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DOMINATION NUMBER AND WATCHING NUMBER OF SUBDIVISION CONSTRUCTION OF GRAPHS

Kamran Mirasheh[1], Ahmad Abbasi[1] and Ebrahim Vatandoost[2]

[1]**Faculty of Mathematical Sciences**
University of Guilan, P. O. Box 41335-19141, Rasht, Iran
[2]**Department of Basic Science**
Imam Khomeini International University, P. O. Box 34148-96818, Qazvin, Iran

ORCID IDs: Kamran Mirasheh https://orcid.org/0009-0006-0041-4352
Ahmad Abbasi N/A
Ebrahim Vatandoost https://orcid.org/0000-0001-5883-7776

Abstract. In light of the results of a domination number of the subdivision of a graph G, we determine an upper bound of the watching number of S(G). In addition, we obtain a condition with which the upper bound becomes sharp.
Keywords: Watching system, Dominating set, Subdivision.

1. Introduction

In this paper, all graphs are assumed to be finite, simple and undirected. We will often use the notation $G = (V, E)$ to denote the graph with non-empty vertex set $V = V(G)$ and edge set $E = E(G)$. 'Order of ' a graph G is the number of vertices in the graph and is denoted by $|G|$. The degree of a vertex v is denoted $deg(v)$. By $\delta(G)$ we denote the minimum degree of G. An edge of G with end vertices u and v is denoted by $u - v$. For every vertex $x \in V(G)$, the open neighborhood of vertex x is denoted by $N_G(x)$ and defined as $N_G(x) = \{y \in V(G) \ : \ x - y\}$, and the close neighborhood of vertex $x \in V(G)$, $N_G[x]$, is $N_G[x] = N_G(x) \cup \{x\}$. For a set $T \subseteq V(G)$, the open neighborhood of T is $N_G(T) = \cup_{x \in T} N_G(x)$ and the closed neighborhood of T is $N_G[T] = N_G(T) \cup T$. For a set $S \subseteq V(G)$, the subgraph

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Corresponding Author: Ebrahim Vatandoost
E-mail addresses: k.mir.1185@gmail.com (K. Mirasheh), aabbasi@guilan.ac.ir (A. Abbasi), vatandoost@sci.ikiu.ac.ir (E. Vatandoost)
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induced by S is denoted by $G[S]$. 'A complete bipartite graph ' is a special kind of bipartite graph where every vertex of the first set is connected to every vertex of the second set. The complete bipartite graph is denoted by $K_{p,q}$, where p and q are the order of bipartition. The 'Cocktail party graph ' denoted by $C_P(s)$ obtained by removing s disjoint edges from complete graph K_{2s}. A subset S of $V(G)$ is a dominating set of G if every vertex in $V(G) \setminus S$ is adjacent to at least one vertex in S. The domination number of a graph G, denoted by $\gamma(G)$, is the minimum size of a dominating set of G. A set C of vertices G is an identifying set of G if for every two vertices x and y the sets $N_G[x] \cap C$ and $N_G[y] \cap C$ are non-empty and different. Given a graph G, the smallest size of an identifying set of G is called the identifying code number of G and denoted by $\gamma^{ID}(G)$. Two vertices x and y are twins when $N_G[x] = N_G[y]$. Graphs with at least two twin vertices are not an identifiable graph. Nowadays, identifying codes are a subject of active research on their own, such as: the structural analysis of RNA proteins [6], error-detection schemes [9] and routing [10], the location of threats in facilities using sensors[12], as well as in networks, terrorist network monitoring [16]. For more details we refer the reader to [3, 4, 8, 11, 15, 17]. are a subject of active research on their own, such as: The Subdivision graph is the graph obtained by inserting an additional vertex in to each edge of G, denoted by $S(G)$. A 'watching system ' was introduced in[1], is a generalization of identifying codes. A 'watcher ' ω of G is a couple of $\omega = (v_i, Z_i)$ where v_i is a vertex and $Z_i \subseteq N_G[v_i]$. We will say that ω is located at v_i and that Z_i is its watching area or watching zone. A watching system in a graph G is a finite set $W = \{\omega_1, \omega_2, \ldots, \omega_k\}$ such that sets $L_W(v) = \{\omega_i \ : \ v \in Z_i \ , \ 1 \leq i \leq k\}$ are non-empty and distinct, for any $v \in V$. The 'watching number ' of G denoted by $\omega(G)$ is the minimum size of watching systems of G.

