FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 39, No 1 (2024), 51–65
https://doi.org/10.22190/FUMI221122004U
Original Scientific Paper

A NEW GLANCE TO THE ASPECTS OF Q-HELICES

Yasin Ünlütürk[1], Cumali Ekici[2] and Doğan Ünal[3]

[1] **Faculty of Science, Kırklareli University**
Department of Mathematics, 39100 Kırklareli, Türkiye
[2] **Faculty of Science, Eskişehir Osmangazi University**
Department of Mathematics and Computer Sciences, 26040 Eskişehir, Türkiye
[3] **Faculty of Engineering, Kırklareli University**
Department of Software Engineering, 39100 Kırklareli, Türkiye

ORCID IDs: Yasin Ünlütürk https://orcid.org/0000-0002-6395-5272
Cumali Ekici https://orcid.org/0000-0002-3247-5727
Dogan Ünal https://orcid.org/0000-0001-5480-2998

Abstract. In this examination, we take q-helices into consideration. By q-helices, we mean curves due to the quasi-frame (abbv. q-frame) whose vector fields make constant angles with a non-zero fixed axis. One by one, all types of these q-helices we study in the work are therefore classified in three dimensional Euclidean space. Additionally, we study Darboux q-helices by using Darboux vector obtained with respect to q-frames fields of a curve. For a curve enclosed with q-frame as a general case, we reach some results for Darboux q-helices.
Keywords: q-frame, q-helices, the relations between q-helices, Darboux q-helices.

1. Introduction

A necessary and sufficient condition for a curve to be of constant slope is that the ratio of curvature to torsion be constant. This expression is a famous theorem characterizing helices which was proposed by M.A. Lancret in 1802, but its first proof was given by B. de Saint Venant in 1845 in his work published at Journal Ec. Polyt. 30, 1845, p. 26. [20].

Slant helices as more general forms of helices were conceptualized by Izumiya and Takeuchi [9]. Several authors introduced different types of helices and investigated their properties [10, 11, 12, 21].

Received November 22, 2022, accepted: January 21, 2023
Communicated by Mića Stanković
Corresponding Author: Yasin Ünlütürk. E-mail addresses: yasinunluturk@klu.edu.tr (Y. Ünlütürk), cekici@ogu.edu.tr (C. Ekici), doganunal@klu.edu.tr (D. Ünal)
2010 *Mathematics Subject Classification.* 53A04

Researches are increasing on $k-$type slant helices with their various aspects [13, 16, 17, 21]. The meaning of $k-$type slant helices is related to the class of curves having a property that the scalar product of frame's vector field and a fixed axis is constant [8]. For example, general helices are $0-$type helices and also 1-type slant helix is one whose normal vector field makes a constant angle with a non zero fixed axis. This subject "*$k-$type slant helices*" has been studied and developed in different types of spaces such as Euclidean, Galilean, and Lorentzian spaces [1, 13, 15, 18]. Another approach called as "k-type Darboux slant helices" is based on the idea that Darboux vector obtained by the frame fields in which curves' behaviour is taken into consideration makes a constant angle with a non-zero fixed axis is seen in the works [13, 16, 17, 23, 25].

The different suggestions to frame a curve such as parallel transport frame, Frenet frame, and etc. are prevalent approaches in differential geometry of curves [2, 3, 4, 5, 6, 22, 24]. The way to establish the quasi-frame was firstly paved with introducing the quasi normal vector of a space curve by Coquillart [3]. Then Shin et al. defined the quasi-normal vector for each point of the curve which lies in the plane perpendicular to the tangent of the curve at this point [19]. The local theory of space curves via q-frame was studied by Dede in [4].

In this research, q-helices by which we mean curves whose q-frame fields make a constant angle with a non-zero fixed axis. We give the necessary and sufficient conditions for curves due to the q-frame to be q-helices. Then we obtain some results of the relations between q-helices and Darboux q-helices. Also we classify Darboux q-helices as special ones whose Darboux vector makes a constant angle with a non-zero fixed axis by choosing the curve as one of the types of q-helices, and also the general case.

2. Preliminaries

The three dimensional Euclidean space $\mathbb{E}^3$ is a real vector space $\mathbb{R}^3$ equipped with

$$g = dx_1^2 + dx_2^2 + dx_3^3, \tag{2.1}$$

where (x_1, x_2, x_3) is a rectangular coordinate system of $\mathbb{E}^3$.

Let $\gamma : I \longrightarrow \mathbb{E}^3$ be an arc-length parametrized curve which has at least four continuous derivatives, then the curve γ has a natural frame called as Frenet frame with the equations below:

$$\begin{aligned} \mathbf{T}' &= \kappa \mathbf{N} \\ \mathbf{N}' &= -\kappa \mathbf{T} + \tau \mathbf{B} \\ \mathbf{B}' &= -\tau \mathbf{N} \end{aligned} \tag{2.2}$$

where κ and τ are the curvature and the torsion functions of the curve γ, respectively. We designate unit tangent vector field with $\mathbf{T}$, unit principle normal vector field with $\mathbf{N}$ and the unit binormal vector field with $\mathbf{B}$. We exclude the condition $\mathbf{T}'(s) = 0$ for some $s \in I$ along with this paper [7].

The quasi-frame (abbv. q-frame) as an alternative frame to Frenet trihedron has been introduced as follows: Given a space curve $\gamma(t)$, the q-frame composes of three orthonormal vectors. These vectors are the unit tangent vector $\mathbf{T}$, the quasi-normal $\mathbf{N}_q$ and the quasi-binormal vector $\mathbf{B}_q$, respectively. The q-frame $\{\mathbf{T}, \mathbf{N}_q, \mathbf{B}_q, \mathbf{k}\}$ is given by

$$\mathbf{T} = \frac{\gamma'}{\|\gamma'\|}, \mathbf{N}_q = \frac{\mathbf{T}\wedge\mathbf{k}}{\|\mathbf{T}\wedge\mathbf{k}\|}, \mathbf{B}_q = \mathbf{T}\wedge\mathbf{N}_q \tag{2.3}$$

where $\mathbf{k}$ is the projection vector.

For clarity, the projection vector $\mathbf{k}$ has been chosen as $\mathbf{k} = (0, 0, 1)$ along with the paper. Nevertheless, the q-frame is singular in all cases where $\mathbf{t}$ and $\mathbf{k}$ become parallel. Hence, in those cases where $\mathbf{t}$ and $\mathbf{k}$ are parallel, the projection vector $\mathbf{k}$ can be chosen as $\mathbf{k} = (0, 1, 0)$ or $\mathbf{k} = (1, 0, 0)$.

