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ON THE COTANGENT BUNDLE AND UNIT COTANGENT BUNDLE WITH A GENERALIZED CHEEGER-GROMOLL METRIC

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Abstract. In this paper, we consider a generalized Cheeger-Gromoll metric on a cotangent bundle over a Riemannian manifold, which is obtained by rescaling the vertical part of the Cheeger-Gromoll metric by a positive differentiable function. Firstly, we investigate the curvature properties on the cotangent bundle with the generalized Cheeger-Gromoll metric. Secondly, we introduce the unit cotangent bundle equipped with this metric, where we present the formulas of the Levi-Civita connection and also all formulas of the Riemannian curvature tensors of this metric. Finally, we study the geodesics on the unit cotangent bundle with respect to this metric.
Keywords: Horizontal lift and vertical lift, cotangent bundles, generalized Cheeger-Gromoll metric, curvature tensor.

1. Introduction

In this field, one of the first works which deal with the cotangent bundles of a manifold as a Riemannian manifold is that of Patterson and Walker [7], who constructed a Riemannian metric on the cotangent bundle from an affine symmetric connection on a manifold, which they called the Riemannian metric the Riemannian extension metric. A generalization of this metric had been given by Sekizawa [10]

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in his classification of natural transformations of affine connections on manifolds to metrics on their cotangent bundles, obtaining the class of natural Riemannian extensions which is a 2-parameter family of metrics, and which had been intensively studied by many authors. On the other hand, inspired by the concept of g-natural metrics on tangent bundles of Riemannian manifolds, Ağca considered another class of metrics on cotangent bundles of Riemannian manifolds, that she called g-natural metrics [2]. Also, there are studies presented by other authors, Salimov and Ağca [3, 8], Yano and Ishihara [11], Ocak [6], Gezer and Altunbas [4]. Note that the deformations of the Sasaki metric on the cotangent bundle are not limited to those mentioned above. We also refer to [12, 13, 14, 15, 16, 17, 18, 19].

In the previous works [16] we gave characterizations of geodesics on the cotangent bundle with a generalized Cheeger-Gromoll metric which rescale the vertical part by a nonzero differentiable function. Also, in [18], we presented some para-complex structures on the cotangent bundle with a generalized Cheeger-Gromoll metric. Here, the generalized Cheeger-Gromoll metric is pure with respect to the para-complex structures. As a continuation of these studies, in this paper, we study the geometry of the cotangent and unit cotangent bundle with the generalized Cheeger-Gromoll metric. Firstly, we find the form of the Riemannian curvature tensor (Theorem 4.1 and Proposition 4.1) of this metric. Then, we characterize the Ricci curvature (Theorem 4.2), the sectional curvature (Theorem 4.3 and Proposition 4.3) and the scalar curvature (Theorem 4.4 and Proposition 4.4) of this metric. In the last section, we present the unit cotangent bundle equipped with the generalized Cheeger-Gromoll metric. We establish the Levi-Civita connection of this metric (Theorem 5.1) and all forms of its Riemannian curvature tensors (Theorem 5.2). Also we study some properties of geodesics on the unit cotangent bundle with this metric (Lemma 6.1, Theorem 6.1, Corollary 6.1 and Corollary 6.2).

2. Preliminaries

Let (M^m, g) be an m-dimensional Riemannian manifold, T^*M be its cotangent bundle and $\pi : T^*M \to M$ be the natural projection. A local chart $(U, x^i)_{i=\overline{1,m}}$ on M^m induces a local chart $(\pi^{-1}(U), x^i, x^{\bar{i}} = p_i)_{i=\overline{1,m},\bar{i}=m+i}$ on T^*M, where p_i is the component of the covector p in each cotangent space T^*_xM, $x \in U$ with respect to the natural coframe dx^i. Let $C^\infty(M^m)$ (resp. $C^\infty(T^*M)$) be the ring of real-valued C^∞ functions on M^m(resp. T^*M) and $\Upsilon^r_s(M^m)$ (resp. $\Upsilon^r_s(T^*M)$) be the module over $C^\infty(M^m)$ (resp. $C^\infty(T^*M)$) of C^∞ tensor fields of type (r, s). Denote by Γ^k_{ij} the Christoffel symbols of g and by ∇ the Levi-Civita connection of g.

We have two complementary distributions on T^*M, the vertical distribution $VT^*M = Ker(d\pi)$ and the horizontal distribution HT^*M that define a direct sum decomposition

$$TT^*M = VT^*M \oplus HT^*M. \tag{2.1}$$

Let $X = X^i \frac{\partial}{\partial x^i}$ and $\omega = \omega_i dx^i$ be a local expressions in $(U, x^i)_{i=\overline{1,m}}$, $U \subset M^m$ of $X \in \Upsilon^1_0(M^m)$ and $\omega \in \Upsilon^0_1(M^m)$, respectively. Then the horizontal lift ${}^HX \in$

$\Upsilon^1_0(T^*M)$ of $X \in \Upsilon^1_0(M^m)$ and the vertical lift ${}^V\omega \in \Upsilon^1_0(T^*M)$ of $\omega \in \Upsilon^0_1(M^m)$ are defined, respectively by

$$ {}^HX = X^i \frac{\partial}{\partial x^i} + p_h \Gamma^h_{ij} X^j \frac{\partial}{\partial x^{\bar{i}}}, \tag{2.2} $$

$$ {}^V\omega = \omega_i \frac{\partial}{\partial x^{\bar{i}}} \tag{2.3} $$

with respect to the natural frame $\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^{\bar{i}}}\}$, (see [11] for more details).

In particular, if $\mathcal{P}$ be a local covector field on each fiber T^*_xM (i.e., $\mathcal{P} = p_i dx^i$), the vertical lift ${}^V\mathcal{P}$ is called the canonical vertical vector field or Liouville vector field on T^*M.

Lemma 2.1. *[11] Let (M^m, g) be a Riemannian manifold. The bracket operations of the vertical and horizontal vector fields on T^*M are given by the formulas*

(1) $[{}^V\omega, {}^V\theta] = 0$,

(2) $[{}^HX, {}^V\theta] = {}^V(\nabla_X \theta)$,

(3) $[{}^HX, {}^HY] = {}^H[X, Y] + {}^V(pR(X, Y))$

for all $X, Y \in \Upsilon^1_0(M^m)$ and $\omega, \theta \in \Upsilon^0_1(M^m)$, where ∇ and R denote the Levi-Civita connection and the Riemannian curvature tensor of (M^m, g), respectively.

Let (M^m, g) be a Riemannian manifold. We can define the following maps

$$ \begin{array}{cccc} \sharp: & \Upsilon^0_1(M^m) & \to & \Upsilon^1_0(M^m) \\ & \omega & \mapsto & \sharp(\omega) \end{array} \quad \textit{and} \quad \begin{array}{cccc} \flat: & \Upsilon^1_0(M^m) & \to & \Upsilon^0_1(M^m) \\ & X & \mapsto & \flat(X) \end{array} $$

by $g(\sharp(\omega), Y) = \omega(Y)$ and $\flat(X)(Y) = g(X, Y)$, respectively. Locally, we have

$$ \sharp(\omega) = g^{ij}\omega_i \frac{\partial}{\partial x^j} \quad \textit{and} \quad \flat(X) = g_{ij} X^i dx^j, $$

where (g^{ij}) is the inverse matrix of the matrix (g_{ij}).

For each $x \in M^m$, the scalar product $g^{-1} = (g^{ij})$ is defined on the cotangent space T^*_xM by

$$ g^{-1}(\omega, \theta) = g(\sharp(\omega), \sharp(\theta)) = g^{ij}\omega_i\theta_j. $$

In this case, we have $\sharp(\omega) = g^{-1} \circ \omega$ and $\flat(X) = g \circ X$.

In the following, we noted $\sharp(\omega)$ by $\widetilde{\omega}$ and $\flat(X)$ by $\widetilde{X}$.

Lemma 2.2. *Let (M^m, g) be a Riemannian manifold. We have the following*

$$ \widetilde{\widetilde{\omega}} = \omega \quad , \quad \widetilde{\widetilde{X}} = X, \tag{2.4} $$

$$ \nabla_X \widetilde{\omega} = \widetilde{\nabla_X \omega}, \tag{2.5} $$

$$ Xg^{-1}(\omega, \theta) = g^{-1}(\nabla_X \omega, \theta) + g^{-1}(\omega, \nabla_X \theta), \tag{2.6} $$

$$ \omega \widetilde{R(Y, X)} = R(X, Y)\widetilde{\omega} \tag{2.7} $$

for any $X \in \Upsilon^1_0(M^m)$ and $\omega, \theta \in \Upsilon^0_1(M^m)$.

Proof. (i) The direct calculations give the results.
(ii) For any $Y \in \Upsilon_0^1(M^m)$, we find

$$\begin{array}{rcl} g(\nabla_X \widetilde{\omega}, Y) &=& X(g(\widetilde{\omega}, Y)) - g(\widetilde{\omega}, \nabla_X Y) \\ &=& X(\omega(Y)) - \omega(\nabla_X Y) \\ &=& (\nabla_X \omega)(Y) \\ &=& g(\widetilde{\nabla_X \omega}, Y). \end{array}$$

(iii) We have

$$\begin{array}{rcl} Xg^{-1}(\omega, \theta) &=& Xg(\widetilde{\omega}, \tilde{\theta}) \\ &=& g(\nabla_X \widetilde{\omega}, \tilde{\theta}) + g(\widetilde{\omega}, \nabla_X \tilde{\theta}) \\ &=& g(\widetilde{\nabla_X \omega}, \tilde{\theta}) + g(\widetilde{\omega}, \widetilde{\nabla_X \theta}) \\ &=& g^{-1}(\nabla_X \omega, \theta) + g^{-1}(\omega, \nabla_X \theta). \end{array}$$

(iv) For any $Z \in \Upsilon_0^1(M^m)$, we have

$$\begin{array}{rcl} g(\omega \widetilde{R(Y, X)}) &=& g_{st}(\omega \widetilde{R(Y, X)})^s Z^t = g_{st} g^{ks} (\omega R(Y, X))_k Z^t \\ &=& \delta_t^k \omega_a R_{ijk}^a Y^i X^j Z^t = g_{ab} \widetilde{\omega}^b R_{ijk}^a Y^i X^j Z^k \\ &=& g(R(Y, X)Z, \widetilde{\omega}) = g(R(X, Y)\widetilde{\omega}, Z). \end{array}$$

□

3. A generalized Cheeger-Gromoll metric

Definition 3.1. [16, 18] Let (M^m, g) be a Riemannian manifold and $f : M^m \to]0, +\infty[$ be a strictly positive smooth function on M^m. On the cotangent bundle T^*M, we define a generalized Cheeger-Gromoll metric noted g^f by

$$\begin{array}{rcl} g^f({}^H X, {}^H Y) &=& {}^V g(X, Y) = g(X, Y) \circ \pi, \\ g^f({}^H X, {}^V \theta) &=& 0, \\ g^f({}^V \omega, {}^V \theta) &=& \dfrac{f}{\alpha}(g^{-1}(\omega, \theta) + g^{-1}(\omega, p) g^{-1}(\theta, p)) \end{array}$$

for all $X, Y \in \Upsilon_0^1(M^m)$, $\omega, \theta \in \Upsilon_1^0(M^m)$, where $\alpha = 1 + |p|^2$ and $|p| = \sqrt{g^{-1}(p, p)}$ is the norm of p with respect to the metric g^{-1}.