In a watching system, the selection of neighbor vertices is favorite as watching area from a watcher. We can place several watchers at the same location, with distinct watching zones. Also watchers enable us to model a monitoring system where monitors could simply tell where they detect a fault, but where the cost of a monitor is proportional to the number of bits needed to send this information.

Auger et al. [2], gave an upper bound on $\omega(G)$ for connected graphs of order n and characterized the trees attaining this bound. In 2014, Maimani et al. [13] studied the watching systems of triangular graphs. They proved watching number of triangular graph $T(n)$ is equal to $\lceil \frac{2n}{3} \rceil$. In 2017, Roozbayani et al. [14] studied identifying codes and watching systems of Keneser graphs.

In this paper, we study the watching number of subdivision of some graphs. Our main results are the following.
Theorem A: Let G be a graph of order $n \geq 3$, of size m and $\delta(G) \geq n-2$. Then $\gamma(S(G)) = n-1$. (Theorem 1 in Section 2.)
Theorem B: Let G be isomorphic to complete bipartite graph $K_{p,q}$. Then $\gamma(S(G)) = p+q-1$. (Theorem 5 in Section 2.)
Theorem C: Let G be a graph of order n. Then $\omega(S(G)) \leq n$. (Theorem 13 in Section 4.)
Theorem D: Let G be a connected graph of order $n \geq 2$. If $\gamma(S(G)) = n-1$, then

$\omega(S(G)) = n$. (Theorem 16 in Section 4.)
We end this paper by the following conjecture.
Conjecture: If G is a graph of order n, then $\omega(S(G)) = n$.

2. Domination number of $S(G)$

In this section, we give some results about dominating number of subdivision of some graphs.

Theorem 2.1. *Let G be a graph of order $n \geq 3$, of size m and $\delta(G) \geq n-2$. Then $\gamma(S(G)) = n-1$.*

Proof. Let $V(G) = \{v_1, v_2, \cdots, v_n\}$ and $V(S(G)) = V(G) \cup B$, where $B = \{v_{ij} : 1 \leq i < j \leq n, N_{S(G)}(v_{ij}) = \{v_i, v_j\}\}$. Without loss of generality, we may assume that v_1 adjacent to v_2 in G. Also let $D = \{v_3, v_4, \cdots, v_n\} \cup \{v_{12}\}$. Then the vertices v_1 and v_2 are dominated by v_{12}. The other vertices $v_{ij} (i \neq j)$ are dominated by v_i or $v_j (i \neq 1, j \neq 2)$. So $\gamma(S(G)) \leq n-1$. Let $n = 3$. Then $\gamma(S(G)) \leq 2$. Since $S(G)$ does not have universal vertex, $\gamma(S(G)) = 2$. Let $n \geq 4$ and D_0 be a dominating set for $S(G)$ with $\mid D_0 \mid = n-2$. Since $n \geq 4$ and $\delta(G) \geq n-2$, so $m \geq n$. Hence, $D_0 \cap V(G) \neq \emptyset$ and $D_0 \cap B \neq \emptyset$. Let $\mid V(G) \cap D_0 \mid = \ell$ and $n - \ell = 2$. Then $D_0 \cap B = \emptyset$, which is not true. So $n - \ell \geq 3$.
We claim that, there exist $\{v_i, v_j, v_t\} \subseteq V(G) \setminus D_0$, $(i < j < t)$ such that $\mid N_{S(G)}[v_i] \cap D_0 \mid = \mid N_{S(G)}[v_j] \cap D_0 \mid = \mid N_{S(G)}[v_t] \cap D_0 \mid = 1$.
For this suppose that $H = S(G)[(B \cap D_0) \cup (V(G) \setminus D_0)]$. It is clear that

