leq n\|\mathbf{L}\|. \tag{3.10}$$

Therefore, we get

$$||\widetilde{\mathbf{M}}||^2 \leq \Big[1 + \frac{3n}{n+1}\Big]\|\mathbf{L}\|^2 \leq 4\|\mathbf{L}\|^2 \tag{3.11}$$

By considering Lemma 3.2, it follows that $||\widetilde{\mathbf{M}}|| < \infty$. □

Here, we study the Landsberg curvature of homogeneous 3-dimensional Landsberg metrics. We show that the Landsberg curvature of homogeneous 3-dimensional Landsberg metrics satisfies an ODE.

Lemma 3.5. *Every 3-dimensional L-reducible Finsler metric satisfies following*

$$\mathbf{g}_{\dot\sigma}(\mathbf{b},\mathbf{J})\mathbf{I} - \mathbf{g}_{\dot\sigma}(\mathbf{b},\mathbf{I})\mathbf{J} - \mathbf{g}_{\dot\sigma}(\mathbf{I},\mathbf{J})\mathbf{b} = \frac{1}{2}||\mathbf{I}||^2 D_{\dot\sigma}\mathbf{b}, \tag{3.12}$$

where $||\mathbf{I}|| := \sqrt{g^{ij}I_iI_j}$.

Proof. By assumption, F is a L-reducible metric and satisfies $\widetilde{\mathbf{M}} = 0$. Thus (3.1) reduces to following

$$\begin{aligned}
&||\mathbf{I}||^2\Big\{\mathbf{h}_y(v,w)D_{\dot\sigma}\mathbf{b}_y(u) + \mathbf{h}_y(u,w)D_{\dot\sigma}\mathbf{b}_y(v) + \mathbf{h}_y(u,v)D_{\dot\sigma}\mathbf{b}_y(w)\Big\}\\
&-4\mathbf{b}_y(u)\Big(\mathbf{J}_y(v)\mathbf{I}_y(w) + \mathbf{J}_y(w)\mathbf{I}_y(v)\Big) - 4\mathbf{b}_y(v)\Big(\mathbf{J}_y(u)\mathbf{I}_y(u) + \mathbf{J}_y(w)\mathbf{I}_y(u)\Big)\\
&-4\mathbf{b}_y(w)\Big(\mathbf{J}_y(u)\mathbf{I}_y(v) + \mathbf{J}_y(v)\mathbf{I}_y(u)\Big)\\
&+2\Big(\mathbf{g}_y(\mathbf{b}, D_{\dot\sigma}\mathbf{I}) + \mathbf{g}_y(D_{\dot\sigma}\mathbf{b},\mathbf{I})\Big)\Big\{\mathbf{I}_y(u)\mathbf{h}_y(v,w) + \mathbf{I}_y(v)\mathbf{h}_y(u,w) + \mathbf{I}_y(w)\mathbf{h}_y(u,v)\Big\}\\
&+2\mathbf{g}_y(D_{\dot\sigma}\mathbf{I},\mathbf{J})\Big\{\mathbf{b}_y(u)\mathbf{h}_y(v,w) + \mathbf{b}_y(v)\mathbf{h}_y(u,w) + \mathbf{b}_y(w)\mathbf{h}_y(u,v)\Big\}\\
&-4\Big\{\mathbf{I}_y(v)\mathbf{I}_y(w)D_{\dot\sigma}\mathbf{b}_y(u) + \mathbf{I}_y(w)\mathbf{I}_y(u)D_{\dot\sigma}\mathbf{b}_y(v) + \mathbf{I}_y(u)\mathbf{I}_y(v)D_{\dot\sigma}\mathbf{b}_y(w)\Big\}\\
&+2\mathbf{g}_y(\mathbf{b},\mathbf{I})\Big\{\mathbf{J}_y(u)\mathbf{h}_y(v,w) + \mathbf{J}_y(v)\mathbf{h}_y(u,w) + \mathbf{J}_y(w)\mathbf{h}_y(u,v)\Big\} = 0.
\end{aligned} \tag{3.13}$$