Note that if $f = 1$, then g^f is the Cheeger-Gromoll metric [3].

Lemma 3.1. *[13] Let (M^m, g) be a Riemannian manifold and $\rho : \mathbb{R} \to \mathbb{R}$ be a smooth function. We have the following*

1. ${}^H X(\rho(r^2)) = 0,$

2. ${}^{V}\omega(\rho(r^2)) = 2\rho'^2 g^{-1}(\omega, p)$,

3. ${}^{H}X(g^{-1}(\theta, p)) = g^{-1}(\nabla_X\theta, p)$,

4. ${}^{V}\omega(g^{-1}(\theta, p)) = g^{-1}(\omega, \theta)$

for any $X \in \Upsilon^1_0(M^m)$ *and* $\omega, \theta \in \Upsilon^0_1(M^m)$, $r^2 = g^{-1}(p, p)$.

Lemma 3.2. *[18] Let* (M^m, g) *be a Riemannian manifold and* (T^*M, g^f) *its cotangent bundle equipped with the generalized Cheeger-Gromoll metric. We get*

$$\begin{array}{rcl}(1)\ {}^{H}Xg^f({}^{V}\theta, {}^{V}\eta) & = & \frac{1}{f}X(f)g^f({}^{V}\theta, {}^{V}\eta) + g^f({}^{V}(\nabla_X\theta), {}^{V}\eta) + g^f({}^{V}\theta, {}^{V}(\nabla_X\eta)),\\ (2)\ {}^{V}\omega g^f({}^{V}\theta, {}^{V}\eta) & = & -\frac{2}{\alpha}g^{-1}(\omega, p)g^f({}^{V}\theta, {}^{V}\eta) + \frac{1}{\alpha}g^{-1}(\omega, \theta)g^f({}^{V}\eta, {}^{V}\mathcal{P}),\\ & & +\frac{1}{\alpha}g^{-1}(\omega, \eta)g^f({}^{V}\theta, {}^{V}\mathcal{P}),\\ (3)\ {}^{H}Xg^f({}^{V}\theta, {}^{V}\mathcal{P}) & = & X(f)g^{-1}(\theta, p) + fg^{-1}(\nabla_X\theta, p),\\ (4)\ {}^{V}\omega g^f({}^{V}\theta, {}^{V}\mathcal{P}) & = & fg^{-1}(\omega, \theta)\end{array}$$

for any $X \in \Upsilon^1_0(M^m)$ *and* $\omega, \theta \in \Upsilon^0_1(M^m)$, *where* ${}^{V}\mathcal{P}$ *is the canonical vertical vector field on* T^*M.

The Levi-Civita connection ∇^f of the generalized Cheeger-Gromoll metric g^f on T^*M is given in the following theorem.

Theorem 3.1. *[18] Let* (M^m, g) *be a Riemannian manifold and* (T^*M, g^f) *its cotangent bundle equipped with the generalized Cheeger-Gromoll metric. We have*

$$\begin{array}{rcl}1.\ \nabla^f_{{}^{H}X}{}^{H}Y & = & {}^{H}(\nabla_X Y) + \frac{1}{2}{}^{V}(pR(X, Y)),\\ 2.\ \nabla^f_{{}^{H}X}{}^{V}\theta & = & {}^{V}(\nabla_X\theta) + \frac{1}{2f}X(f){}^{V}\theta + \frac{f}{2\alpha}{}^{H}(R(\tilde{p}, \tilde{\theta})X),\\ 3.\ \nabla^f_{{}^{V}\omega}{}^{H}Y & = & \frac{1}{2f}Y(f){}^{V}\omega + \frac{f}{2\alpha}{}^{H}(R(\tilde{p}, \widetilde{\omega})Y),\\ 4.\ \nabla^f_{{}^{V}\omega}{}^{V}\theta & = & -\frac{1}{2f}g^f({}^{V}\omega, {}^{V}\theta){}^{H}(grad f) - \frac{1}{\alpha f}\big(g^f({}^{V}\omega, {}^{V}\mathcal{P}){}^{V}\theta + g^f({}^{V}\theta, {}^{V}\mathcal{P}){}^{V}\omega\big)\\ & & +\big(\frac{\alpha+1}{\alpha f}g^f({}^{V}\omega, {}^{V}\theta) - \frac{1}{\alpha f^2}g^f({}^{V}\omega, {}^{V}\mathcal{P})g^f({}^{V}\theta, {}^{V}\mathcal{P})\big){}^{V}\mathcal{P}\end{array}$$

for all $X, Y \in \Upsilon^1_0(M^m)$ *and* $\omega, \theta \in \Upsilon^0_1(M^m)$.

As a consequence of Theorem 3.1, we get the following Lemma.

Lemma 3.3. *Let (M^m, g) be a Riemannian manifold and (T^*M, g^f) its cotangent bundle equipped with the generalized Cheeger-Gromoll metric. Then*

$$\begin{aligned}
(1)\ \nabla^f_{{}^HX}{}^V\mathcal{P} &= \frac{1}{2f}X(f)^V\mathcal{P},\\
(2)\ \nabla^f_{{}^V\mathcal{P}}{}^HX &= \frac{1}{2f}X(f)^V\mathcal{P},\\
(3)\ \nabla^f_{{}^V\omega}{}^V\mathcal{P} &= -\frac{1}{2}g^{-1}(\omega,p)^H(grad f) + \frac{1}{\alpha}{}^V\omega + \frac{1}{\alpha}g^{-1}(\omega,p)^V\mathcal{P},\\
(4)\ \nabla^f_{{}^V\mathcal{P}}{}^V\omega &= -\frac{1}{2}g^{-1}(\omega,p)^H(grad f) - \frac{\alpha-1}{\alpha}{}^V\omega + \frac{1}{\alpha}g^{-1}(\omega,p)^V\mathcal{P},\\
(5)\ \nabla^f_{{}^V\mathcal{P}}{}^V\mathcal{P} &= -\frac{\alpha-1}{2}{}^H(grad f) + {}^V\mathcal{P}
\end{aligned}$$

for $X \in \Upsilon^1_0(M)$, $\omega \in \Upsilon^0_1(M)$ and $\alpha = 1 + g^{-1}(p,p)$.

Definition 3.2. Let (M^m, g) be a Riemannian manifold and F be a tensor field of type $(1,1)$ on M^m. Then the vertical and horizontal vector fields ${}^VF$ and ${}^HF$ respectively are defined on T^*M by

$$\begin{aligned}
{}^VF : T^*M &\to TT^*M\\
(x,p) &\mapsto {}^VF(x,p) = {}^V(pF),
\end{aligned}$$

$$\begin{aligned}
{}^HF : T^*M &\to TT^*M\\
(x,p) &\mapsto {}^HF(x,p) = {}^H(F(\tilde{p})),
\end{aligned}$$

locally we have

$$ {}^VF = p_i{}^V(dx^iF), \tag{3.1}$$

$$ {}^HF = \tilde{p}^{i\,H}(F(\frac{\partial}{\partial x^i})), \tag{3.2}$$

where $p = p_j dx^j$ and $\tilde{p} = \tilde{p}^i \frac{\partial}{\partial x^i} = p_j g^{ij} \frac{\partial}{\partial x^i}$.

Proposition 3.1. *Let (M^m, g) be a Riemannian manifold, (T^*M, g^f) its tangent bundle equipped with the generalized Cheeger-Gromoll metric and F be a tensor field of type $(1,1)$ on M^m. Then we have the following formulas*

$$\begin{aligned}
1.\ \nabla^f_{{}^HX}{}^HF &= {}^H(\nabla_XF) + \frac{1}{2}{}^V(pR(X,F(\tilde{p}))),\\
2.\ \nabla^f_{{}^HX}{}^VF &= {}^V(\nabla_XF) + \frac{1}{2f}X(f)^VF + \frac{f}{2\alpha}{}^H(R(\tilde{p},\widetilde{pF})X),\\
3.\ \nabla^f_{{}^V\omega}{}^HF &= {}^H(F(\widetilde{\omega})) + \frac{1}{2f}g(F(\tilde{p}), grad f)^V\omega + \frac{f}{2\alpha}{}^H(R(\tilde{p},\widetilde{\omega})F(\tilde{p})),\\
4.\ \nabla^f_{{}^V\omega}{}^VF &= {}^V(\omega F) - \frac{1}{2f}g^f({}^V\omega, {}^VF)^H(grad f)
\end{aligned}$$

$$-\frac{1}{\alpha f}\big(g^f({}^V\omega,{}^V\mathcal{P}){}^VF+g^f({}^VF,{}^V\mathcal{P}){}^V\omega\big)$$
$$+\big(\frac{\alpha+1}{\alpha f}g^f({}^V\omega,{}^VF)-\frac{1}{\alpha f^2}g^f({}^V\omega,{}^V\mathcal{P})g^f({}^VF,{}^V\mathcal{P})\big){}^V\mathcal{P}$$

for $X\in\Upsilon^1_0(M^m)$, $\omega\in\Upsilon^0_1(M^m)$ *where* $\widetilde{pF}=g^{-1}\circ(pF)$.

Proof. The results come directly from Theorem 3.1. □

4. Curvatures of the generalized Cheeger-Gromoll metric

We shall calculate the Riemannian curvature tensor R^f of T^*M with the generalized Cheeger-Gromoll metric g^f. This curvature tensor is characterized by the formula

$$R^f(U,V)W=\nabla^f_U\nabla^f_VW-\nabla^f_V\nabla^f_UW-\nabla^f_{[U,V]}W \tag{4.1}$$

for all $U,V,W\in\Upsilon^1_0(T^*M)$.