$$\sum_{x \in D_0 \cap B} deg_H(x) = \sum_{y \in V(G) \setminus D_0} deg_H(y) \quad and \quad \sum_{x \in D_0 \cap B} deg_H(x) \leq 2\mid D_0 \cap B \mid.$$

If for every $v_i \in V(G) \setminus D_0$, $\mid N_{S(G)}[v_i] \cap D_0 \mid \geq 2$, then

$$\sum_{y \in V(G) \setminus D_0} deg_H(y) \geq 2\mid V(G) \setminus D_0 \mid.$$

Hence, $2\mid V(G) \setminus D_0 \mid \leq 2\mid D_0 \cap B \mid$ that is false.
If for every $v_i \in V(G) \setminus (D_0 \cup \{v_j\})$, $\mid N_{S(G)}[v_i] \cap D_0 \mid \geq 2$ and $\mid N_{S(G)}[v_j] \cap D_0 \mid = 1$, $(v_j \notin D_0)$, then

$$\sum_{y \in V(G) \setminus D_0} deg_H(y) = \sum_{y \in V(G) \setminus (D_0 \cup \{v_j\})} deg_H(y) + deg_H(v_j)$$
$$\geq 2\mid V(G) \setminus (D_0 \cup \{v_j\}) \mid + 1.$$

Hence, $2(n - \ell - 1) + 1 \leq 2\mid D_0 \cap B \mid = 2(n - \ell - 2)$ that is not true.
If for every $v_i \in V(G) \setminus (D_0 \cup \{v_j, v_t\})$, $\mid N_{S(G)}[v_i] \cap D_0 \mid \geq 2$ and $\mid N_{S(G)}[v_j] \cap D_0 \mid = \mid N_{S(G)}[v_t] \cap D_0 \mid = 1$, then

$$\sum_{y \in V(G) \setminus D_0} deg_H(y) = \sum_{y \in V(G) \setminus (D_0 \cup \{v_j, v_t\})} deg_H(y) + deg_H(v_j) + deg_H(v_t)$$

$$\geq 2|\ V(G) \setminus (D_0 \cup \{v_j, v_t\})\ | + 2.$$

Hence, $2(n-\ell-2)+2 \leq 2|\ D_0 \cap B\ | = 2(n-\ell-2)$, which is a contradiction. So we can assume that there exist $i<j<t$ such that $\{v_i, v_j, v_t\} \subseteq V(G) \setminus D_0$, $N_{S(G)}[v_i] \cap D_0 = \{v_{i\alpha}\}$, $N_{S(G)}[v_j] \cap D_0 = \{v_{j\beta}\}$ and $N_{S(G)}[v_t] \cap D_0 = \{v_{t\sigma}\}$.
We have two following cases.
Case 1. Let $|\{v_{i\alpha}, v_{j\beta}, v_{t\sigma}\}| = 3$. Since $\delta(G) \geq n-2$, so v_i adjacent to v_j or v_t in G. Let $v_j \in N_G(v_i)$. Then $N_{S(G)}[v_{ij}] = \{v_i, v_j\}$ and $N_{S(G)}[v_{ij}] \cap D_0 = \emptyset$. This is a contradiction with this fact that D_0 is a dominating set for $S(G)$.
Case 2. Let $|\ \{v_{i\alpha}, v_{j\beta}, v_{t\sigma}\}| = 2$. Without loss of generality, suppose that $N_{S(G)}[v_i] \cap D_0 = N_{S(G)}[v_j] \cap D_0$ and $N_{S(G)}[v_i] \cap D_0 \neq N_{S(G)}[v_t] \cap D_0$. Then $N_{S(G)}[v_i] \cap D_0 = N_{S(G)}[v_j] \cap D_0 = \{v_{ij}\}$. Since $\delta(G) \geq n-2$, v_t is adjacent to v_i or v_j. If $v_t \in N_G(v_i)$, then $N_{S(G)}[v_{it}] \cap D_0 = \emptyset$ and similarly if $v_t \in N_G(v_j)$, then $N_{S(G)}[v_{jt}] \cap D_0 = \emptyset$.
However, it is a contradiction whit this fact that D_0 is a dominating set for $S(G)$. □