Let $\gamma(s)$ be a curve that is parameterized by arc length s. The variation equations of the q-frame is given ([4]) as

$$\begin{bmatrix} \mathbf{T}' \\ \mathbf{N}_q' \\ \mathbf{B}_q' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & k_3 \\ -k_2 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N}_q \\ \mathbf{B}_q \end{bmatrix}, \tag{2.4}$$

where the q-curvatures are

$$k_1 = \frac{\langle \mathbf{T}', \mathbf{N}_q\rangle}{\|\gamma'\|}, \quad k_2 = \frac{\langle \mathbf{T}', \mathbf{B}_q\rangle}{\|\gamma'\|}, \quad k_3 = -\frac{\langle \mathbf{N}_q, \mathbf{B}_q'\rangle}{\|\gamma'\|}. \tag{2.5}$$

3. The q-helices

In this section, we study different types of q-helices which means $k-$type slant helices of curves via q-frame in Euclidean 3-space $\mathbb{E}^3$. By q-helices, we intend the curves whose q-frame vector fields make a constant angle with a non-zero fixed axis. These types of helices within the q-frame are enclosed as depending on a constant angle between the tangent vector field $\mathbf{T}$ and the fixed vector $\mathbf{U}$, the quasi-normal vector field $\mathbf{N}_q$ and the fixed vector $\mathbf{U}$, and the quasi-binormal vector field $\mathbf{B}_q$ and the fixed vector $\mathbf{U}$.

Definition 3.1. A curve γ in $\mathbb{E}^3$ given by the q-frame $\{\mathbf{T}, \mathbf{N}_q, \mathbf{B}_q\}$ is called a slant helix of type-0, a slant helix of type-1 and a slant helix of type-2 if there exists a non zero fixed direction $\mathbf{U} \in \mathbb{E}^3$ such that satisfies, respectively,

$$\langle \mathbf{T}, \mathbf{U}\rangle = \cos\theta_1, \quad \langle \mathbf{N}_q, \mathbf{U}\rangle = \cos\theta_2, \quad \langle \mathbf{B}_q, \mathbf{U}\rangle = \cos\theta_3, \tag{3.1}$$

where θ_1, θ_2 and θ_3 are constant angles. The fixed direction $\mathbf{U}$ is called axis of the q-helices.

The vector $\mathbf{U}$ can be written as a combination of q-frame fields as subsequent

$$\mathbf{U} = \lambda_1\mathbf{T} + \lambda_2\mathbf{N}_q + \lambda_3\mathbf{B}_q, \tag{3.2}$$

where

$$\lambda_1 = \langle \mathbf{T}, \mathbf{U}\rangle, \lambda_2 = \langle \mathbf{N}_q, \mathbf{U}\rangle, \lambda_3 = \langle \mathbf{B}_q, \mathbf{U}\rangle.$$

Since U is a fixed vector field, its differentiation vanishes, that is,

$$\begin{aligned}\mathbf{U}' &= (\lambda_1' - \lambda_2 k_1 - \lambda_3 k_2)\mathbf{T} + (\lambda_2' + \lambda_1 k_1 - \lambda_3 k_3)\mathbf{N}_q + (\lambda_3' + \lambda_1 k_2 + \lambda_2 k_3)\mathbf{B}_q \\ &= 0.\end{aligned} \tag{3.3}$$

By (3.3), the following system is obtained as

$$\begin{aligned}\lambda_1' - \lambda_2 k_1 - \lambda_3 k_2 &= 0,\\ \lambda_2' + \lambda_1 k_1 - \lambda_3 k_3 &= 0,\\ \lambda_3' + \lambda_1 k_2 + \lambda_2 k_3 &= 0.\end{aligned} \tag{3.4}$$

In the following subsections, we study q-helices based on the system of differential equations (3.4).

3.1. The q-helices of type-0

Theorem 3.1. *Let γ be a curve due to the q-frame in $\mathbb{E}^3$. Then γ is a q-helix of type-0 if and only if*

$$\left(e^{-\int \frac{k_1 k_3}{k_2} ds} \int k_1 e^{\int \frac{k_1 k_3}{k_2} ds} ds\right) k_1 + \left(e^{\int \frac{k_2 k_3}{k_1} ds} \int k_2 e^{-\int \frac{k_2 k_3}{k_1} ds} ds\right) k_2 = 0. \tag{3.5}$$

Proof. A q-helix of type-0 satisfies the condition

$$\lambda_1 = \langle \mathbf{T}, \mathbf{U} \rangle = \cos\theta_1, \tag{3.6}$$

where θ_1 is a constant angle. Therefore, by substituting $\lambda_1 = \cos\theta_1$ into the system (3.4), it turns into

$$\begin{aligned}\lambda_2 k_1 + \lambda_3 k_2 &= 0,\\ \lambda_2' - \lambda_3 k_3 + \cos\theta_1 k_1 &= 0,\\ \lambda_3' + \lambda_2 k_3 + \cos\theta_1 k_2 &= 0.\end{aligned} \tag{3.7}$$

From $(3.7)_1$,

$$\lambda_3 = -\frac{k_1}{k_2}\lambda_2, \qquad \lambda_2 = -\frac{k_2}{k_1}\lambda_3. \tag{3.8}$$

By using (3.8) in the equations $(3.7)_1$, and $(3.7)_2$, we get the following linear differential equations of first order:

$$\lambda_2' + \frac{k_1 k_3}{k_2}\lambda_2 = -\cos\theta_1 k_1, \tag{3.9}$$

$$\lambda_3' - \frac{k_2 k_3}{k_1}\lambda_3 = -\cos\theta_1 k_2. \tag{3.10}$$

The solution of (3.9) is

$$\lambda_2 = -\cos\theta_1 e^{-\int \frac{k_1 k_3}{k_2} ds} \int k_1 e^{\int \frac{k_1 k_3}{k_2} ds} ds, \tag{3.11}$$

and the solution of (3.10) is

$$\lambda_3 = -\cos\theta_1 e^{\int \frac{k_2 k_3}{k_1} ds} \int k_2 e^{-\int \frac{k_2 k_3}{k_1} ds} ds. \tag{3.12}$$