Theorem 4.1. *Let (M^m,g) be a Riemannian manifold and (T^*M,g^f) its cotangent bundle equipped with the generalized Cheeger-Gromoll metric. Then we have the following formulas*

$$\begin{aligned}R^f({}^HX,{}^HY){}^HZ &= {}^H(R(X,Y)Z)+\frac{f}{4\alpha}{}^H(R(\tilde p,R(Z,Y)\tilde p)X)\\&-\frac{f}{4\alpha}{}^H(R(\tilde p,R(Z,X)\tilde p)Y)+\frac{f}{2\alpha}{}^H(R(\tilde p,R(X,Y)\tilde p)Z)\\&+\frac{1}{4f}X(f){}^V(pR(Y,Z))-\frac{1}{4f}Y(f){}^V(pR(X,Z))\\&-\frac{1}{2f}Z(f){}^V(pR(X,Y))-\frac{1}{2}{}^V\big(p(\nabla_ZR)(X,Y),\end{aligned}\tag{4.2}$$

$$\begin{aligned}R^f({}^HX,{}^V\theta){}^HZ &= \frac{1}{2\alpha}X(f){}^H(R(\tilde p,\tilde\theta)Z)+\frac{1}{4\alpha}Z(f){}^H(R(\tilde p,\tilde\theta)X)\\&+\frac{f}{2\alpha}{}^H((\nabla_XR)(\tilde p,\tilde\theta)Z)+\frac{1}{4\alpha}g^{-1}(pR(X,Z),\theta){}^H(grad f)\\&-\frac{1}{2}{}^V(\theta R(X,Z))+\frac{f}{4\alpha}{}^V(pR(X,R(\tilde p,\tilde\theta)Z))\\&+\big(\frac{1}{2f}Hess^f(X,Z)-\frac{1}{4f^2}X(f)Z(f)\big){}^V\theta\\&+\frac{1}{2\alpha}g^{-1}(\theta,p){}^V(pR(X,Z))-\frac{\alpha+1}{2\alpha^2}g^{-1}(pR(X,Z),\theta){}^V\mathcal{P},\end{aligned}\tag{4.3}$$

$$R^f({}^HX,{}^HY){}^V\eta = \frac{f}{2\alpha}{}^H((\nabla_XR)(\tilde p,\tilde\eta)Y)-\frac{f}{2\alpha}{}^H((\nabla_YR)(\tilde p,\tilde\eta)X)$$

$$\begin{aligned}
&+\frac{1}{4\alpha}X(f)^H(R(\tilde{p},\tilde{\eta})Y)-\frac{1}{4\alpha}Y(f)^H(R(\tilde{p},\tilde{\eta})X)\\
&+\frac{1}{2\alpha}g^{-1}(pR(X,Y),\eta)^H(grad f)-{}^V(\eta R(X,Y))\\
&+\frac{f}{4\alpha}{}^V(pR(X,R(\tilde{p},\tilde{\eta})Y))-\frac{f}{4\alpha}{}^V(pR(Y,R(\tilde{p},\tilde{\eta})X))\\
&+\frac{1}{\alpha}g^{-1}(\eta,p)^V(pR(X,Y))-\frac{\alpha+1}{\alpha^2}g^{-1}(pR(X,Y),\eta)^V\mathcal{P},
\end{aligned}\tag{4.4}$$

$$\begin{aligned}
R^f({}^HX,{}^V\theta)^V\eta \;=\;& -\frac{1}{2\alpha}\big(g^{-1}(\theta,\eta)+g^{-1}(\theta,p)g^{-1}(\eta,p)\big)^H(\nabla_X grad f)\\
&+\frac{1}{4\alpha f}X(f)\big(g^{-1}(\theta,\eta)+g^{-1}(\theta,p)g^{-1}(\eta,p)\big)^H(grad f)\\
&-\frac{f}{2\alpha}{}^H(R(\tilde{\theta},\tilde{\eta})X)-\frac{f^2}{4\alpha^2}{}^H(R(\tilde{p},\tilde{\theta})R(\tilde{p},\tilde{\eta})X)\\
&+\frac{f}{2\alpha^2}\big(g^{-1}(\theta,p)^H(R(\tilde{p},\tilde{\eta})X)-g^{-1}(\eta,p)^H(R(\tilde{p},\tilde{\theta})X)\big)\\
&+\frac{1}{4\alpha}\big(g^{-1}(\theta,\eta)+g^{-1}(\theta,p)g^{-1}(\eta,p)\big)^V(pR(X,grad f))\\
&-\frac{1}{4\alpha}g(R(\tilde{p},\tilde{\eta})X,grad f)^V\theta,
\end{aligned}\tag{4.5}$$

$$\begin{aligned}
R^f({}^V\omega,{}^V\theta)^HZ \;=\;& \frac{f}{\alpha}{}^H(R(\widetilde{\omega},\tilde{\theta})Z)\\
&+\frac{f^2}{4\alpha^2}\big({}^H(R(\tilde{p},\widetilde{\omega})R(\tilde{p},\tilde{\theta})Z)-{}^H(R(\tilde{p},\tilde{\theta})R(\tilde{p},\widetilde{\omega})Z)\big)\\
&+\frac{f}{\alpha^2}\big(g^{-1}(\theta,p)^H(R(\tilde{p},\widetilde{\omega})Z)-g^{-1}(\omega,p)^H(R(\tilde{p},\tilde{\theta})Z)\big)\\
&+\frac{1}{4\alpha}\big(g(R(\tilde{p},\tilde{\theta})Z,grad f)^V\omega-g(R(\tilde{p},\widetilde{\omega})Z,grad f)^V\theta\big),
\end{aligned}\tag{4.6}$$

$$\begin{aligned}
R^f({}^V\omega,{}^V\theta)^V\eta \;=\;& \frac{f}{4\alpha^2}\big(g^{-1}(\omega,\eta)^H(R(\tilde{p},\tilde{\theta})grad f)-g^{-1}(\theta,\eta)^H(R(\tilde{p},\widetilde{\omega})grad f)\big)\\
&+\frac{f}{4\alpha^2}g^{-1}(\eta,p)\big(g^{-1}(\omega,p)^H(R(\tilde{p},\tilde{\theta})grad f)\\
&-g^{-1}(\theta,p)^H(R(\tilde{p},\widetilde{\omega})grad f)\big)\\
&+\big(\frac{\alpha^2+\alpha+1}{\alpha^3}-\frac{|grad f|^2}{4\alpha f}\big)\big(g^{-1}(\theta,\eta)^V\omega-g^{-1}(\omega,\eta)^V\theta\big)\\
&+\big(\frac{1-\alpha}{\alpha^3}-\frac{|grad f|^2}{4\alpha f}\big)g^{-1}(\eta,p)\big(g^{-1}(\theta,p)^V\omega-g^{-1}(\omega,p)^V\theta\big)\\
&+\frac{\alpha+2}{\alpha^3}\big(g^{-1}(\theta,p)g^{-1}(\omega,\eta)-g^{-1}(\omega,p)g^{-1}(\theta,\eta)\big)^V\mathcal{P},
\end{aligned}\tag{4.7}$$

for all $X,Y,Z\in\Upsilon^1_0(M^m)$ *and* $\omega,\theta,\eta\in\Upsilon^0_1(M^m)$, *where* $Hess^f(X,Z)=g(\nabla_X grad f,Z)$.

Proof. By applying Lemma 3.2, Theorem 3.1, Proposition 3.1 we have

(1) $R^f({}^HX, {}^HY){}^HZ = \nabla^f_{{}^HX}\nabla^f_{{}^HY}{}^HZ - \nabla^f_{{}^HY}\nabla^f_{{}^HX}{}^HZ - \nabla^f_{[{}^HX,{}^HY]}{}^HZ.$

(i) Let $F : T^*M \to T^*M$ be the bundle map given by $pF = pR(Y, Z)$. Then

$$\begin{aligned}
\nabla^f_{{}^HX}\nabla^f_{{}^HY}{}^HZ &= \nabla^f_{{}^HX}\big({}^H(\nabla_Y Z) + \frac{1}{2}{}^VF\big) \\
&= {}^H(\nabla_X\nabla_Y Z) + \frac{1}{2}{}^V(pR(X, \nabla_Y Z)) + \frac{1}{2}{}^V(\nabla_X(pR(Y, Z))) \\
&\quad -\frac{1}{2}{}^V((\nabla_X p)R(Y, Z)) + \frac{1}{4f}X(f){}^V(pR(Y, Z)) \\
&\quad +\frac{f}{4\alpha}{}^H(R(\tilde{p}, \widetilde{pR(Y, Z)})X)
\end{aligned}$$

and using (2.4), we have

$$\begin{aligned}
\nabla^f_{{}^HX}\nabla^f_{{}^HY}{}^HZ &= {}^H(\nabla_X\nabla_Y Z) + \frac{1}{2}(pR(X, \nabla_Y Z))^V + \frac{1}{2}{}^V(\nabla_X(pR(Y, Z))) \\
&\quad -\frac{1}{2}{}^V((\nabla_X p)R(Y, Z)) + \frac{1}{4f}X(f){}^V(pR(Y, Z)) \\
&\quad +\frac{f}{4\alpha}{}^H(R(\tilde{p}, R(Z, Y)\tilde{p})X).
\end{aligned}$$

(ii) With permutation of X by Y, we have

$$\begin{aligned}
\nabla^f_{{}^HY}\nabla^f_{{}^HX}{}^HZ &= {}^H(\nabla_Y\nabla_X Z) + \frac{1}{2}{}^V(pR(Y, \nabla_X Z)) + \frac{1}{2}{}^V(\nabla_Y(pR(X, Z))) \\
&\quad -\frac{1}{2}{}^V((\nabla_Y p)R(X, Z)) + \frac{1}{4f}Y(f){}^V(pR(X, Z)) \\
&\quad +\frac{f}{4\alpha}{}^H(R(\tilde{p}, R(Z, X)\tilde{p})Y).
\end{aligned}$$

(iii) Direct calculations give

$$\begin{aligned}
\nabla^f_{[{}^HX,{}^HY]}{}^HZ &= \nabla^f_{{}^H[X,Y]}{}^HZ + \nabla^f_{{}^V(pR(X,Y))}{}^HZ \\
&= {}^H(\nabla_{[X,Y]}Z) + \frac{1}{2}{}^V(pR([X, Y], Z)) + \frac{1}{2f}Z(f){}^V(pR(X, Y)) \\
&\quad +\frac{f}{2\alpha}{}^H(R(\tilde{p}, R(Y, X)\tilde{p})Z).
\end{aligned}$$