Corollary 2.1. *If $n \geq 3$, then $\gamma(S(K_n)) = n-1$.*

Proof. By Theorem 2.1, the proof is straightforward. □

Corollary 2.2. *If $s \geq 2$, then $\gamma(S(C_P(s))) = 2s-1$.*

Proof. By Theorem 2.1, the proof is straightforward. □

Example 2.1. In this example we show that the graph $S(Cp(2))$ has domination number of 3.(see Figure 2.1).

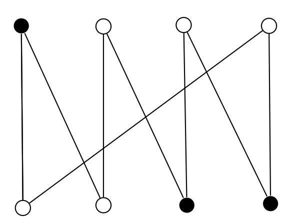


FIG. 2.1

Theorem 2.2. *Let G be isomorphic to the complete bipartite graph $K_{p,q}$. Then $\gamma(S(G)) = p+q-1$.*

Proof. Let $Y_1 = \{v_1, v_2, \cdots, v_p\}$ and $Y_2 = \{u_1, u_2, \cdots, u_q\}$. Let $V(S(G)) = Y_1 \cup Y_2 \cup \{x_{ij}\ :\ 1 \leq i \leq p,\ 1 \leq j \leq q,\ N_{S(G)}(x_{ij}) = \{v_i, u_j\}\}$, where Y_1 and Y_2 are partitions of graph G. Let $D = (V(G) \cup \{x_{11}\}) \setminus \{v_1, u_1\}$. Then v_1 and u_1 are dominated by x_{11}. All of the vertices in $V(S(G)) \setminus (V(G) \cup \{x_{11}\})$ are dominated by $D \setminus \{x_{11}\}$. So D is a dominating set for $S(G)$. Hence $\gamma(S(G)) \leq |\ D\ | = p+q-1$.

Let $\gamma(S(G)) = p+q-2$. We define set $L = \{| D \cap V(G) | : | D |= p+q-2,\ N_{S(G)}[D] = V(S(G))\}$. Let D_0 is a dominating set of $S(G)$ such that $|D_0| = p+q-2$ and $|D_0 \cap V(G)| = Max(L)$. If $x_{ij} \in D_0$ for some $1 \leq i \leq p$, $1 \leq j \leq q$, then $\{v_i, u_j\} \cap D_0 = \emptyset$, because $|D_0 \cap V(G)|$ is the maximum of L (if $v_i \in D_0$, then $(D_0 \cup \{u_j\}) \setminus \{x_{ij}\}$ is a dominating set for $S(G)$). Let $\{x_{ij}, x_{rs}\} \subseteq D_0$ and $j = s$. Then $\{v_i, v_r, u_s = u_j\} \cap D_0 = \emptyset$. It is easy to see that $(D_0 \cup \{v_i\}) \setminus \{x_{ij}\}$ is a dominating set for $S(G)$, this is contradiction with this fact that $| D_0 \cap V(G) |$ is the maximum of L. Similarly, if $\{x_{ij}, x_{rs}\} \subseteq D_0$, and $i = r$, then we have a contradiction. Now let $| D_0 \cap \{x_{ij} \ :\ 1 \leq i \leq p,\ 1 \leq j \leq q,\ N_{S(G)}(x_{ij}) = \{v_i, u_j\}\} | = t \geq 2$. Then $| V(G) \setminus D_0 | = 2t$ and so $t = 2$. Let $D_0 \cap \{x_{ij} \ :\ 1 \leq i \leq p,\ 1 \leq j \leq q,\ N_{S(G)}(x_{ij}) = \{v_i, u_j\}\} = \{x_{ij}\ ,\ x_{rs}\}$, where i, r and j, s are distinct. Since $N_{S(G)}[x_{rj}] = \{v_r, u_j, x_{rj}\}$ and $\{v_r, u_j, x_{rj}\} \cap D_0 = \emptyset$, so x_{ij} is not dominated by any vertex in D_0., which is a contradiction. Therefore, $\gamma(S(G)) = p+q-1$. □