Substituting (3.11) and (3.12) into $(3.7)_1$ gives the condition to be q-helices of type-0 as follows:

$$\left(e^{-\int \frac{k_1 k_3}{k_2} ds} \int k_1 e^{\int \frac{k_1 k_3}{k_2} ds} ds\right) k_1 + \left(e^{\int \frac{k_2 k_3}{k_1} ds} \int k_2 e^{-\int \frac{k_2 k_3}{k_1} ds} ds\right) k_2 = 0. \tag{3.13}$$

Conversely, suppose that the relation (3.5) holds, also the fixed vector filed $\mathbf{U}$ can be composed of

$$\begin{aligned} \mathbf{U} = \cos\theta_1 \mathbf{T} &- \left(\cos\theta_1 e^{-\int \frac{k_1 k_3}{k_2} ds} \int k_1 e^{\int \frac{k_1 k_3}{k_2} ds} ds\right) \mathbf{N}_q \\ &- \left(\cos\theta_1 e^{\int \frac{k_2 k_3}{k_1} ds} \int k_2 e^{-\int \frac{k_2 k_3}{k_1} ds} ds\right) \mathbf{B}_q. \end{aligned} \tag{3.14}$$

We obtain $\mathbf{U}' = \mathbf{0}$ by using (3.6). Hence γ is a q-helices of type-0. □

Corollary 3.1. *If γ is a q-helix of type-0, an axis of γ is*

$$\begin{aligned} \mathbf{D}_0 = \cos\theta_1 \mathbf{T} &+ \left(-\cos\theta_1 e^{-\int \frac{k_1 k_3}{k_2} ds} \int k_1 e^{\int \frac{k_1 k_3}{k_2} ds} ds\right) \mathbf{N}_q \\ &+ \left(-\cos\theta_1 e^{\int \frac{k_2 k_3}{k_1} ds} \int k_2 e^{-\int \frac{k_2 k_3}{k_1} ds} ds\right) \mathbf{B}_q. \end{aligned} \tag{3.15}$$

Remark 3.1. If the tangent vector field $\mathbf{T}$ of the curve γ and the fixed axis $\mathbf{D}_0$ are orthogonal to each other, that is, $\cos\theta_1 = \cos\frac{\pi}{2} = 0$, then the q-helix of type-0 can not occur since the vanishing of the axis $\mathbf{D}_0$.

3.2. The q-helices of type-1

Theorem 3.2. *Let γ be a curve due to the q-frame in $\mathbb{E}^3$. Then γ is a q-helix of type-1 if and only if*

$$\left(e^{\int \frac{k_1 k_2}{k_3} ds} \int k_1 e^{-\int \frac{k_1 k_2}{k_3} ds} ds\right) k_1 + \left(e^{-\int \frac{k_2 k_3}{k_1} ds} \int k_3 e^{\int \frac{k_2 k_3}{k_1} ds} ds\right) k_3 = 0. \tag{3.16}$$

Proof. A q-helix of type-1 satisfies the condition

$$\lambda_2 = \langle \mathbf{N}_q, \mathbf{U} \rangle = \cos\theta_2, \tag{3.17}$$

where θ_2 is a constant angle. Therefore, by substituting $\lambda_2 = \cos\theta_2$ into the system (3.4), it turns into

$$\begin{aligned} &\lambda_1' - \lambda_2 k_1 - \lambda_3 k_2 = 0, \\ &\lambda_1 k_1 - \lambda_3 k_3 = 0, \\ &\lambda_3' + \lambda_1 k_2 + \lambda_2 k_3 = 0. \end{aligned} \tag{3.18}$$

From $(3.18)_2$,

$$\lambda_3 = \tfrac{k_1}{k_3}\lambda_1, \qquad \lambda_1 = \tfrac{k_3}{k_1}\lambda_3. \tag{3.19}$$

By using (3.19) in the equations $(3.18)_1$ and $(3.18)_3$, we get the following linear differential equations of first order:

$$\lambda_1' - \tfrac{k_1 k_2}{k_3}\lambda_1 = \cos\theta_2 k_1, \tag{3.20}$$

$$\lambda_3' + \frac{k_2 k_3}{k_1}\lambda_3 = -\cos\theta_2 k_3. \tag{3.21}$$

The solution of (3.20) is

$$\lambda_1 = \cos\theta_2 e^{\int \frac{k_1 k_2}{k_3} ds} \int k_1 e^{-\int \frac{k_1 k_2}{k_3} ds} ds, \tag{3.22}$$

and the solution of (3.21) is

$$\lambda_3 = -\cos\theta_2 e^{-\int \frac{k_2 k_3}{k_1} ds} \int k_3 e^{\int \frac{k_2 k_3}{k_1} ds} ds. \tag{3.23}$$

Substituting (3.22), and (3.23) into $(3.18)_2$ gives the condition to be q-helices of type-1 as follows:

$$\left(e^{\int \frac{k_1 k_2}{k_3} ds} \int k_1 e^{-\int \frac{k_1 k_2}{k_3} ds} ds\right) k_1 + \left(e^{-\int \frac{k_2 k_3}{k_1} ds} \int k_3 e^{\int \frac{k_2 k_3}{k_1} ds} ds\right) k_3 = 0. \tag{3.24}$$

Conversely, suppose that the relation (3.16) holds, also the fixed vector field $\mathbf{U}$ can be composed of

$$\begin{aligned} \mathbf{U} = &\left(\cos\theta_2 e^{\int \frac{k_1 k_2}{k_3} ds} \int k_1 e^{-\int \frac{k_1 k_2}{k_3} ds} ds\right) \mathbf{T} + \cos\theta_2 \mathbf{N}_q \\ &- \left(\cos\theta_2 e^{-\int \frac{k_2 k_3}{k_1} ds} \int k_3 e^{\int \frac{k_2 k_3}{k_1} ds} ds\right) \mathbf{B}_q. \end{aligned} \tag{3.25}$$

We obtain $\mathbf{U}' = \mathbf{0}$ by using (3.17). Hence γ is a q-helix of type-1. □

Corollary 3.2. *If γ is a q-helix of type-1, an axis of γ is*

$$\begin{aligned} \mathbf{D}_1 = &\left(\cos\theta_2 e^{\int \frac{k_1 k_2}{k_3} ds} \int k_1 e^{-\int \frac{k_1 k_2}{k_3} ds} ds\right) \mathbf{T} + \cos\theta_2 \mathbf{N}_q \\ &+ \left(-\cos\theta_2 e^{-\int \frac{k_2 k_3}{k_1} ds} \int k_3 e^{\int \frac{k_2 k_3}{k_1} ds} ds\right) \mathbf{B}_q. \end{aligned} \tag{3.26}$$

Remark 3.2. If the tangent vector field $\mathbf{N}_q$ of the curve γ and the fixed axis $\mathbf{D}_1$ are orthogonal to each other, that is, $\cos\theta_2 = \cos\frac{\pi}{2} = 0$, then the q-helix of type-1 can not occur since the vanishing of the axis $\mathbf{D}_1$.