Hence, we have

$$\begin{aligned}
R^f({}^HX, {}^HY){}^HZ &= {}^H(R(X, Y)Z) + \frac{f}{4\alpha}{}^H(R(\tilde{p}, R(Z, Y)\tilde{p})X) \\
&\quad -\frac{f}{4\alpha}{}^H(R(\tilde{p}, R(Z, X)\tilde{p})Y) + \frac{f}{2\alpha}{}^H(R(\tilde{p}, R(X, Y)\tilde{p})Z)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{4f}X(f)^V(pR(Y,Z)) - \frac{1}{4f}Y(f)^V(pR(X,Z)) \\
&-\frac{1}{2f}Z(f)^V(pR(X,Y)) + \frac{1}{2}{}^V\big(p(\nabla_X R)(Y,Z) \\
&-\frac{1}{2}{}^V\big(p(\nabla_Y R)(X,Z),
\end{aligned}
$$

where

$$
\begin{aligned}
{}^V\big(p(\nabla_X R)(Y,Z)\big) &= {}^V\big(p(\nabla_X(R(Y,Z)) - R(\nabla_X Y,Z) - R(Y,\nabla_X Z))\big) \\
&= {}^V\big(p\nabla_X(R(Y,Z)) - pR(\nabla_X Y,Z) - pR(Y,\nabla_X Z)\big) \\
&= {}^V\big(\nabla_X(pR(Y,Z)) - (\nabla_X p)R(Y,Z) - pR(\nabla_X Y,Z) \\
&\quad -pR(Y,\nabla_X Z)\big).
\end{aligned}
$$

Using the second Bianchi identity, we obtain

$$
{}^V\big(p(\nabla_X R)(Y,Z)\big) - {}^V\big(p(\nabla_Y R)(X,Z)\big) = -{}^V\big(p(\nabla_Z R)(X,Y)\big),
$$

which gives the formula (4.2). The other formulas are obtained by a similar calculation. □

Proposition 4.1. *Let (M^m, g) be a Riemannian manifold and (T^*M, g^f) its cotangent bundle equipped with the generalized Cheeger-Gromoll metric. If (T^*M, g^f) is flat, then (M^m, g) is flat.*

Proof. It is easy to see from (4.2) If we assume that $R^f = 0$ and calculate the Riemannian curvature tensor for three horizontal vector fields at $(x, 0)$ we get

$$
R^f_{(x,0)}({}^HX, {}^HY)^HZ = {}^H(R_x(X,Y)Z) = 0.
$$

□

Let $(x, p) \in T^*M$ with $p \neq 0$, $\{E_i\}_{i=\overline{1,m}}$ and $\{\omega^i\}_{i=\overline{1,m}}$ be a local orthonormal frame and coframe on M^m, respectively, such that $\omega^1 = \frac{p}{|p|}$, then

$$
\big\{F_i = E_i^H, F_{m+1} = \frac{1}{\sqrt{f}}{}^V\omega^1, F_{m+j} = \sqrt{\frac{\alpha}{f}}{}^V\omega^j\big\}_{i=\overline{1,m}, j=\overline{2,m}} \tag{4.8}
$$

is a local orthonormal frame on T^*M.

Theorem 4.2. *Let (M^m, g) be a Riemannian manifold and (T^*M, g^f) its cotangent bundle equipped with the generalized Cheeger-Gromoll metric. If Ric (resp. Ric^f) denotes the Ricci curvature of (M^m, g) (resp. (T^*M, g^f)), then we have*

$$
\begin{aligned}
Ric^f({}^HX, {}^HY) &= Ric(X,Y) - \frac{f}{2\alpha}\sum_{a=1}^{m} g(R(E_a,X)\tilde{p}, R(E_a,Y)\tilde{p}) \\
&\quad -\frac{m}{2f}Hess^f(X,Y) + \frac{m}{4f^2}X(f)Y(f), \qquad (4.9)
\end{aligned}
$$

$$
\begin{aligned}
Ric^f({}^HX, {}^V\theta) &= \frac{f}{2\alpha}\sum_{a=1}^{m} g((\nabla_{E_a}R)(\tilde{p},\tilde{\theta})X, E_a) \\
&\quad + \frac{m+4}{4\alpha} g(R(\tilde{p},\tilde{\theta})X, grad f),
\end{aligned} \tag{4.10}
$$

$$
\begin{aligned}
Ric^f({}^V\omega, {}^V\theta) &= \frac{f^2}{4\alpha^2}\sum_{a=1}^{m} g(R(\tilde{p},\widetilde{\omega})E_a, R(\tilde{p},\tilde{\theta})E_a) \\
&\quad -\Big(\frac{\Delta(f)}{2f} + \frac{(m-2)|grad f|^2}{4f^2} + \frac{\alpha(m-2)-m-1}{f\alpha^2}\Big) g^f({}^V\omega, {}^V\theta) \\
&\quad + \frac{(\alpha+2)(m-2)}{\alpha^2} g^{-1}(\omega,\theta)
\end{aligned} \tag{4.11}
$$

for all $X, Y \in \Upsilon^1_0(M^m)$ *and* $\omega, \theta \in \Upsilon^0_1(M^m)$.

Proof. In here, we use the local orthonormal frame (4.8) on T^*M.
$i)$ From the formula (4.2), we have

$$
\begin{aligned}
Ric^f({}^HX, {}^HY) &= \sum_{a=1}^{m} g^f(R^f({}^HE_a, {}^HX){}^HY, {}^HE_a) + \frac{1}{f} g^f(R^f({}^V\omega^1, {}^HX){}^HY, {}^V\omega^1) \\
&\quad + \frac{\alpha}{f}\sum_{a=2}^{m} g^f(R^f({}^V\omega^a, {}^HX){}^HY, {}^V\omega^a) \\
&= \sum_{a=1}^{m}\Big(g(R(E_a, X)Y, E_a) - \frac{f}{4\alpha} g(R(\tilde{p}, R(Y, E_a)\tilde{p})X, E_a) \\
&\quad + \frac{f}{2\alpha} g(R(\tilde{p}, R(E_a, X)\tilde{p})Y, E_a)\Big) \\
&\quad - \frac{1}{2f^2} Hess^f(X,Y) g^f({}^V\omega^1, {}^V\omega^1) + \frac{1}{4f^3} X(f)Y(f) g^f({}^V\omega^1, {}^V\omega^1) \\
&\quad + \sum_{a=2}^{m}\Big(-\frac{1}{4} g^f({}^V(pR(X, R(\tilde{p}, \widetilde{\omega^a})Y)), {}^V\omega^a) \\
&\quad - \frac{\alpha}{2f^2} Hess^f(X,Y) g^f({}^V\omega^a, {}^V\omega^a) + \frac{\alpha}{4f^3} X(f)Y(f) g^f({}^V\omega^a, {}^V\omega^a)\Big).
\end{aligned}
$$

In order to simplify the last expression, we have

$$
\begin{aligned}
\widetilde{\omega^a} &= \sum_{i=1}^{m} g(\widetilde{\omega^a}, E_i)E_i = \sum_{i,h,k=1}^{m} g_{hk}\widetilde{\omega^a}^h E_i^k E_i = \sum_{i,h,k,j=1}^{m} g_{hk} g^{jh}\omega^a_j E_i^k E_i \\
&= \sum_{i,k,j=1}^{m} \delta^j_k \omega^a_j E_i^k E_i = \sum_{i,j=1}^{m} \omega^a_j E_i^j E_i = \sum_{i=1}^{m} \omega^a(E_i)E_i = \sum_{i=1}^{m} \delta^a_i E_i \\
&= E_a.
\end{aligned} \tag{4.12}
$$

With simple calculation we have

$$\begin{aligned} g^f({}^V(pR(X,R(\tilde{p},\widetilde{\omega^a})Y)),{}^V\omega^a) &= \frac{f}{\alpha}g^{-1}(pR(X,R(\tilde{p},\widetilde{\omega^a})Y),\omega^a) \\ &= \frac{f}{\alpha}g(\widetilde{pR(X,R(\tilde{p},\widetilde{\omega^a})Y)},\widetilde{\omega^a}), \end{aligned}$$

using (2.4) and (4.12)

$$\begin{aligned} g^f({}^V(pR(X,R(\tilde{p},\widetilde{\omega^a})Y)),{}^V\omega^a) &= \frac{f}{\alpha}g(R(R(\tilde{p},\widetilde{\omega^a})Y,X)\tilde{p},\widetilde{\omega^a}) \\ &= \frac{f}{\alpha}g(R(\tilde{p},\widetilde{\omega^a})R(\tilde{p},\widetilde{\omega^a})Y,X) \\ &= -\frac{f}{\alpha}g(R(\tilde{p},E_a)X,R(\tilde{p},E_a)Y), \end{aligned}$$

then we find

$$\begin{aligned} Ric^f({}^HX,{}^HY) &= Ric(X,Y)-\frac{3f}{4\alpha}\sum_{a=1}^{m}g(R(E_a,X)\tilde{p},R(E_a,Y)\tilde{p}) \\ &\quad -\frac{m}{2f}Hess^f(X,Y)+\frac{m}{4f^2}X(f)Y(f) \\ &\quad +\frac{f}{4\alpha}\sum_{a=1}^{m}g(R(\tilde{p},E_a)X,R(\tilde{p},E_a)Y). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \sum_{a=1}^{m}g(R(E_a,X)\tilde{p},R(E_a,Y)\tilde{p}) &= \sum_{a,s=1}^{m}g(R(E_a,X)\tilde{p},E_s)g(R(E_a,Y)\tilde{p},E_s) \\ &= \sum_{a,s=1}^{m}g(R(\tilde{p},E_s)E_a,X)g(R(\tilde{p},E_s)E_a,Y) \\ &= \sum_{a,s=1}^{m}g(R(\tilde{p},E_s)X,E_a)g(R(\tilde{p},E_s)Y,E_a) \\ &= \sum_{s=1}^{m}g(R(\tilde{p},E_s)X,R(\tilde{p},E_s)Y) \\ &= \sum_{a=1}^{m}g(R(\tilde{p},E_a)X,R(\tilde{p},E_a)Y). \end{aligned} \tag{4.13}$$

Hence, we get

$$\begin{aligned} Ric^f({}^HX,{}^HY) &= Ric(X,Y)-\frac{f}{2\alpha}\sum_{a=1}^{m}g(R(E_a,X)\tilde{p},R(E_a,Y)\tilde{p}) \\ &\quad -\frac{m}{2f}Hess^f(X,Y)+\frac{m}{4f^2}X(f)Y(f). \end{aligned}$$

The other formulas are obtained by a similar calculation. □

It is known that the sectional curvature K^f on $(T^*M, \tilde{g})$ for P is given by

$$K^f(V,W) = \frac{g^f(R^f(V,W)W,V)}{g^f(V,V)g^f(W,W) - g^f(V,W)^2}, \tag{4.14}$$

where $P = P(V,W)$ denotes the plane spanned by $\{V,W\}$, for all linearly independent vector fields $V,W \in \Upsilon_0^1(T^*M)$.

Let $K^f({}^HX, {}^HY)$, $K^f({}^HX, {}^V\theta)$ and $K^f({}^V\omega, {}^V\theta)$ denote the sectional curvature of the plane spanned by $\{{}^HX, {}^HY\}$, $\{{}^HX, {}^V\theta\}$ and $\{{}^V\omega, {}^V\theta\}$ on $(T^*M, \tilde{g})$, respectively, where X,Y are orthonormal vector fields and ω, θ are orthonormal covector fields on M^m.