Example 2.2. In this example we show that the graph $S(K_{2,6})$ has domination number of 7.(see figure 2.2).

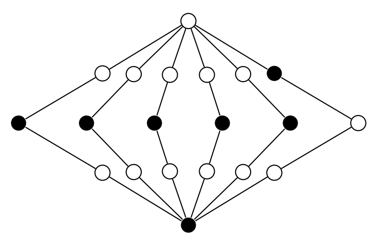


FIG. 2.2

It is well known, that for each $n \geqslant 2$, $\gamma(P_n) = \lceil \frac{n}{3} \rceil$. It seem that if the graph G is of order of n, then the domination number of $S(G) = n-1$. But this is not true, because $S(P_5) \cong P_9$ and $\gamma(S(P_5)) = 3$.

3. Watching number of some special graphs

In this Section, we obtain the watching number of Complete graphs, Star graph $K_{1,n-1}$ and Cocktail party graphs.

Theorem 3.1. *[1] Let G be a graph of order n. Then*

i) If G is twin free graph, then $\gamma(G) \leqslant \omega(G) \leqslant \gamma^{ID}(G)$,

ii) $\lceil log_2(n+1) \rceil \leqslant \omega(G) \leqslant \gamma(G) \lceil log_2(\Delta(G)+2) \rceil$.

Lemma 3.1. *If $n \geq 3$, then $\omega(K_n) = \omega(K_{1,n-1}) = \lceil log_2(n+1) \rceil$.*

Proof. By Theorem 3.1 *(ii)*, the proof is straightforward. □

Theorem 3.2. *If $s \geq 2$, then $\omega(C_P(s)) = \lceil log_2(2s+1) \rceil$.*

Proof. Let $V(G) = \{v_1, v_2, \cdots, v_n\}$, $X = \{v_2, v_3, \cdots, v_s\}$ and $Y = \{v_{s+2}, v_{s+3}, \cdots, v_{2s}\}$. Let X_1 be induced subgraph on $X \cup \{v_1\}$. By Theorem 3.1, $\omega(X_1) = \lceil log_2^{(s+1)} \rceil$. Let $W_1 = \{(v_1\ ,\ Z_i)\ :\ 1 \leq i \leq \lceil log_2(s+1) \rceil,\ Z_i \subseteq N_G[v_1]\}$ be a watching system for X_1 and $Z_{i+s} = \{v_{t+s}\ :\ v_t \in Z_i\}$. Also let

$$W_2 = \{(v_1\ ,\ Z_i \cup Z_{i+s})\ :\ 1 \leq i \leq \lceil log_2(s+1) \rceil\}$$

and

$$W = W_2\ \cup\ \{\omega_{s+1} = (v_{s+1}\ ,\ Y \cup \{v_{s+1}\})\ \}.$$

We have:

$$L_W(v_i) = L_{W_1}(v_i) \qquad 1 \leq i \leq s$$
$$L_W(v_i) = L_{W_1}(v_i) \cup \{\omega_{s+1}\} \qquad s+1 \leq i \leq 2s.$$