3.3. The q-helices of type-2

Theorem 3.3. *Let γ be a curve due to the q-frame in $\mathbb{E}^3$. Then γ is a q-helix of type-2 if and only if*

$$\left(e^{-\int \frac{k_1 k_2}{k_3} ds} \int k_2 e^{\int \frac{k_1 k_2}{k_3} ds} ds\right) k_2 + \left(e^{\int \frac{k_1 k_3}{k_2} ds} \int k_3 e^{-\int \frac{k_1 k_3}{k_2} ds} ds\right) k_3 = 0. \tag{3.27}$$

Proof. A q-helix of type-2 satisfies the condition

$$\lambda_3 = \langle \mathbf{B}_q, \mathbf{U} \rangle = \cos\theta_3, \tag{3.28}$$

where θ_3 is a constant angle. Therefore, by substituting $\lambda_3 = \cos\theta_3$ into the system (3.4), it turns into

$$\begin{aligned} &\lambda_1 k_2 + \lambda_2 k_3 = 0. \\ &\lambda_1' - \lambda_2 k_1 - \lambda_3 k_2 = 0, \\ &\lambda_2' + \lambda_1 k_1 - \lambda_3 k_3 = 0. \end{aligned} \tag{3.29}$$

From $(3.29)_1$,

$$\lambda_2 = -\tfrac{k_2}{k_3}\lambda_1, \qquad \lambda_1 = -\tfrac{k_3}{k_2}\lambda_2. \tag{3.30}$$

By using (3.30) in the equations $(3.29)_2$, and $(3.29)_3$, we get the following linear differential equations of first order:

$$\lambda_1' + \tfrac{k_1 k_2}{k_3}\lambda_1 = \cos\theta_3 k_2, \tag{3.31}$$

$$\lambda_2' - \tfrac{k_1 k_3}{k_2}\lambda_2 = \cos\theta_3 k_3. \tag{3.32}$$

The solutions of (3.31) and (3.32) are

$$\lambda_1 = \cos\theta_3 e^{-\int \frac{k_1 k_2}{k_3} ds} \int k_2 e^{\int \frac{k_1 k_2}{k_3} ds} ds, \tag{3.33}$$

$$\lambda_2 = \cos\theta_3 e^{\int \frac{k_1 k_3}{k_2} ds} \int k_3 e^{-\int \frac{k_1 k_3}{k_2} ds} ds, \tag{3.34}$$

respectively.

Substituting (3.33) and (3.34) into $(3.29)_1$ gives the condition to be q-helices of type-2 as follows:

$$\left(e^{-\int \frac{k_1 k_2}{k_3} ds} \int k_2 e^{\int \frac{k_1 k_2}{k_3} ds} ds\right) k_2 + \left(e^{\int \frac{k_1 k_3}{k_2} ds} \int k_3 e^{-\int \frac{k_1 k_3}{k_2} ds} ds\right) k_3 = 0. \tag{3.35}$$

Conversely, suppose that the relation (3.27) holds, also the fixed vector field $\mathbf{U}$ can be composed of

$$\begin{aligned} \mathbf{U} = &\left(\cos\theta_3 e^{-\int \frac{k_1 k_2}{k_3} ds} \int k_2 e^{\int \frac{k_1 k_2}{k_3} ds} ds\right) \mathbf{T} \\ &+ \left(\cos\theta_3 e^{\int \frac{k_1 k_3}{k_2} ds} \int k_3 e^{-\int \frac{k_1 k_3}{k_2} ds} ds\right) \mathbf{N}_q + \cos\theta_3 \mathbf{B}_q. \end{aligned} \tag{3.36}$$

We obtain $\mathbf{U}' = \mathbf{0}$ by using (3.27) and (3.28). Hence γ is a q-helix of type-2. □

Corollary 3.3. *If γ is a q-helix of type-2, an axis of γ is*

$$\begin{aligned} \mathbf{D}_2 = &\left(\cos\theta_3 e^{-\int \frac{k_1 k_2}{k_3} ds} \int k_2 e^{\int \frac{k_1 k_2}{k_3} ds} ds\right) \mathbf{T} \\ &+ \left(\cos\theta_3 e^{\int \frac{k_1 k_3}{k_2} ds} \int k_3 e^{-\int \frac{k_1 k_3}{k_2} ds} ds\right) \mathbf{N}_q + \cos\theta_3 \mathbf{B}_q. \end{aligned} \tag{3.37}$$

Remark 3.3. If the tangent vector field $\mathbf{B}_q$ of the curve γ and the fixed axis $\mathbf{D}_2$ are orthogonal to each other, that is, $\cos\theta_3 = \cos\frac{\pi}{2} = 0$, then the q-helix of type-2 can not occur since the vanishing of the axis $\mathbf{D}_2$.

3.4. The relations of q-helices to each other

In this part, we give the relations of q-helices to each other based on the consequences of Theorems 3.1, 3.2 and 3.3.

Corollary 3.4. *Let γ be a q-helix of type-0 in $\mathbb{E}^3$. Then γ is a q-helix of type-1 if and only if*

$$k_1 = 0 \quad or \quad k_2 = ck_3, \tag{3.38}$$

where c is a constant.

Proof. Using (3.14) at the condition to be a q-helix of type-1 as follows:

$$\langle \mathbf{N}_q, \mathbf{U} \rangle = -\cos\theta_1 e^{-\int \frac{k_1 k_3}{k_2} ds} \int k_1 e^{\int \frac{k_1 k_3}{k_2} ds} ds. \tag{3.39}$$

The expression in (3.39) becomes constant if the cases (3.38) are satisified. □

Corollary 3.5. *Let γ be a q-helix of type-0 in $\mathbb{E}^3$. Then γ is a q-helix of type-2 if and only if*

$$k_2 = 0 \quad or \quad k_1 = -ck_3 \tag{3.40}$$

where c is a constant.