Proposition 4.2. *Let (M^m, g) be a Riemannian manifold and (T^*M, g^f) its cotangent bundle equipped with the generalized Cheeger-Gromoll metric. Then we have the following*

$$\begin{aligned}
g^f(R^f({}^HX, {}^HY){}^HY, {}^HX) &= g(R(X,Y)Y,X) - \frac{3f}{4\alpha}|R(X,Y)\tilde{p}|^2, \\
g^f(R^f({}^HX, {}^V\theta){}^V\theta, {}^HX) &= \big(\frac{X(f)^2}{4\alpha f} - \frac{1}{2\alpha}Hess^f(X,X)\big)\big(1 + g^{-1}(\theta,p)^2\big) \\
&\quad + \frac{f^2}{4\alpha^2}|R(\tilde{p},\tilde{\theta})X|^2, \\
g^f(R^f({}^V\omega, {}^V\theta){}^V\theta, {}^V\omega) &= \big(\frac{(1-\alpha)f}{\alpha^4} - \frac{|grad f|^2}{4\alpha^2}\big)\big(1 + g^{-1}(\omega,p)^2 + g^{-1}(\theta,p)^2\big) \\
&\quad + \frac{(\alpha+2)f}{\alpha^3}
\end{aligned}$$

for all $X,Y \in \Upsilon_0^1(M^m)$ and $\omega, \theta \in \Upsilon_1^0(M^m)$.

Proof. i) From the formula (4.2), we have

$$\begin{aligned}
\tilde{g}(R^f({}^HX, {}^HY){}^HY, {}^HX) &= g(R(X,Y)Y,X) - \frac{f}{4\alpha}g(R(\tilde{p}, R(Y,X)\tilde{p})Y,X) \\
&\quad + \frac{f}{2\alpha}g(R(\tilde{p}, R(X,Y)\tilde{p})Y,X) \\
&= g(R(X,Y)Y,X) - \frac{3f}{4\alpha}|R(X,Y)\tilde{p}|^2.
\end{aligned}$$

ii) From the formula (4.5), we have

$$\begin{aligned}
g^f(R^f({}^HX, {}^V\theta){}^V\theta, {}^HX) &= -\frac{1}{2\alpha}\big(1 + g^{-1}(\theta,p)^2\big)Hess^f(X,X) \\
&\quad + \frac{X(f)^2}{4\alpha f}\big(1 + g^{-1}(\theta,p)^2\big) + \frac{f^2}{4\alpha^2}|R(\tilde{p},\tilde{\theta})X|^2
\end{aligned}$$

$$= \big(\frac{X(f)^2}{4\alpha f} - \frac{1}{2\alpha}Hess^f(X,X)\big)\big(1+g^{-1}(\theta,p)^2\big) + \frac{f^2}{4\alpha^2}|R(\tilde{p},\tilde{\theta})X|^2.$$

iii) The result follows immediately from the formula (4.7)

$$\begin{aligned} g^f(R^f({}^V\omega,{}^V\theta){}^V\theta,{}^V\omega) &= \big(\frac{\alpha^2+\alpha+1}{\alpha^3} - \frac{|grad f|^2}{4\alpha f}\big)\frac{f}{\alpha}\big(1+g^{-1}(\omega,p)^2\big) \\ &\quad +\big(\frac{1-\alpha}{\alpha^3} - \frac{|grad f|^2}{4\alpha f}\big)\frac{f}{\alpha}g^{-1}(\theta,p)^2 - \frac{(\alpha+2)f}{\alpha^3}g^{-1}(\omega,p)^2 \\ &= \big(\frac{(1-\alpha)f}{\alpha^4} - \frac{|grad f|^2}{4\alpha^2}\big)\big(1+g^{-1}(\omega,p)^2+g^{-1}(\theta,p)^2\big) \\ &\quad +\frac{(\alpha+2)f}{\alpha^3}. \end{aligned}$$

□

Theorem 4.3. *Let (M^m,g) be a Riemannian manifold and (T^*M,g^f) its cotangent bundle equipped with the generalized Cheeger-Gromoll metric. The sectional curvature K^f satisfies the following equations*

$$\begin{aligned} K^f({}^HX,{}^HY) &= K(X,Y) - \frac{3f}{4\alpha}|R(X,Y)\tilde{p}|^2, \\ K^f({}^HX,{}^V\theta) &= \frac{f|R(\tilde{p},\tilde{\theta})X|^2}{4\alpha\big(1+g^{-1}(\theta,p)^2\big)} + \frac{X(f)^2}{4f^2} - \frac{1}{2f}Hess^f(X,X), \\ K^f({}^V\omega,{}^V\theta) &= \frac{1-\alpha}{f\alpha^2} + \frac{\alpha+2}{\alpha f}\frac{1}{1+g^{-1}(\omega,p)^2+g^{-1}(\theta,p)^2} - \frac{|grad f|^2}{4f^2} \end{aligned}$$

for all $X,Y\in\Upsilon^1_0(M^m)$ and $\omega,\theta\in\Upsilon^0_1(M^m)$, where K denotes the sectional curvature tensor of (M^m,g).

Proof. Using the Proposition 4.2 and direct calculations we have

$$\begin{aligned} (1)\, K^f({}^HX,{}^HY) &= \frac{g^f(R^f({}^HX,{}^HY){}^HY,{}^HX)}{g^f({}^HX,{}^HX)g^f({}^HY,{}^HY) - g^f({}^HX,{}^HY)^2} \\ &= K(X,Y) - \frac{3f}{4}|pR(X,Y)|^2, \end{aligned}$$

$$\begin{aligned} (2)\, K^f({}^HX,{}^V\theta) &= \frac{g^f(R^f({}^HX,{}^V\theta){}^V\theta,{}^HX)}{g^f({}^HX,{}^HX)g^f({}^V\theta,{}^V\theta) - g^f({}^HX,{}^V\theta)^2} \\ &= \frac{f|R(\tilde{p},\tilde{\theta})X|^2}{4\alpha\big(1+g^{-1}(\theta,p)^2\big)} + \frac{X(f)^2}{4f^2} - \frac{1}{2f}Hess^f(X,X), \end{aligned}$$

$$
\begin{aligned}
(3)\, K^f({}^V\omega,{}^V\theta) &= \frac{g^f(R^f({}^V\omega,{}^V\theta){}^V\theta,{}^V\omega)}{g^f({}^V\omega,{}^V\omega)g^f({}^V\theta,{}^V\theta)-g^f({}^V\omega,{}^V\theta)^2}\\
&= \frac{1-\alpha}{f\alpha^2}+\frac{\alpha+2}{\alpha f}\frac{1}{1+g^{-1}(\omega,p)^2+g^{-1}(\theta,p)^2}-\frac{|grad f|^2}{4f^2}.
\end{aligned}
$$

□

Proposition 4.3. *Let (M^m, g) be a Riemannian manifold of constant sectional curvature κ and (T^*M, g^f) its cotangent bundle equipped with the generalized Cheeger-Gromoll metric. Then the sectional curvature K^f satisfies the following equations*

$$
\begin{aligned}
K^f({}^HX,{}^HY) &= \kappa-\frac{3f\kappa^2}{4\alpha}\big(g(X,\tilde p)^2+g(Y,\tilde p)^2\big),\\
K^f({}^HX,{}^V\theta) &= \frac{f\kappa^2\big(g(X,\tilde\theta)^2|p|^2-2g(X,\tilde\theta)g(X,\tilde p)g(\tilde\theta,\tilde p)+g(X,\tilde p)^2\big)}{4\alpha(1+g^{-1}(\omega,p)^2)}\\
&\quad+\frac{X(f)^2}{4f^2}-\frac{1}{2f}Hess^f(X,X),\\
K^f({}^V\omega,{}^V\theta) &= \frac{1-\alpha}{f\alpha^2}+\frac{\alpha+2}{\alpha f}\frac{1}{1+g^{-1}(\omega,p)^2+g^{-1}(\theta,p)^2}-\frac{|grad f|^2}{4f^2}
\end{aligned}
$$

for all $X, Y \in \Upsilon^1_0(M^m)$ and $\omega,\theta \in \Upsilon^0_1(M^m)$.

Proof. If M^m has constant curvature κ, then for all $U, V, W \in \Upsilon^1_0(M^m)$ we get

$$R(U,V)W = \kappa\big(g(V,W)U - g(U,W)V\big).$$

From direct calculations we get

$$
\begin{aligned}
|R(Y,X)\tilde p|^2 &= \kappa^2\big(g(X,\tilde p)^2+g(Y,\tilde p)^2\big),\\
|R(\tilde p,\tilde\theta)X|^2 &= \kappa^2\big(g(X,\tilde\theta)^2|p|^2-2g(X,\tilde\theta)g(X,\tilde p)g(\tilde\theta,\tilde p)+g(X,\tilde p)^2\big),
\end{aligned}
$$

which completes the proof. □

Theorem 4.4. *Let (M^m, g) be a Riemannian manifold and (T^*M, g^f) its cotangent bundle equipped with the generalized Cheeger-Gromoll metric. If σ (resp. σ^f) denotes the scalar curvature of (M^m, g) (resp. (T^*M, g^f)), then we have*

$$
\begin{aligned}
\sigma^f &= \sigma-\frac{f}{4\alpha}\sum_{a,b=1}^{m}|R(E_a,E_b)\tilde p|^2-\frac{m(m-3)}{4f^2}|grad f|^2-\frac{m}{f}\Delta(f)\\
&\quad+\frac{(m-1)}{f\alpha^2}\Big(6+(m-2)(\alpha^2+\alpha+1)\Big),
\end{aligned}
\tag{4.15}
$$

where $\Delta(f)$ is the Laplacian of f.