Thus W is a watching system for $C_P(s)$. Hence

$$\omega(C_P(s)) \leq |\ W\ | = \lceil log_2(s+1) \rceil + 1 = \lceil log_2(2s+2) \rceil = \lceil log_2(2s+1) \rceil.$$

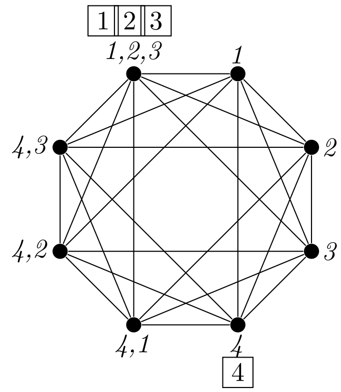
By Theorem 3.1, we have

$$\lceil log_2(2s+1) \rceil \leq \omega(CP(s)).$$

So $\lceil log_2(2s+1) \rceil = \omega(C_P(s))$. □

Example 3.1. We use this example to consider the graph $C_P(4)$(see figure 3.1). Watchers' locations are written down inside squares and labels nearby vertices, in italics. The watching number of $C_P(4)$ is 3.

Let $\omega_1 = (v_1, \{v_1, v_2, v_6\})$, $\omega_2 = (v_1, \{v_1, v_3, v_7\})$, $\omega_3 = (v_1, \{v_1, v_4, v_8\})$ and $\omega_4 = (v_5, \{v_5, v_6, v_7, v_8\})$. Then $W = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ is a watching system of $C_P(4)$. So we have $\omega(C_P(4)) \leq 4$. On the other hand by Theorem 3.1, $\lceil log_2 9 \rceil \leq \omega(CP(4))$. Therefore $\omega(C_P(4)) = 4$.



$C_P(4)$

Fig. 3.1

4. Watching number for some of $S(G)$

In this section, we obtain an upper bound for subdivision of a graph. Also we show that this upper bound is sharp. Specially, the watching number of graphs $P_n,\ C_n,\ K_{p,q},\ K_n,\ C_p(s)$ and watching number of their subdivision is calculated.

Theorem 4.1. *[7] Let $n \geqslant 2$ be a positive integer. Then*

i) $\omega(P_n) = \lceil \frac{n+1}{2} \rceil$

ii) $\omega(C_n) = \begin{cases} 3 & \text{if } n = 4 \\ \lceil \frac{n}{2} \rceil & \text{if } n \neq 4 \end{cases}$

Corollary 4.1. *If $n \geq 2$, then $\omega(S(P_n)) = \omega(S(C_n)) = n$.*

Proof. Since $S(P_n) \cong P_{2n-1}$ and $S(C_n) \cong C_{2n}$, by Theorem 4.1, $\omega(S(P_n)) = \omega(P_{2n-1}) = \lceil \frac{2n}{2} \rceil = n$ also $\omega(S(C_n)) = \omega(C_{2n}) = \lceil \frac{2n}{2} \rceil = n$. □

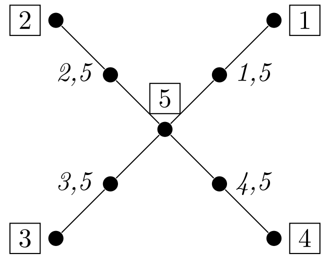
Theorem 4.2. *Let G be a graph of order n. Then $\omega(S(G)) \leq n$.*

Proof. Let $V(G) = \{v_1, v_2, \cdots, v_n\}$ and $V(S(G)) = V(G) \cup \{v_{ij} : v_i \in N_G(v(j)),\ 1 \leq i < j \leq n,\ N_{S(G))}(v_{ij}) = \{v_i, v_j\}\}$. Also let $\omega_i = (v_i\ ,\ N_{S(G)}(v_i))$ for $1 \leq i \leq n$ and $W = \{\omega_1, \omega_2, \cdots, \omega_n\}$. It is clear that $L_W(v_i) = \{\omega_i\},\ L_W(v_{ij}) = \{\omega_i, \omega_j\}$ where $1 \leq i \neq j \leq n$. Thus W is a watching system for $S(G)$. Hence, $\omega(S(G)) \leq n$. □