Proof. Using (3.14) at the condition to be a q-helix of type-2 as follows:

$$\langle \mathbf{B}_q, \mathbf{U} \rangle = -\cos\theta_1 e^{\int \frac{k_2 k_3}{k_1} ds} \int k_2 e^{-\int \frac{k_2 k_3}{k_1} ds} ds. \tag{3.41}$$

The expression in (3.41) becomes constant if the cases (3.40) are satisified. □

Corollary 3.6. *Let γ be a q-helix of type-1 in $\mathbb{E}^3$. Then γ is a q-helix of type-0 if and only if*

$$k_1 = 0 \quad or \quad k_3 = -ck_2 \tag{3.42}$$

where c is a constant.

Proof. Using (3.25) at the condition to be a q-helix of type-0 as follows:

$$\langle \mathbf{T}, \mathbf{U} \rangle = \cos\theta_2 e^{\int \frac{k_1 k_2}{k_3} ds} \int k_1 e^{-\int \frac{k_1 k_2}{k_3} ds} ds. \tag{3.43}$$

The expression in (3.43) becomes constant if the cases (3.42) are satisified. □

Corollary 3.7. *Let γ be a q-helix of type-1 in $\mathbb{E}^3$. Then γ is a q-helix of type-2 if and only if*

$$k_3 = 0 \quad or \quad k_1 = -ck_2 \tag{3.44}$$

where c is a constant.

Proof. Using (3.25) at the condition to be a q-helix of type-2 as follows:

$$\langle \mathbf{B}_q, \mathbf{U} \rangle = -\cos\theta_2 e^{-\int \frac{k_2 k_3}{k_1} ds} \int k_3 e^{\int \frac{k_2 k_3}{k_1} ds} ds. \tag{3.45}$$

The expression in (3.45) becomes constant if the cases (3.44) are satisified. □

Corollary 3.8. *Let γ be a q-helix of type-2 in $\mathbb{E}^3$. Then γ is a q-helix of type-0 if and only if*

$$k_2 = 0 \quad or \quad k_3 = ck_1. \tag{3.46}$$

where c is a constant.

Proof. Using (3.36) at the condition to be a q-helix of type-0 as follows:

$$\langle \mathbf{T}, \mathbf{U} \rangle = \cos\theta_3 e^{-\int \frac{k_1 k_2}{k_3} ds} \int k_2 e^{\int \frac{k_1 k_2}{k_3} ds} ds. \tag{3.47}$$

The expression in (3.47) becomes constant if the cases (3.46) are satisified. □

Corollary 3.9. *Let γ be a q-helix of type-2 in $\mathbb{E}^3$. Then γ is a q-helix of type-1 if and only if*

$$k_3 = 0 \quad or \quad k_2 = -ck_1. \tag{3.48}$$

where c is a constant.

Proof. Using (3.36) at the condition to be a q-helix of type-1 as follows:

$$\langle \mathbf{N}_q, \mathbf{U} \rangle = \cos\theta_3 e^{\int \frac{k_1 k_2}{k_3} ds} \int k_1 e^{-\int \frac{k_1 k_2}{k_3} ds} ds. \tag{3.49}$$

The expression in (3.49) becomes constant if the cases (3.48) are satisified. □

The above results can be put together with the following corollary:

Corollary 3.10. *Let γ be a curve via q-frame in $\mathbb{E}^3$. Then*

(i) *The curve γ is both a q-helix of type-0 and type-1 provided that*

$$k_1 = 0 \quad or \quad k_2 = Ak_3,$$

where A is an arbitrary constant.

(ii) *The curve γ is both a q-helix of type-0 and type-2 provided that*

$$k_2 = 0 \quad or \quad k_1 = Bk_3,$$

where B is an arbitrary constant.

(iii) *The curve γ is both a q-helix of type-1 and type-2 provided that*

$$k_3 = 0 \quad or \quad k_2 = Ck_1,$$

where C is an arbitrary constant.

4. The Darboux q-helices

In this part of our research, we classify the Darboux q-helices. First we study the conditions of q-helices of type-0, type-1 and type-2 to be Darboux q-helices, respectively. Finally, we obtain the general case for q-helices to be Darboux helices.

From [4], the Darboux vector of a curve due to the q-frame is as follows:

$$\partial = k_3\mathbf{T} - k_2\mathbf{N}_q + k_1\mathbf{B}_q. \tag{4.1}$$

We have to give the description of Darboux q-helices as follows:

Definition 4.1. A unit speed curve γ framed by q-frame whose Darboux vector ∂ is said to be a Darboux helix provided that there exists a non-zero fixed direction $\mathbf{U} \in \mathbb{E}^3$ such that satisfies

$$\langle \partial, \mathbf{U} \rangle = \cos\varphi, \tag{4.2}$$

φ is a constant angle between the vectors ∂ and $\mathbf{U}$.

Differentiating (4.2) and rearranging the equation gives the system

$$\begin{aligned} &\lambda_1' - \lambda_2 k_1 - \lambda_3 k_2 = 0, \\ &\lambda_2' + \lambda_1 k_1 - \lambda_3 k_3 = 0, \\ &\lambda_3' + \lambda_1 k_2 + \lambda_2 k_3 = 0. \end{aligned} \tag{4.3}$$

Based upon the system (4.3), we take the q-helices of type-0, type-1 and type-2, and a curve frame by a q-frame to be Darboux helices into consideration, respectively, in the subsequent four cases:

Case 1: Let γ be a q-helix of type-0. Hence the equation (3.6) holds. Then we have the equation

$$\langle \partial', \mathbf{U} \rangle = \lambda_1 k_3' - \lambda_2 k_2' + \lambda_3 k_1' = 0. \tag{4.4}$$

Using (3.6) and (4.4) in the system (4.3) results in the following system:

$$\begin{aligned} &\cos\theta_1 k_3' - \lambda_2 k_2' + \lambda_3 k_1' = 0, \\ &\lambda_2 k_1 + \lambda_3 k_2 = 0, \\ &\lambda_2' + \cos\theta_1 k_1 - \lambda_3 k_3 = 0, \\ &\lambda_3' + \cos\theta_1 k_2 + \lambda_2 k_3 = 0. \end{aligned} \tag{4.5}$$

Applying $(4.5)_2$ into the equations $(4.5)_3$ and $(4.5)_4$, the functions λ_2 and λ_3 are found as in (3.11) and (3.12). If the values obtained are substituted to the equation $(4.5)_1$, then it follows that

$$k_3' + k_2' e^{-\int \frac{k_1 k_3}{k_2} ds} \int k_1 e^{\int \frac{k_1 k_3}{k_2} ds} ds - k_1' e^{\int \frac{k_2 k_3}{k_1} ds} \int k_2 e^{-\int \frac{k_2 k_3}{k_1} ds} ds = 0. \tag{4.6}$$

Also from (3.13), we have

$$\left(e^{\int \frac{k_2 k_3}{k_1} ds} \int k_2 e^{-\int \frac{k_2 k_3}{k_1} ds} ds \right) = -\frac{k_1}{k_2} \left(e^{-\int \frac{k_1 k_3}{k_2} ds} \int k_1 e^{\int \frac{k_1 k_3}{k_2} ds} ds \right). \tag{4.7}$$

Substituting (4.7) into (4.6) gives

$$k_3' + \left(k_2' + \frac{k_1k_1'}{k_2}\right)\left(e^{-\int \frac{k_1k_3}{k_2}ds}\int k_1 e^{\int \frac{k_1k_3}{k_2}ds}ds\right) = 0 \tag{4.8}$$

which is the condition for a q-helix of type-0 to be a Darboux helix.