Applying $\mathbf{I}^\sharp_y(u)$ on (3.13) yields

$$
\begin{aligned}
&||\mathbf{I}||^2\Big\{\mathbf{g}_{\dot\sigma}(D_{\dot\sigma}\mathbf{b},\mathbf{I})\mathbf{h}_y(v,w)+D_{\dot\sigma}\mathbf{b}_y(v)\mathbf{I}_y(w)+D_{\dot\sigma}\mathbf{b}_y(w)\mathbf{I}_y(v)\Big\}\\
&-4\Big\{\mathbf{g}_{\dot\sigma}(\mathbf{b},\mathbf{I})\big(\mathbf{J}_y(v)\mathbf{I}_y(w)+\mathbf{I}_y(v)\mathbf{J}_y(w)\big)+\mathbf{b}_y(v)\big(\mathbf{g}_{\dot\sigma}(\mathbf{I},\mathbf{J})\mathbf{I}_y(w)+||\mathbf{I}||^2\mathbf{J}_y(w)\big)\\
&\qquad+\mathbf{b}_y(w)(\mathbf{g}_{\dot\sigma}(\mathbf{I},\mathbf{J})\mathbf{I}_y(v)+||\mathbf{I}||^2\mathbf{J}_y(v))\Big\}\\
&+2\Big(\mathbf{g}_{\dot\sigma}(\mathbf{b},\mathbf{J})+\mathbf{g}_{\dot\sigma}(D_{\dot\sigma}\mathbf{b},\mathbf{I})\Big)\Big\{||\mathbf{I}||^2\mathbf{h}_y(v,w)+2\mathbf{I}_y(v)\mathbf{I}_y(w)\Big\}\\
&+2\mathbf{g}_{\dot\sigma}(\mathbf{I},\mathbf{J})\Big\{\mathbf{g}_{\dot\sigma}(\mathbf{b},\mathbf{I})\mathbf{h}_y(v,w)+\mathbf{b}_y(v)\mathbf{I}_y(w)+\mathbf{b}_y(w)\mathbf{I}_y(v)\Big\}\\
&+2\mathbf{g}_{\dot\sigma}(\mathbf{b},\mathbf{I})\Big\{\mathbf{g}_{\dot\sigma}(\mathbf{I},\mathbf{J})\mathbf{h}_y(v,w)+\mathbf{J}_y(v)\mathbf{I}_y(w)+\mathbf{J}_y(w)\mathbf{I}_y(v)\Big\}\\
&-4\Big\{\mathbf{g}_{\dot\sigma}(D_{\dot\sigma}\mathbf{b},\mathbf{I})\mathbf{I}_y(v)\mathbf{I}_y(w)+||\mathbf{I}||^2D_{\dot\sigma}\mathbf{b}_y(v)\mathbf{I}_y(w)\\
&\qquad+||\mathbf{I}||^2D_{\dot\sigma}\mathbf{b}_y(w)\mathbf{I}_y(v)\Big\}=0.
\end{aligned}
\tag{3.14}
$$

Contracting (3.14) with $\mathbf{I}^\sharp_y(v)$ implies (3.12). □

Proof of Theorem 1.1: We have two main cases. If $c=0$, then F is a weakly Landsberg metric $\mathbf{J}=0$ and by considering $\widetilde{\mathbf{M}}=0$ it follows that F is a Landsberg metric. Now, suppose that $c\neq 0$. By assumption, F has constant relatively isotropic mean Landsberg curvature $\mathbf{J}=cF\mathbf{I}$. Then (3.12) reduces to following

$$||\mathbf{I}||^2(D_{\dot\sigma}\mathbf{b}+2cF\mathbf{b})=0. \tag{3.15}$$

If $||\mathbf{I}||=0$, then by Deicke theorem F is Riemannian. Suppose that F is not a Riemannian metric. On Finslerian geodesics, (3.15) is written as follows

$$\mathbf{b}'+2c\mathbf{b}=0. \tag{3.16}$$

Then the solution of (3.16) is given by

$$\mathbf{b}(t)=\exp(-2ct)\mathbf{b}(0). \tag{3.17}$$

Since the main scalars are bounded then $||\mathbf{b}||<\infty$. Therefore, letting $t\to-\infty$ implies that $\mathbf{b}=0$. Putting this in (3.3) implies that

$$\mathbf{C}_y(u,v,w)=\mathbf{a}_y(u)\mathbf{h}_y(v,w)+\mathbf{a}_y(v)\mathbf{h}_y(w,u)+\mathbf{a}_y(w)\mathbf{h}_y(u,v). \tag{3.18}$$

Taking a trace from (3.18) give us the following

$$\mathbf{a}_y(w)=\frac{1}{4}\mathbf{I}_y(w). \tag{3.19}$$

Putting (3.19) into (3.18) yields

$$\mathbf{C}_y(u,v,w)=\frac{1}{4}\Big\{\mathbf{I}_y(u)\mathbf{h}_y(v,w)+\mathbf{I}_y(v)\mathbf{h}_y(w,u)+\mathbf{I}_y(w)\mathbf{h}_y(u,v)\Big\}. \tag{3.20}$$

By (3.20), the Matumoto torsion of F vanishes, i.e., $\mathbf{M} = 0$. Then, according to Matsumoto-Hōjō's lemma, F is a Randers metric. It is easy to see that for a Randers metric, $\mathbf{J} = cF\mathbf{I}$ if and only if $\mathbf{L} = cF\mathbf{C}$. In [4], it is proved that a Randers metric $F = \alpha + \beta$ satisfies $\mathbf{J} = cF\mathbf{I}$ if and only if it has isotropic S-curvature $\mathbf{S} = 4cF$ and β is closed. It is proved that every homogeneous Finsler metric of isotropic S-curvature has vanishing S-curvature $\mathbf{S} = 0$ (Corollary 4.3. in [8]). Thus F is a Landsberg metric. It is well-known that every Randers metric with vanishing Landsberg curvature is a Berwald metric. This completes the proof. □

By Shen's theorem, every regular (α, β)-metric with vanishing Landsberg curvature is a Berwald metric [20]. Then, by Theorem 1.1 we conclude the following.

Corollary 3.1. *Let $F = F(x, y)$ be a homogeneous 3-dimensional L-reducible (α, β)-metric on a manifold M. Then F has relatively isotropic mean Landsberg curvature if and only if it is a a Berwald metric.*

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