Theorem 4.3. *Let $n \geq 2$ and G be isomorphic to $K_{1,n-1}$. Then $\omega(S(G)) = n$.*

Proof. Let $V(G) = \{v_1, v_2, \cdots, v_n\}$ and $V(S(G)) = V(G) \cup \{v_{in} \;:\; 1 \le i < n\}$, where $deg_G(v_n) = n-1$ and $N_{S(G)}(v_{in}) = \{v_i, v_n\}$. By Theorem 4.2, $\omega(S(G)) \le n$. Let $\omega(S(G)) < n$ and $W = \{\omega_i = (x_i, Z_i) \;:\; Z_i \subseteq N_{S(G)}[x_i],\ 1 \le i \le n-1\}$ be a watching system for $S(G)$ with minimum cardinality. Since $D = \{x_1, x_2, \cdots, x_{n-1}\}$ is a dominating set of $S(G)$, $|D \cap \{v_i, v_{in}\}| = 1$ for $1 \le i \le n-1$. Also there is $1 \le j \le n-1$ such that $v_{jn} \in D$. It is easy to see that $L_W(v_j) = \{\omega_j\} = L_W(v_{jn})$, which is a contradiction. Hence, $\omega(S(G)) = n$. □

Example 4.1. Consider the graph $S(K_{1,4})$ (see figure 4.1). Watchers' locations are written down inside squares, hence by Theorem 4.3, the watching number of $S(K_{1,4})$ is 5.



$S(K_{1,4})$

FIG. 4.1

Theorem 4.4. *Let G be a connected graph of order $n \ge 2$. If $\gamma(S(G)) = n-1$, then $\omega(S(G)) = n$.*

Proof. Let $V(G) = \{v_1, \cdots, v_n\}$ and $V(S(G)) = V(G) \cup B$, where $B = \{v_{ij} \;:\; 1 \le i < j \le n\ ,\ v_iv_j \in E(G)\}$. By Theorems 4.2 and 3.1, $\gamma(S(G)) \le \omega(S(G)) \le n$. So $\omega(S(G)) \in \{n-1, n\}$. On the contrary, we assume that $\omega(S(G)) = n-1$ and W is a watching system for $S(G)$ with $|\ W\ | = n-1$. Also let $V_W = \{x_i \;:\; \omega_i = (x_i, Z_i) \in W\}$. Since $\gamma(S(G)) = n-1$ and V_W is a minimum dominating set for $S(G)$, so there is at most one watcher on each vertex of $S(G)$. Suppose that $V_W \cap V(G) = C_1$ and $V_W \cap B = C_2$.
Since $C_1 \cup C_2$ is a dominating set for bipartite graph $S(G)$, so $C_2 \ne \emptyset$.
If $C_1 = \emptyset$, then $C_2 = B$. Since $\gamma(S(G)) = n-1$ and V_W is a dominating set for $S(G)$, so $|\ B\ | = n-1$. Hence, the size of G is $n-1$. Thus G is a tree. Hence, G has a vertex v_t of degree 1. Let v_j be adjacent to v_t in G. Without loss of generality, we assume that $j < t$. Then $L_W(v_{jt}) = L_W(v_t) = \{\omega_{jt}\}$. This is not true.
Since $C_1 \cup C_2$ is a minimum dominating set for $S(G)$, for every $v_i \in V(G) \setminus C_1$ there exist $v_j \in V(G)$ such that $v_{ij} \in C_2$. It is clear that $|C_2| = |V(G) \setminus C_1| - 1$ and $|N_{S(G)}(x) \cap C_1| \le 1$ for $x \in C_2$. We define $T = \{x \in C_2 \;:\; |N_{S(G)}(x) \cap C_1| = 1\}$. We now consider the cases $T = \emptyset$ and $T \ne \emptyset$ separately.
Case 1. $T = \emptyset$. Let induced subgraph on $C_2 \cup (V(G) \setminus C_1)$ in $S(G)$ be H. It is