Proof. Let $(F_k)_{k=\overline{1,2m}}$ be a local orthonormal frame on (T^*M, g^f) defined by (4.8). Using Theorem 4.2 and the definition of the scalar curvature, we have

$$\sigma^f = \sum_{b=1}^{m} Ric^f(F_b, F_b) + Ric^f(F_{m+1}, F_{m+1}) + \sum_{b=2}^{m} Ric^f(F_{m+b}, F_{m+b}).$$

Using (4.9), we have

$$\begin{aligned}\sum_{b=1}^{m} Ric^f(F_b, F_b) &= \sum_{b=1}^{m} Ric^f({}^H E_b, {}^H E_b) \\ &= \sigma - \frac{f}{2\alpha}\sum_{a,b=1}^{m} |R(E_a, E_b)\tilde{p}|^2 + \frac{m}{4f^2}|grad f|^2 - \frac{m}{2f}\Delta(f).\end{aligned}$$

Using (4.11), we have

$$\begin{aligned}Ric^f(F_{m+1}, F_{m+1}) &= \frac{1}{f} Ric^f({}^V\omega^1, {}^V\omega^1) \\ &= -\frac{m-2}{4f^2}|grad f|^2 - \frac{1}{2f}\Delta(f) + \frac{3(m-1)}{f\alpha^2}.\end{aligned}$$

and

$$\begin{aligned}\sum_{b=2}^{m} Ric^f(F_{m+b}, F_{m+b}) &= \sum_{b=2}^{m} \frac{\alpha}{f} Ric^f({}^V\omega^b, {}^V\omega^b) \\ &= \frac{f}{4\alpha}\sum_{a,b=1}^{m} |R(\tilde{p}, E_b)E_a|^2 - \frac{(m-2)(m-1)}{4f^2}|grad f|^2 \\ &\quad -\frac{m-1}{2f}\Delta(f) + \frac{m-1}{f\alpha^2}\big((\alpha^2+\alpha)(m-2) + m + 1\big).\end{aligned}$$

$$\begin{aligned}\sigma^f &= \sigma - \frac{f}{2\alpha}\sum_{a,b=1}^{m} |R(E_a, E_b)\tilde{p}|^2 + \frac{m}{4f^2}|grad f|^2 - \frac{m}{2f}\Delta(f) \\ &\quad -\frac{m-2}{4f^2}|grad f|^2 - \frac{1}{2f}\Delta(f) + \frac{3(m-1)}{f\alpha^2} \\ &\quad +\frac{f}{4\alpha}\sum_{a,b=1}^{m} |R(\tilde{p}, E_b)E_a|^2 - \frac{(m-2)(m-1)}{4f^2}|grad f|^2 \\ &\quad -\frac{m-1}{2f}\Delta(f) + \frac{m-1}{f\alpha^2}\big((\alpha^2+\alpha)(m-2) + m + 1\big).\end{aligned}$$

In order to simplify the last expression, we have

$$\sum_{a,b=1}^{m} |R(\tilde{p}, E_b)E_a|^2 = \sum_{a,b=1}^{m} |R(E_a, E_b)\tilde{p}|^2,$$

by (4.13), hence

$$\sigma^f = \sigma - \frac{f}{4\alpha}\sum_{a,b=1}^{m}|R(E_a,E_b)\tilde{p}|^2 - \frac{m(m-3)}{4f^2}|grad f|^2 - \frac{m}{f}\Delta(f) + \frac{m-1}{f\alpha^2}\Big(6+(m-2)(\alpha^2+\alpha+1)\Big).$$

□

Proposition 4.4. *Let (M^m, g) be a Riemannian manifold of constant sectional curvature κ and (T^*M, g^f) its cotangent bundle equipped with the generalized Cheeger-Gromoll metric. If σ^f denotes the scalar curvature of (T^*M, g^f), then we have*

$$\sigma^f = (m-1)\kappa\Big(m - \frac{\kappa(\alpha-1)f}{2\alpha}\Big) - \frac{m(m-3)}{4f^2}|grad f|^2 - \frac{m}{f}\Delta(f) + \frac{m-1}{f\alpha^2}\Big(6+(m-2)(\alpha^2+\alpha+1)\Big). \tag{4.16}$$

Proof. If M^m has constant curvature κ, then for all $U,V,W \in \Upsilon_0^1(M^m)$,

$$R(U,V)W = \kappa\big(g(V,W)U - g(U,W)V\big),$$

$$\sigma = m(m-1)\kappa$$

and

$$\sum_{i,j=1}^{m}|R(E_i,E_j)\tilde{p}|^2 = 2\kappa^2(m-1)|p|^2 = 2\kappa^2(m-1)(\alpha-1).$$

This completes the proof. □

5. The generalized Cheeger-Gromoll metric on the unit cotangent bundle T_1^*M

The cotangent sphere bundle of radius $r > 0$ over a Riemannian manifold (M^m, g) is the hypersurface

$$T_r^*M = \big\{(x,p) \in T^*M,\ g^{-1}(p,p) = r^2\big\}.$$

When $r = 1$, T_1^*M is called the unit cotangent (sphere) bundle such that

$$T_1^*M = \big\{(x,p) \in T^*M,\ g^{-1}(p,p) = 1\big\}. \tag{5.1}$$

If we set

$$\begin{aligned} F : T^*M &\to \mathbb{R} \\ (x,p) &\mapsto F(x,p) = g^{-1}(p,p) - 1, \end{aligned}$$

then the hypersurface T_1^*M is given by

$$T_1^*M = \{(x,p) \in T^*M, \quad F(x,p) = 0\}$$

and $grad^f F$ (the gradient of F with respect to g^f) is a normal vector field to T_1^*M. From the Lemma 3.1, for any $X \in \Upsilon_0^1(M^m)$ and $\omega \in \Upsilon_1^0(M^m)$, we get

$$g^f({}^HX, grad^f F) = {}^HX(F) = {}^HX(g^{-1}(p,p) - 1) = 0,$$
$$g^f({}^V\omega, grad^f F) = {}^V\omega(F) = {}^V\omega(g^{-1}(p,p) - 1) = 2g^{-1}(\omega,p) = \frac{2}{f} g^f({}^V\omega, {}^V\mathcal{P}).$$

So

$$grad^f F = \frac{2}{f} {}^V\mathcal{P}.$$

Then the unit normal vector field to T_1^*M is given by

$$\mathcal{N} = \frac{grad^f F}{\sqrt{g^f(grad^f F, grad^f F)}} = \frac{{}^V\mathcal{P}}{\sqrt{g^f({}^V\mathcal{P}, {}^V\mathcal{P})}} = \frac{1}{\sqrt{f}} {}^V\mathcal{P}.$$

The tangential lift ${}^T\omega$ with respect to g^f of a covector $\omega \in T_x^*M$ to $(x,p) \in T_1^*M$ as the tangential projection of the vertical lift of ω to (x,p) with respect to $\mathcal{N}$, that is,

$${}^T\omega = {}^V\omega - g^f_{(x,p)}({}^V\omega, \mathcal{N}_{(x,p)})\mathcal{N}_{(x,p)} = {}^V\omega - g_x^{-1}(\omega,p) {}^V\mathcal{P}_{(x,p)}.$$

For the sake of notational clarity, we will use $\overline{\omega} = \omega - g^{-1}(\omega,p)p$, then ${}^T\omega = {}^V\overline{\omega}$. From the above, we get the direct sum decomposition

$$T_{(x,p)}T^*M = T_{(x,p)}T_1^*M \oplus span\{\mathcal{N}_{(x,p)}\} = T_{(x,p)}T_1^*M \oplus span\{{}^V\mathcal{P}_{(x,p)}\}, \tag{5.2}$$

where $(x,p) \in T_1^*M$.

Indeed, if $W \in T_{(x,p)}T^*M$, then there exist $X \in T_xM$ and $\omega \in T_x^*M$ such that

$$\begin{aligned} W &= {}^HX + {}^V\omega \\ &= {}^HX + {}^T\omega + g^f_{(x,p)}({}^V\omega, \mathcal{N}_{(x,p)})\mathcal{N}_{(x,p)} \\ &= {}^HX + {}^T\omega + g_x^{-1}(\omega,p) {}^V\mathcal{P}_{(x,p)}. \end{aligned} \tag{5.3}$$

From (5.3) we can say that the tangent space $T_{(x,p)}T_1^*M$ of T_1^*M at (x,p) is given by

$$T_{(x,p)}T_1^*M = \{{}^HX + {}^T\omega \,/\, X \in T_xM, \omega \in \{p\}^\perp \subset T_x^*M\},$$

where $\{p\}^\perp = \{\omega \in T_x^*M,\ g^{-1}(\omega,p) = 0\}$. Hence $T_{(x,p)}T_1^*M$ is spanned by the vectors of the form ${}^HX$ and ${}^T\omega$.

Given a covector field ω on M^m, the tangential lift ${}^T\omega$ of ω is given by

$${}^T\omega_{(x,p)} = \left({}^V\omega - g^f({}^V\omega, \mathcal{N})\mathcal{N}\right)_{(x,p)} = {}^V\omega_{(x,p)} - g_x^{-1}(\omega_x, p) {}^V\mathcal{P}_{(x,p)}. \tag{5.4}$$

For any $X \in \Upsilon_0^1(M^m)$ and $\omega \in \Upsilon_1^0(M^m)$, we have the following

(1) $g^f({}^H X, \mathcal{N}) = 0$,

(2) $g^f({}^T\omega, \mathcal{N}) = 0$,

(3) ${}^T\omega = {}^V\omega$ if and only if $g^{-1}(\omega, p) = 0$,

(4) ${}^T\mathcal{P} = 0$,

(5) $g(\overline{\omega}, \mathcal{P}) = 0$.

Definition 5.1. Let (M^m, g) be a Riemannian manifold and (T^*M, g^f) its cotangent bundle equipped with the generalized Cheeger-Gromoll metric. The Riemannian metric $\hat{g}^f$ on T_1^*M, induced by $\hat{g}^f$, is completely determined by the identities

$$\begin{aligned} \hat{g}^f({}^H X, {}^H Y) &= g(X, Y), \\ \hat{g}^f({}^H X, {}^T\theta) &= 0, \\ \hat{g}^f({}^T\omega, {}^T\theta) &= \frac{f}{2}\big(g^{-1}(\omega, \theta) - g^{-1}(\omega, p)g^{-1}(\theta, p)\big) \end{aligned}$$

for all $X, Y \in \Upsilon_0^1(M^m)$ and $\omega, \theta \in \Upsilon_1^0(M^m)$.

We shall calculate the Levi-Civita connection $\widehat{\nabla}$ of T_1^*M with the generalized Cheeger-Gromoll metric $\hat{g}^f$. This connection is characterized by the formula

$$\widehat{\nabla}_U V = \nabla^f_U V - g^f(\nabla^f_U V, \mathcal{N})\mathcal{N} \tag{5.5}$$

for all $U, V \in \Upsilon_0^1(T^*M)$.

Theorem 5.1. *Let (M^m, g) be a Riemannian manifold and $(T_1^*M, \hat{g}^f)$ its unit cotangent bundle equipped with the generalized Cheeger-Gromoll metric. Then we have the following formulas*

$$\begin{aligned} 1.\ \widehat{\nabla}_{{}^H X}{}^H Y &= {}^H(\nabla_X Y) + \frac{1}{2}{}^T(pR(X, Y)), \\ 2.\ \widehat{\nabla}_{{}^H X}{}^T\theta &= {}^T(\nabla_X \theta) + \frac{1}{2f}X(f){}^T\theta + \frac{f}{4}{}^H(R(\tilde{p}, \tilde{\theta})X), \\ 3.\ \widehat{\nabla}_{{}^T\omega}{}^H Y &= \frac{1}{2f}Y(f){}^T\omega + \frac{f}{4}{}^H(R(\tilde{p}, \widetilde{\omega})Y), \\ 4.\ \widehat{\nabla}_{{}^T\omega}{}^T\theta &= -\frac{1}{4}\big(g^{-1}(\omega, \theta) - g^{-1}(\omega, p)g^{-1}(\theta, p)\big){}^H grad f - g^{-1}(\theta, p){}^T\omega \end{aligned}$$

for all $X, Y \in \Upsilon_0^1(M^m)$ and $\omega, \theta \in \Upsilon_1^0(M^m)$.