Conversely, suppose that the relation (4.8) holds, it can be seen that the axis given in (3.14) is a fixed one.

Case 2: Let γ be a q-helix of type-1. Hence the equation (3.17) holds. Using (3.17), and (4.4) in the system (4.3), we find the system

$$\begin{aligned} &\lambda_1 k_3' - \cos\theta_2 k_2' + \lambda_3 k_1' = 0,\\ &\lambda_1' - \lambda_2 k_1 - \lambda_3 k_2 = 0,\\ &\lambda_1 k_1 - \lambda_3 k_3 = 0,\\ &\lambda_3' + \lambda_1 k_2 + \lambda_2 k_3 = 0. \end{aligned} \tag{4.9}$$

Applying $(4.9)_3$ into the equations $(4.9)_2$ and $(4.9)_4$, the functions λ_1 and λ_3 are found as in (3.22) and (3.23). If the values obtained are substituted to the equation $(4.9)_1$, then it follows that

$$k_3' e^{\int \frac{k_1k_2}{k_3}ds}\int k_1 e^{-\int \frac{k_1k_2}{k_3}ds}ds - k_2' - k_1' e^{-\int \frac{k_2k_3}{k_1}ds}\int k_3 e^{\int \frac{k_2k_3}{k_1}ds}ds = 0 \tag{4.10}$$

Also from (3.24), we obtain

$$\left(e^{-\int \frac{k_2k_3}{k_1}ds}\int k_3 e^{\int \frac{k_2k_3}{k_1}ds}ds\right) = -\frac{k_1}{k_3}\left(e^{\int \frac{k_1k_2}{k_3}ds}\int k_1 e^{-\int \frac{k_1k_2}{k_3}ds}ds\right) \tag{4.11}$$

Substituting (4.11) into (4.10), we attain the equation

$$k_2' + \left(\frac{k_1k_1'}{k_3} - k_3'\right)\left(e^{\int \frac{k_1k_2}{k_3}ds}\int k_1 e^{-\int \frac{k_1k_2}{k_3}ds}ds\right) = 0 \tag{4.12}$$

which is the condition for a q-helix of type-1 to be a Darboux helix.

Conversely, suppose that the relation (4.12) holds. It can be seen that the axis given in (3.25) is a fixed one.

Case 3: Let γ be a q-helix of type-2. So the equation (3.28) holds. Using (3.28) and (4.4) in the system (4.3), the system is as follows:

$$\begin{aligned} &\lambda_1 k_3' - \lambda_2 k_2' + \cos\theta_3 k_1' = 0,\\ &\lambda_1' - \lambda_2 k_1 - \lambda_3 k_2 = 0,\\ &\lambda_2' + \lambda_1 k_1 - \lambda_3 k_3 = 0,\\ &\lambda_1 k_2 + \lambda_2 k_3 = 0. \end{aligned} \tag{4.13}$$

Applying $(4.13)_4$ into the equations $(4.13)_2$ and $(4.13)_3$, the functions λ_1 and λ_2 are obtained as in (3.33) and (3.34). If the values obtained are put into the equation $(4.13)_1$, then it is followed that

$$k_3' e^{-\int \frac{k_1 k_2}{k_3} ds} \int k_2 e^{\int \frac{k_1 k_2}{k_3} ds} ds - k_2' e^{\int \frac{k_1 k_3}{k_2} ds} \int k_3 e^{-\int \frac{k_1 k_3}{k_2} ds} ds + k_1' = 0. \tag{4.14}$$

Also from (3.35), we obtain

$$\left(e^{-\int \frac{k_1 k_2}{k_3} ds} \int k_2 e^{\int \frac{k_1 k_2}{k_3} ds} ds\right) = -\frac{k_3}{k_2}\left(e^{\int \frac{k_1 k_3}{k_2} ds} \int k_3 e^{-\int \frac{k_1 k_3}{k_2} ds} ds\right). \tag{4.15}$$

Replacing (4.15) into (4.14), we reach the result

$$k_1' - \left(k_2' + \frac{k_3 k_3'}{k_2}\right)\left(e^{\int \frac{k_1 k_3}{k_2} ds} \int k_3 e^{-\int \frac{k_1 k_3}{k_2} ds} ds\right) = 0 \tag{4.16}$$

which is the condition for a q-helix of type-2 to be a Darboux helix.

Conversely, suppose that the relation (4.16) holds. It can be seen that the axis given in (3.36) is a fixed one.

Case 4 (General Case): Let γ be a curve due to the q-frame in $\mathbb{E}^3$. From (4.2), we obtain

$$\lambda_1 k_3 - \lambda_2 k_2 + \lambda_3 k_1 = \cos\varphi. \tag{4.17}$$

Differentiating (4.17) and after some arrangements, we find

$$\lambda_1 k_3' - \lambda_2 k_2' + \lambda_3 k_1' = 0. \tag{4.18}$$

With the aid of (4.17) and (4.18), we arrive

$$\lambda_3 = \frac{\left(k_2' k_3 - k_2 k_3'\right)\lambda_2 - \cos\varphi k_3'}{k_1' k_3 - k_1 k_3'}, \tag{4.19}$$

and

$$\lambda_1 = \frac{\left(k_2' k_1 - k_2 k_1'\right)\lambda_2 - \cos\varphi k_1'}{k_3' k_1 - k_3 k_1'}, \tag{4.20}$$

respectively. Substituting (4.19) and (4.20) into $(4.3)_2$ delivers the linear DE as

$$\lambda_2' + \left(\frac{k_2' k_1^2 - k_2 k_1 k_1' + k_2' k_3^2 - k_2 k_3 k_3'}{k_3' k_1 - k_3 k_1'}\right)\lambda_2 = \frac{\cos\varphi\left(k_1 k_1' + k_3 k_3'\right)}{k_3' k_1 - k_3 k_1'}. \tag{4.21}$$