clear that $| E(H) | = 2| C_2 |$. On the contrary, let every vertex in H has degree at least two. Then we have

$$2|E(H)| = \sum_{x\in C_2} deg_H(x) + \sum_{y\in V(G)\setminus C_1} deg_H(y) \geq 2| C_2 | + 2(| C_2 | + 1) = 4| C_2 | + 1.$$

Which is a contradiction.Thus there exist $v_i \in V(G) \setminus C_1$ such that $| N_{S(G)}(v_i) \cap C_2 | = 1$. Suppose that $N_{S(G)}(v_i) \cap C_2 = \{v_{i\ell}\}$, where $v_\ell \in V(G) \setminus C_1$. Hence $L_W(v_i) = L_W(v_{i\ell}) = \{\omega_{i\ell}\}$. Which is false.
Case 2. Let $T \neq \emptyset$, $| T | = t$ and $N_{S(G)}(T) \setminus C_1 = F$. Then $| F | = t$ and so $| V(G) \setminus (C_1 \cup F) | > | C_2 \setminus T |$. Hence, there exist $v_i \in V(G) \setminus (C_1 \cup F)$ such that $| N_{S(G)}(v_i) \cap (C_2 \setminus T) | = 1$. Suppose that $N_{S(G)}(v_i) \cap (C_2 \setminus T) = \{v_{i\ell}\}$, where $v_\ell \in V(G) \setminus (C_1 \cup F)$. Hence, $L_W(v_i) = L_W(v_{i\ell}) = \{\omega_{i\ell}\}$. Which is false.
Therefore $\omega(S(G)) \neq n - 1$, and the theorem is proved. □

Corollary 4.2. *If $n \geq 3$, then $\omega(S(K_n)) = n$.*

Proof. By Corollary 2.1, $\gamma(S(K_n)) = n - 1$. By Theorem 4.4, $\omega(S(K_n)) = n$. □

Corollary 4.3. *If $s \geq 2$, then $\omega(S(C_P(s))) = 2s$.*

Proof. By Corollary 2.2, $\gamma(S(C_P(s))) = 2s - 1$. By Theorem 4.4, $\omega(S(C_P(s))) = 2s$. □

Corollary 4.4. *Let $p \geq 2$ and $q \geq 2$ be two integers. Then $\gamma(S(K_{p,q})) = p + q$.*

Proof. By Theorem 2.2, $\gamma(S(K_{p,q})) = p + q - 1$. By Theorem 4.4, $\omega(S(K_{p,q})) = p + q$. □

We end this section by the following conjecture.

Conjecture 1. *If G is a graph of order n, then $\omega(S(G)) = n$.*

5. Conclusion

The subdivision operation of G is an operation that replaces any edge by a path of order at least two. If each edge is replaced by a path of order three (and length two), then the subdivision graph is denoted by $S(G)$. In Sec. 2, the domination number of $S(G)$ have been investigated. The results showed that $\gamma(S(G)) \leq |G| - 1$ and this bound is sharp. It is shown that $\gamma(S(G)) = |G| - 1$, for $G \in \{K_n, K_{p,q}, CP(s)\}$. But there exist graph G such that $\gamma(S(G)) \neq |G| - 1$.
In Sec. 3, the watching number of $S(G)$ have been investigated. We have shown that for every graph G, the watching number of $S(G)$ is at most $|G|$ and this bound is sharp. It is shown that $\omega(S(G)) = |G|$, for $G \in \{P_n, C_n, K_n, K_{p,q}, CP(s)\}$. Finally, we conjecture that for each graph G, $\omega(S(G)) = |G|$.

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