Proof. In the proof, we will use the Theorem 3.1, Lemma 3.3 and the formula (5.5).
1. By direct calculation, we have

$$\begin{aligned} \widehat{\nabla}_{{}^H X}{}^H Y &= \nabla^f_{{}^H X}{}^H Y - g^f(\nabla^f_{{}^H X}{}^H Y, \mathcal{N})\mathcal{N} \\ &= {}^H(\nabla_X Y) + \frac{1}{2}{}^V(pR(X, Y)) - g^f(\frac{1}{2}{}^V(pR(X, Y)), \mathcal{N})\mathcal{N} \\ &= {}^H(\nabla_X Y) + \frac{1}{2}{}^T(pR(X, Y)). \end{aligned}$$

2. We have $\widehat{\nabla}_{{}^{H}X}{}^{T}\theta = \nabla^{f}_{{}^{H}X}{}^{T}\theta - g^{f}(\nabla^{f}_{{}^{H}X}{}^{T}\theta, \mathcal{N})\mathcal{N}$, by direct calculation, we get

$$\nabla^{f}_{{}^{H}X}{}^{T}\theta = {}^{T}(\nabla_X\theta) + \frac{1}{2f}X(f)^{T}\theta + \frac{f}{4}{}^{H}(R(\tilde{p},\tilde{\theta})X) \ \ and \quad g^{f}(\nabla^{f}_{{}^{H}X}{}^{T}\theta, \mathcal{N})\mathcal{N} = 0.$$

Hence

$$\widehat{\nabla}_{{}^{H}X}{}^{T}\theta = {}^{T}(\nabla_X\theta) + \frac{1}{2f}X(f)^{T}\theta + \frac{f}{4}{}^{H}(R(\tilde{p},\tilde{\theta})X).$$

3. Also, we have $\widehat{\nabla}_{{}^{T}\omega}{}^{H}Y = \nabla^{f}_{{}^{T}\omega}{}^{H}Y - g^{f}(\nabla^{f}_{{}^{T}\omega}{}^{H}Y, \mathcal{N})\mathcal{N}$, by direct calculation, we get

$$\nabla^{f}_{{}^{T}\omega}{}^{H}Y = \frac{1}{2f}Y(f)^{T}\omega + \frac{f}{4}{}^{H}(R(\tilde{p},\widetilde{\omega})Y) \ \ and \ \ g^{f}(\nabla^{f}_{{}^{T}\omega}{}^{H}Y, \mathcal{N})\mathcal{N} = 0.$$

Hence

$$\widehat{\nabla}_{{}^{T}\omega}{}^{H}X = \frac{1}{2f}Y(f)^{T}\omega + \frac{f}{4}{}^{H}(R(\tilde{p},\widetilde{\omega})Y).$$

4. In the same way above, we have $\widehat{\nabla}_{{}^{T}\omega}{}^{T}\theta = \nabla^{f}_{{}^{T}\omega}{}^{T}\theta - g^{f}(\nabla^{f}_{{}^{T}\omega}{}^{T}\theta, \mathcal{N})\mathcal{N}$,

$$\begin{aligned}\nabla^{f}_{{}^{T}\omega}{}^{T}\theta &= -\frac{1}{4}\big(g^{-1}(\omega,\theta) - g^{-1}(\omega,p)g^{-1}(\theta,p)\big)^{H}grad f - g^{-1}(\theta,p)^{V}\omega \\ &\quad + \big(\frac{-1}{4}g^{-1}(\omega,\theta) + \frac{5}{4}g^{-1}(\omega,p)g^{-1}(\theta,p)\big)^{V}\mathcal{P}\end{aligned}$$

and

$$g^{f}(\nabla^{f}_{{}^{T}X}{}^{T}Y, \mathcal{N})\mathcal{N} = \big(-g^{-1}(\omega,p)g^{-1}(\theta,p) - \frac{1}{4}g^{-1}(\omega,\theta) + \frac{5}{4}g^{-1}(\omega,p)g^{-1}(\theta,p)\big)^{V}\mathcal{P}.$$

Hence

$$\widehat{\nabla}_{{}^{T}\omega}{}^{T}\theta = -\frac{1}{4}\big(g^{-1}(\omega,\theta) - g^{-1}(\omega,p)g^{-1}(\theta,p)\big)^{H}grad f - g^{-1}(\theta,p)^{T}\omega.$$

□

Now, we shall calculate the Riemannian curvature tensor of T_1^*M with the generalized Cheeger-Gromoll metric $\hat{g}^f$.

Denoting by $\widehat{R}$ the Riemannian curvature tensor of $(T_1^*M, \hat{g}^f)$, from the Gauss equation for hypersurfaces we deduce that $\widehat{R}(U,V)W$ satisfies

$$\widehat{R}(U,V)W = {}^{t}(R^{f}(U,V)W) - B(U,W).A_{\mathcal{N}}V + B(V,W).A_{\mathcal{N}}U, \tag{5.6}$$

for all $U, V, W \in \Upsilon^1_0(T^*M)$, where ${}^{t}(R^f(U,V)W)$ is the tangential component of $R^f(U,V)W$ with respect to the direct sum decomposition (5.2), $A_{\mathcal{N}}$ is the shape operator of T_1^*M in (T^*M, g^f) derived from $\mathcal{N}$, and B is the second fundamental form of T_1^*M (as a hypersurface immersed in T^*M), associated with $\mathcal{N}$ on T_1^*M.

$A_{\mathcal{N}}U$ is the tangential component of $(-\nabla^f_U\mathcal{N})$, i.e.,

$$A_{\mathcal{N}}U = -{}^t(\nabla^f_U\mathcal{N}).$$

$B(U,V)$ is given by Gauss formula, $\nabla^f_U V = \widehat{\nabla}_U V + B(U,V).\mathcal{N}$. So

$$B(U,V) = g^f(\nabla^f_U V, \mathcal{N}).$$

Theorem 5.2. *Let (M^m, g) be a Riemannian manifold and $(T^*_1 M, \hat{g}^f)$ its unit cotangent bundle equipped with the generalized Cheeger-Gromoll metric. We have the following formulas*

$$\begin{aligned}
\widehat{R}({}^HX, {}^HY){}^HZ &= {}^H(R(X,Y)Z) + \frac{f}{8}{}^H(R(\tilde{p}, R(Z,Y)\tilde{p})X) \\
&\quad - \frac{f}{8}{}^H(R(\tilde{p}, R(Z,X)\tilde{p})Y) + \frac{f}{4}{}^H(R(\tilde{p}, R(X,Y)\tilde{p})Z) \\
&\quad + \frac{1}{4f}X(f){}^T(pR(Y,Z)) - \frac{1}{4f}Y(f){}^T(pR(X,Z)) \\
&\quad - \frac{1}{2f}Z(f){}^T(pR(X,Y)) - \frac{1}{2}{}^T\big(p(\nabla_Z R)(X,Y),
\end{aligned}$$

$$\begin{aligned}
\widehat{R}({}^HX, {}^T\theta){}^HZ &= \frac{1}{4}X(f){}^H(R(\tilde{p},\tilde{\theta})Z) + \frac{1}{8}Z(f){}^H(R(\tilde{p},\tilde{\theta})X) \\
&\quad + \frac{f}{4}{}^H((\nabla_X R)(\tilde{p},\tilde{\theta})Z) + \frac{1}{8}g^{-1}(pR(X,Z),\theta){}^H(grad f) \\
&\quad - \frac{1}{2}{}^V(\overline{\theta}R(X,Z)) + \frac{f}{8}{}^V(pR(X, R(\tilde{p},\tilde{\theta})Z)) \\
&\quad + \big(\frac{1}{2f}Hess^f(X,Z) - \frac{1}{4f^2}X(f)Z(f)\big)^T\theta,
\end{aligned}$$

$$\begin{aligned}
\widehat{R}({}^HX, {}^HY){}^T\eta &= \frac{f}{4}{}^H((\nabla_X R)(\tilde{p},\tilde{\eta})Y) - \frac{f}{4}{}^H((\nabla_Y R)(\tilde{p},\tilde{\eta})X) \\
&\quad + \frac{1}{8}X(f){}^H(R(\tilde{p},\tilde{\eta})Y) - \frac{1}{8}Y(f){}^H(R(\tilde{p},\tilde{\eta})X) \\
&\quad + \frac{1}{4}g^{-1}(pR(X,Y),\eta){}^H(grad f) - {}^V(\overline{\eta}R(X,Y)) \\
&\quad + \frac{f}{8}{}^V(pR(X, R(\tilde{p},\tilde{\eta})Y)) - \frac{f}{8}{}^V(pR(Y, R(\tilde{p},\tilde{\eta})X)),
\end{aligned}$$

$$\begin{aligned}
\widehat{R}({}^HX, {}^T\theta){}^T\eta &= -\frac{1}{4}g^{-1}(\overline{\theta},\overline{\eta}){}^H(\nabla_X grad f) + \frac{1}{8f}X(f)g^{-1}(\overline{\theta},\overline{\eta}){}^H(grad f) \\
&\quad - \frac{f}{4}{}^H(R(\tilde{\overline{\theta}},\tilde{\overline{\eta}})X) - \frac{f^2}{16}{}^H(R(\tilde{p},\tilde{\theta})R(\tilde{p},\tilde{\eta})X) \\
&\quad + \frac{1}{8}g^{-1}(\overline{\theta},\overline{\eta}){}^V(pR(X, grad f)) - \frac{1}{8}g(R(\tilde{p},\tilde{\eta})X, grad f)^T\theta,
\end{aligned}$$

$$\begin{aligned}\widehat{R}({}^T\omega, {}^T\theta)^H Z &= \frac{f}{2}{}^H(R(\widetilde{\overline{\omega}}, \widetilde{\overline{\theta}})Z) \\ &+\frac{f^2}{16}\big({}^H(R(\tilde{p},\widetilde{\omega})R(\tilde{p},\tilde{\theta})Z) - {}^H(R(\tilde{p},\tilde{\theta})R(\tilde{p},\widetilde{\omega})Z)\big) \\ &+\frac{1}{8}\big(g(R(\tilde{p},\tilde{\theta})Z, grad f)^T\omega - g(R(\tilde{p},\widetilde{\omega})Z, grad f)^T\theta\big),\end{aligned}$$

$$\begin{aligned}\widehat{R}({}^T\omega, {}^T\theta)^T\eta &= \frac{f}{16}\big(g^{-1}(\overline{\omega},\overline{\eta})^H(R(\tilde{p},\tilde{\theta})grad f) - g^{-1}(\overline{\theta},\overline{\eta})^H(R(\tilde{p},\widetilde{\omega})grad f)\big) \\ &+\big(1-\frac{|grad f|^2}{4\alpha f}\big)\big(g^{-1}(\overline{\theta},\overline{\eta})^T\omega - g^{-1}(\omega,\eta)^T\theta\big)\end{aligned}$$

for all $X, Y \in \Upsilon^1_0(M)$ *and* $\omega, \theta \in \Upsilon^0_1(M)$*, where* $\overline{\omega} = \omega - g^{-1}(\omega, p)p$ *and* $\widetilde{\overline{\omega}} = g^{-1}\circ\overline{\omega}$.