The solution of (4.21) is

$$\lambda_2 = \cos\varphi e^{\int \frac{k_2 k_1 k_1' + k_2 k_3 k_3' - k_2' k_1^2 - k_2' k_3^2}{k_3' k_1 - k_3 k_1'} ds} \int \frac{\left(k_1 k_1' + k_3 k_3'\right)}{k_3' k_1 - k_3 k_1'} e^{\int \frac{k_2' k_1^2 - k_2 k_1 k_1' + k_2' k_3^2 - k_2 k_3 k_3'}{k_3' k_1 - k_3 k_1'} ds} ds. \tag{4.22}$$

Using (4.17) and (4.18), we obtain

$$\lambda_1 = \frac{\left(k_1' k_2 - k_1 k_2'\right)\lambda_3 + \cos\varphi k_2'}{k_2' k_3 - k_2 k_3'}, \tag{4.23}$$

and

$$\lambda_2 = \frac{\left(k_1' k_3 - k_1 k_3'\right)\lambda_3 + \cos\varphi k_3'}{k_2' k_3 - k_2 k_3'}, \tag{4.24}$$

respectively. Replacing (4.23) and (4.24) into $(4.3)_3$, we have the following differential equation

$$\lambda_3' + \left(\frac{k_1'k_2^2 - k_1k_2k_2' + k_1'k_3^2 - k_1k_3k_3'}{k_2'k_3 - k_2k_3'}\right)\lambda_3 = \frac{\cos\varphi k_2k_2' + \cos\varphi k_3k_3'}{k_2k_3' - k_2'k_3}. \tag{4.25}$$

The solution of (4.25) is

$$\lambda_3 = \cos\varphi e^{\int \frac{k_1k_2k_2' + k_1k_3k_3' - k_1'k_3^2 - k_1'k_2^2}{k_2'k_3 - k_2k_3'}ds} \int \frac{k_2k_2' + k_3k_3'}{k_2k_3' - k_2'k_3} e^{\int \frac{k_1'k_2^2 - k_1k_2k_2' + k_1'k_3^2 - k_1k_3k_3'}{k_2'k_3 - k_2k_3'}ds} ds. \tag{4.26}$$

From (4.17) and (4.18), we attain

$$\lambda_2 = \frac{\left(k_3'k_1 - k_3k_1'\right)\lambda_1 + \cos\varphi k_1'}{k_2'k_1 - k_2k_1'}, \tag{4.27}$$

and

$$\lambda_3 = \frac{\left(k_2'k_3 - k_2k_3'\right)\lambda_1 - \cos\varphi k_2'}{\left(k_1'k_2 - k_1k_2'\right)}, \tag{4.28}$$

respectively. Usage of the equations (4.27) and (4.28) at $(4.3)_1$ allows the equation

$$\lambda_1' + \left(\frac{k_3k_1' - k_3'k_1 + k_2'k_3 - k_2k_3'}{k_1k_2' - k_1'k_2}\right)\lambda_1 = \frac{\cos\varphi k_1k_1' + \cos\varphi k_2k_2'}{k_1k_2' - k_1'k_2}. \tag{4.29}$$

The solution of (4.29) is

$$\lambda_1 = \cos\varphi e^{\int \frac{k_3'k_1 - k_3k_1' - k_2'k_3 + k_2k_3'}{k_1k_2' - k_1'k_2}ds} \int \frac{\cos\varphi k_1k_1' + \cos\varphi k_2k_2'}{k_1k_2' - k_1'k_2} e^{\int \frac{k_3k_1' - k_3'k_1 + k_2'k_3 - k_2k_3'}{k_1k_2' - k_1'k_2}ds} ds. \tag{4.30}$$

Substituting (4.22), (4.26) and (4.30) into (4.18) gives the condition for a curve to be a Darboux q-helix as follows:

$$\begin{aligned} &\left(e^{\int \frac{k_1k_2k_2' + k_1k_3k_3' - k_1'k_3^2 - k_1'k_2^2}{k_2'k_3 - k_2k_3'}ds} \int \frac{k_2k_2' + k_3k_3'}{k_2k_3' - k_2'k_3} e^{\int \frac{k_1'k_2^2 - k_1k_2k_2' + k_1'k_3^2 - k_1k_3k_3'}{k_2'k_3 - k_2k_3'}ds} ds\right) k_1' \\ &+ \left(e^{\int \frac{k_3'k_1 - k_3k_1' - k_2'k_3 + k_2k_3'}{k_1k_2' - k_1'k_2}ds} \int \frac{k_1k_1' + k_2k_2'}{k_1k_2' - k_1'k_2} e^{\int \frac{k_3k_1' - k_3'k_1 + k_2'k_3 - k_2k_3'}{k_1k_2' - k_1'k_2}ds} ds\right) k_3' \\ &= \left(e^{\int \frac{k_2k_1k_1' + k_2k_3k_3' - k_2'k_1^2 - k_2'k_3^2}{k_3'k_1 - k_3k_1'}ds} \int \frac{\left(k_1k_1' + k_3k_3'\right)}{k_3'k_1 - k_3k_1'} e^{\int \frac{k_2'k_1^2 - k_2k_1k_1' + k_2'k_3^2 - k_2k_3k_3'}{k_3'k_1 - k_3k_1'}ds} ds\right) k_2'. \end{aligned} \tag{4.31}$$

Conversely, suppose that the relation (4.31) holds, the fixed vector filed $\mathbf{U}$ can also be composed of

$$\begin{aligned} \mathbf{U} = &\left(\cos\varphi e^{\int \frac{k_3'k_1 - k_3k_1' - k_2'k_3 + k_2k_3'}{k_1k_2' - k_1'k_2}ds} \int \frac{\cos\varphi k_1k_1' + \cos\varphi k_2k_2'}{k_1k_2' - k_1'k_2} e^{\int \frac{k_3k_1' - k_3'k_1 + k_2'k_3 - k_2k_3'}{k_1k_2' - k_1'k_2}ds} ds\right) \mathbf{T} \\ &+ \left(\cos\varphi e^{\int \frac{k_2k_1k_1' + k_2k_3k_3' - k_2'k_1^2 - k_2'k_3^2}{k_3'k_1 - k_3k_1'}ds} \int \frac{\left(k_1k_1' + k_3k_3'\right)}{k_3'k_1 - k_3k_1'} e^{\int \frac{k_2'k_1^2 - k_2k_1k_1' + k_2'k_3^2 - k_2k_3k_3'}{k_3'k_1 - k_3k_1'}ds} ds\right) \mathbf{N}_q \\ &+ \left(\cos\varphi e^{\int \frac{k_1k_2k_2' + k_1k_3k_3' - k_1'k_3^2 - k_1'k_2^2}{k_2'k_3 - k_2k_3'}ds} \int \frac{k_2k_2' + k_3k_3'}{k_2k_3' - k_2'k_3} e^{\int \frac{k_1'k_2^2 - k_1k_2k_2' + k_1'k_3^2 - k_1k_3k_3'}{k_2'k_3 - k_2k_3'}ds} ds\right) \mathbf{B}_q \end{aligned} \tag{4.32}$$

We obtain $\mathbf{U}' = \mathbf{0}$ by using (4.17) and (4.31). Hence γ is a Darboux q-helix.