Proof. Using the Theorem 3.1 and Lemma 3.1, we obtain

$$A_{\mathcal{N}}{}^H X = 0\ , \quad A_{\mathcal{N}}{}^T\omega = -\frac{1}{2\sqrt{f}}{}^T\omega, \tag{5.7}$$

$$B({}^HX, {}^HY) = B({}^HX, {}^T\theta) = B({}^T\omega, {}^HY) = 0 \tag{5.8}$$

and

$$B({}^T\omega, {}^T\theta) = -\frac{\sqrt{f}}{4}g^{-1}(\overline{\omega}, \overline{\theta}). \tag{5.9}$$

It is sufficient to use the Theorem 4.1 and (5.6)-(5.9) for obtaining the required formulas for the curvature tensor (see [1]). □

6. Geodesics of the generalized Cheeger-Gromoll metric on the unit cotangent bundle T^*_1M

Let C be a parameterized curve on the cotangent bundle T^*M. Geometrically $C(t) = (x(t), \vartheta(t))$, where $x(t)$ is a curve on M^m and $\vartheta(t)$ is a covector field along this curve. Denote by $x' = \frac{dx}{dt} = \dot{x}$, $x'' = \nabla_{x'}x'$, $\vartheta' = \nabla_{x'}\vartheta$, $\vartheta'' = \nabla_{x'}\vartheta'$ and $C' = \frac{dC}{dt} = \dot{C}$. The following formula will be useful [13]:

$$C' = {}^Hx' + {}^V\vartheta'. \tag{6.1}$$

Lemma 6.1. *Let* (M^m, g) *be a Riemannian manifold,* $(T^*_1M, \hat{g}^f)$ *its unit cotangent bundle equipped with the generalized Cheeger-Gromoll metric and* $C(t) = (x(t), \vartheta(t))$ *be a curve on* T^*_1M*. Then we have*

$$C' = {}^Hx' + {}^T\vartheta'. \tag{6.2}$$

Proof. Using (6.1), we have

$$C' = {}^Hx' + {}^V\vartheta' = {}^Hx' + {}^T\vartheta' + g^{-1}(\vartheta', \vartheta)^V\vartheta.$$

Since $C(t) = (x(t), \vartheta(t)) \in T_1^*M$, $g^{-1}(\vartheta, \vartheta) = 1$. On the other hand

$$0 = x'\left(g^{-1}(\vartheta, \vartheta)\right) = 2g^{-1}(\vartheta', \vartheta),$$

i.e.,

$$g^{-1}(\vartheta', \vartheta) = 0.$$

Hence, the proof of the lemma is completed. □

Subsequently, let t be an arc length parameter on $C(t)$, from 6.2, we have

$$1 = |x'|^2 + \frac{f}{2}|\vartheta'|^2. \tag{6.3}$$

Theorem 6.1. *Let (M^m, g) be a Riemannian manifold, $(T_1^*M, \hat{g}^f)$ its unit cotangent bundle equipped with the generalized Cheeger-Gromoll metric and $C(t) = (x(t), \vartheta(t))$ be a curve on T_1^*M. Then $C(t)$ is a geodesic on T_1^*M if and only if*

$$\begin{cases} x'' &= \frac{f}{2}R(\tilde{\vartheta}', \tilde{\vartheta})x' + \frac{1}{4}|\vartheta'|^2 grad f, \\ \vartheta'' &= -\frac{1}{f}x'(f)\vartheta'. \end{cases} \tag{6.4}$$

Moreover,

$$\begin{cases} |\vartheta'| &= \frac{\kappa}{f}, \\ |x'| &= \sqrt{1 - \frac{\kappa}{2f}}, \end{cases} \tag{6.5}$$

where $\kappa = const. \geq 0$.

Proof. Using (6.2) and the Theorem 5.1, when we compute the derivative $\widehat{\nabla}_{C'}C'$, we find

$$\begin{aligned} \widehat{\nabla}_{C'}C' &= \widehat{\nabla}_{({}^Hx' + {}^T\vartheta')}({}^Hx' + {}^T\vartheta') \\ &= \widehat{\nabla}_{{}^Hx'}{}^Hx' + \widehat{\nabla}_{{}^Hx'}{}^T\vartheta' + \widehat{\nabla}_{{}^T\vartheta'}{}^Hx' + \widehat{\nabla}_{{}^T\vartheta'}{}^T\vartheta' \\ &= {}^Hx'' + {}^T\vartheta'' + \frac{1}{f}x'(f){}^T\vartheta' + \frac{f}{2}{}^H(R(\tilde{\vartheta}, \tilde{\vartheta}')x') - \frac{1}{4}g^{-1}(\vartheta', \vartheta')^H grad f \\ &= {}^H\big(x'' + \frac{f}{2}R(\tilde{\vartheta}, \tilde{\vartheta}')x' - \frac{1}{4}|\vartheta'|^2 grad f\big) + {}^T\big(\vartheta'' + \frac{1}{f}x'(f)\vartheta'\big). \end{aligned}$$

If $\widehat{\nabla}_{C'}C' = 0$, we obtain (6.4). Moreover, $x'\left(|\vartheta'|^2\right) = x'g(\vartheta', \vartheta') = 2g(\vartheta'', \vartheta')$. Using the second equation of the formula (6.4) we obtain

$$x'\left(|\vartheta'|^2\right) = -\frac{2}{f}x'(f)|\vartheta'|^2 \Rightarrow x'\left(\ln|\vartheta'|^2\right) = -2x'\left(\ln f\right) \Rightarrow f^2|\vartheta'|^2 = \kappa^2 = const.,$$

i.e., $|\vartheta'| = \frac{\kappa}{f}$, using (6.3) we find $|x'| = \sqrt{1 - \frac{\kappa}{2f}}$, where $\kappa = const \geq 0$. □

Corollary 6.1. *Let (M^m, g) be a Riemannian manifold and $(T_1^*M, \hat{g}^f)$ its unit cotangent bundle equipped with the generalized Cheeger-Gromoll metric. The curve $C(t) = (x(t), \widetilde{x'(t)})_{/T_1^*M}$ is a geodesic on $(T_1^*M, \hat{g}^f)$ if and only if $x(t)$ is a geodesic on (M^m, g).*

Proof. If $x'(t)$ is a vector field along the curve $x(t)$, then $\vartheta(t) = \widetilde{x'(t)}$ is a covector field along the curve $x(t)$. From (6.1), we find $\vartheta'(t) = \widetilde{x''(t)}$. By virtue of Theorem 6.1 we deduce $C(t) = (x(t), \vartheta(t))_{/T_1^*M}$ (the restriction of a curve $(x(t), \vartheta(t))$ on T_1^*M) is a geodesic on $(T_1^*M, \hat{g}^f)$ if and only if $x(t)$ is a geodesic on (M^m, g). □

A curve $C(t) = (x(t), \vartheta(t))$ on T^*M is said to be the horizontal lift of the curve $x(t)$ on M if and only if $\vartheta' = 0$ [11]. Using Theorem 6.1 we deduce the following corollary.

Corollary 6.2. *Let (M^m, g) be a Riemannian manifold and $(T_1^*M, \hat{g}^f)$ its unit cotangent bundle equipped with the generalized Cheeger-Gromoll metric. Let $C(t) = (x(t), \vartheta(t))$ be the horizontal lift of the curve $x(t)$. $C(t)_{/T_1^*M}$ is a geodesic on $(T_1^*M, \hat{g}^f)$ if and only if $x(t)$ is a geodesic on (M^m, g).*

Example 6.1. Let $\mathbb{R}$ be equipped with the Riemannian metric:

$$g = e^x dx^2.$$

The Christoffel symbol of the Riemannian connection of g is given by

$$\Gamma_{11}^1 = \frac{1}{2} g^{11} \left(\frac{\partial g_{11}}{\partial x^1} + \frac{\partial g_{11}}{\partial x^1} - \frac{\partial g_{11}}{\partial x^1}\right) = \frac{1}{2}.$$

The geodesics $x(t)$ such that $x(0) = a \in \mathbb{R}$, $x'(0) = v \in \mathbb{R}$ satisfy the equation

$$\frac{d^2x^k}{dt^2} + \sum_{i,j=1}^m \frac{dx^i}{dt}\frac{dx^j}{dt}\Gamma_{ij}^k = 0 \Leftrightarrow x'' + \frac{1}{2}(x')^2 = 0.$$

Then $x'(t) = \dfrac{2v}{2+vt}$ and $x(t) = a + 2\ln(1 + \dfrac{vt}{2})$.

1) $\widetilde{x'(t)} = \sum_{i,j=1}^m g_{ij} x'^j(t) dx^i = g_{11} x'(t) dx = \dfrac{e^a v(2+vt)}{2} dx$.

From Corollary 6.1, the curve $C_1(t)_{/T_1^*\mathbb{R}} = (x(t), \widetilde{x'(t)})_{/T_1^*\mathbb{R}}$ is a geodesic on $T_1^*\mathbb{R}$.

2) If $C_2(t) = (x(t), \vartheta(t))$ is the horizontal lift of the curve $x(t)$ and $\vartheta(t) = \vartheta_1(t)dx$, then

$$\frac{d\vartheta_h}{dt} - \Gamma_{jh}^i \frac{dx^j}{dt}\vartheta_i = 0 \Leftrightarrow \vartheta_1' - \frac{1}{2}\vartheta_1 x' = 0 \Leftrightarrow \vartheta_1(t) = k.\exp\left(\frac{1}{2}x'(t)\right).$$

Then $\vartheta_1(t) = k.\exp(\frac{v}{2+vt})$ and $\vartheta(t) = k.\exp(\frac{v}{2+vt})dx$.

From Corollary 6.2, the curve $C_2(t)_{/T_1^*\mathbb{R}}$ is a geodesic on $T_1^*\mathbb{R}$.

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