We can give the following theorem containing the cases above:

Theorem 4.1. *Let γ be a curve due to the q-frame in Euclidean 3-space $\mathbb{E}^3$. Then*

(i) γ is a Darboux q-helix satisfying the condition to be q-helix of type-0 if and only if the equation (4.8) is satisfied.

(ii) γ is a Darboux q-helix satisfying the condition to be q-helix of type-1 if and only if the equation (4.12) is satisfied.

(iii) γ is a Darboux q-helix satisfying the condition to be q-helix of type-2 if and only if the equation (4.16) is satisfied.

(iv) γ is a Darboux q-helix if and only if the equation (4.31) is satisfied, and the fixed axis is given as in (4.32).

5. Conclusion

Helices are very special curves by which many patterns can be modelled in nature. In the present examination, we considered these special curves from the point of view of frame fields which describe the behaviour of the curves. The original aspect of our research is to deal quasi-frame (abbv. q-frame) in Euclidean 3-space. For all vector fields of the mentioned frame, slant helices, which are recalled, in the context of the paper, as q-helices, have been worked out in Euclidean 3-space. Additionally, the Darboux q-helices are obtained by Darboux vector which has been formed by q-frame fields.

REFERENCES

1. A. Alİ, R. Lopez and M. Turgut: *$k-$type partially null and pseudo null slant helices in Minkowski 4-space.* Math. Commun. **17** (2012), 93–103.
2. R. L. Bishop: *There is more than one way to frame a curve.* Am. Math. Mon. **82** (1975), 246–251.
3. S. Coquillart: *Computing offsets of B-spline curves.* Computer-Aided Design **19** (1987), 305–309.
4. M. Dede, C. Ekİcİ and A. Görgülü: *Directional q-frame along a space curve.* IJARCSSE **5** (2015), 775–780.
5. M. Dede, C. Ekİcİ and İ. Güven: *Directional Bertrand Curves.* GU. J. Sci. **31** (2018), 202–211.
6. M. Dede, M. Ç. Aslan, and C. Ekİcİ: *On a variational problem due to the B-Darboux frame in Euclidean 3-spaces.* Math. Meth. Appl. Sci. **44** (2021), 12630–12639.
7. P. M. Do Carmo: *Differential Geometry of Curves and Surfaces.* Prentice-Hall Inc. Englewood Cliffs, New Jersey, 1976.

8. M. Ergüt, H. B. Öztekİn and S. Aykurt: *Non-null k−slant helices and their spherical indicatrices in Minkowski 3-space.* J. Adv. Res. Dyn. Control Syst. **2** (2021), 1–12.

9. S. Izumİya and N. Takeuchİ: *New special curves and developable surfaces.* Turk. J. Math. **28** (2004), 531–537.

10. L. Kula and Y. Yayli: *On slant helix and its spherical indicatrix.* Appl. Math. Comput. **169** (2005), 600–607.

11. L. Kula, N. Ekmekçİ, Y. Yayli and K. İlarslan: *Characterizations of slant helices in Euclidean 3-space.* Turk. J. Math. **34** (2010), 261–273.

12. R. S. Millman and G. D. Parker: *Elements of Differential Geometry.* Prentice-Hall Inc. Englewood Cliffs, New Jersey, 1977.

13. E. Nesovic, U. Öztürk and E. B. Koç Öztürk: *On k−type pseudo null Darboux helices in Minkowski 3-space.* Math. Anal. Appl. **439** (2016), 690–700.

14. E. Nesovic, E. B. Koç Öztürk and U. Öztürk: *On k−type null Cartan slant helices in Minkowski 3-space.* Math. Meth. Appl. Sci. **41** (2018), 7583–7598.

15. E. Nesovic, U. Öztürk and E. B. Koç Öztürk: *On T-Slant, N-Slant and B-Slant helices in Galilean space* G_3. J. Dyn. Syst. Geom. Theor. **16** (2018), 187–199.

16. U. Öztürk and E. Nesovic: *On pseudo null and null Cartan Darboux helices in Minkowski 3-space.* Kuwait J. Sci. **43** (2016), 64-82.

17. U. Öztürk and Z. B. Alkan: *Darboux helices in three dimensional Lie groups.* AIMS Mathematics **5** (2020), 3169–3181.

18. J. Quian and Y. Ho Kim: *Null helix and k-type null slant helices in* E_1^4. Rev. Un. Mat. Argentina **57** (2016), 71–83.

19. H. Shin, S. K. Yoo, S. K. Cho and W. H. Chung: *Directional Offset of a Spatial Curve for Practical Engineering Design.* ICCSA **3** (2003), 711–720.

20. D. J. Struik: *Lectures on Classical Differential Geometry.* Dover, New York, 1988.

21. M. Turgut and S. Yilmaz: *Characterizations of some special helices in* E^4. Sci. Magna **4** (2008), 51–55.

22. G. Uğur Kaymanli, C. Ekİcİ and M. Dede: *Directional evolution of the ruled surfaces via the evolution of their directrix using q-frame along a timelike space curve.* EJOSAT **20** (2020), 392–396.

23. Y. Ünlütürk and T. Körpinar: *On Darboux helices in Complex space* C^3. JOSA **4** (2019), 851–858.

24. S. Yilmaz, and M. Turgut: *A new version of Bishop frame and an application to spherical images.* J. Math. Anal. Appl. **371** (2010), 764–776.

25. E. Ziplar, A. Şenol and Y. Yayli: *On Darboux helices in Euclidean 3-space.* Glob. J. Sci. Front. Res. **12** (2012), 73–80.