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QUALITATIVE ANALYSIS OF SOLUTIONS FOR A TIMOSHENKO TYPE EQUATION WITH LOGARITHMIC SOURCE TERM

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Abstract. This paper deals with a Timoshenko type equation with strong damping and logarithmic source terms. The global existence and the decay estimate of the solutions have been obtained. We reproduce the finite time blow up results of weak solutions by the combining of the concavity method, perturbation energy method and differential–integral inequality technique. These results extend and improve some recent results in logarithmic nonlinearity.
Keywords: Timoshenko type equation, concavity method, perturbation energy method, logarithmic nonlinearity.

1. Introduction

We study the following Timoshenko type equation with strong damping and logarithmic source terms

$$(1.1)\quad \begin{cases} u_{tt} - M\left(\|\nabla u\|^2\right)\Delta u + \Delta^2 u - \Delta u_t = |u|^{p-2}\, u \ln|u|\,, & x \in \Omega,\ t > 0, \\ u\,(x,0) = u_0\,(x)\,, \quad u_t\,(x,0) = u_1\,(x)\,, & x \in \Omega, \\ u\,(x,t) = \frac{\partial}{\partial \nu} u\,(x,t) = 0, & x \in \partial\Omega,\ t \geq 0, \end{cases}$$

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where $\Omega \subset R^n$ $(n \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$, $M(s) = 1 + s^\gamma$, $\gamma > 0$; ν is the outer normal, and

$$\begin{cases} 2\gamma + 2 \leq p < \infty, & n \leq 4, \\ 2\gamma + 2 \leq p \leq \frac{2n}{n-4}, & n \geq 5. \end{cases}$$

This type of equation is derived from the extensible beam equation of Woinowsky-Krieger [30],

$$u_{tt} + u_{xxxx} - \left(\alpha_1 + \beta_1 \int_0^L |u_t| \, d\tau \right) u_{xx} + g(u_t) = 0,$$

for $g = 0$, where $u(x,t)$ is the deflection of the point x of the beam at the time t and $\alpha_1, \beta_1 > 0$ are constants.

Many authors have considered the following equation

$$u_{tt} + \Delta^2 u - M\left(\|\nabla u\|^2\right) \Delta u + |u_t|^{p-2} u_t = |u|^{q-2} u. \tag{1.2}$$

In [7, 8], Esquivel-Avila studied the attractor, unboundedness and convergence of solutions for the equation (1.2). In [23], Pişkin investigated the existence, nonexistence and decay estimates of solutions for the equation (1.2). Also, we note that many authors [10, 22, 24, 26–28] have considered (1.2).

In absent the $\Delta^2 u$ term the equation (1.1) can be named Kirchhoff type equation. This type equation is introduced by Kirchhoff [14]. The following form of Kirchhoff type equation

$$u_{tt} - M\left(\|\nabla u\|^2\right) \Delta u + f(u_t) = g(u), \tag{1.3}$$

was studied by a lot of authors [2, 20, 31, 32].

If we ser $g(u) = u \ln|u|$ and $M(s) = 1$, equation (1.3) becomes the classical wave equation with a logarithmic source term. This type of problems have many applications in many branches physics, such as quantum mechanics, nuclear physics, supersymmetric field theories, optics [3, 4].

In [5], Cazenave and Haraux considered the following equation

$$u_{tt} - \Delta u + u = u \ln|u|^k, \tag{1.4}$$

and they showed the existence of solutions in R^3. Numerous studies related to logarithmic nonlinearity can be found in the literature [1, 6, 9, 11, 12, 25].

Yang et. al [34] investigated the equation

$$u_{tt} - M\left(\|\nabla u\|^2\right) \Delta u + |u_t|^{p-1} u_t - \Delta u_t = u^{k-1} \ln|u|. \tag{1.5}$$

They studied the local existence, decay and finite time blow up of solutions. In [25], Pişkin and Irkıl discussed the problem (1.5) without strong damping term for negative initial energy by using modified energy functional method. In [12], the same authors considered the following equation

$$u_{tt} + M\left(\|\Delta u\|^2\right)\Delta^2 u + a_0 u_t + a_1 |u_t|^{r-1} u_t = |u|^{p-1} \ln |u|^k ,$$

where $a_0,\ a_1,\ k$ are a positive real number. They proved that the solution exists globally. Furthermore, they studied decay estimates result of the solutions.

On the other hand, problems involving the Kirchhoff term ($M\left(\|\Delta u\|^2\right)$) or the logarithmic source term ($\ln|u|$) have drawn significant interest [16–19, 21, 29, 33].

Upon examining the existing literature, although many studies address Kirchhoff-type equations with logarithmic source terms, very few studies focus on Timoshenko-type equations with such terms. Therefore, we considered the Timoshenko equation with a logarithmic source term.

This paper is organized as follows: In Section 2, we present some notations and lemmas that will be used in our proofs. In Section 3, we prove the global existence of the solution to problem (1.1) using the Faedo-Galerkin method. In Sections 4 and 5, we establish decay estimates and determine the upper and lower bounds for the blow-up time, respectively.

2. Preliminaries

In this work, we denote

$$\begin{cases} W^{2,2}(\Omega) = H^2(\Omega), \\ W^{0,p}(\Omega) = L^p(\Omega), \\ \int\limits_\Omega uv dx = \langle u, v\rangle, \\ \|u\| = \|u\|_{L^2(\Omega)} = \left(\int_\Omega |u|^2 dx\right)^{\frac{1}{2}}, \\ \|\cdot\|_p = \|\cdot\|_{L^p(\Omega)} = \left(\int_\Omega |u|^p dx\right)^{\frac{1}{p}}. \end{cases}$$

Moreover, $C_i\ (i = 1, 2, ...)$ are arbitrary constants.

We define the energy functional $E(t)$ of problem (1.1) as:

$$\begin{aligned} E(t) &= \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\Delta u\|^2 + \frac{1}{2}\|\nabla u\|^2 + \frac{1}{2\gamma+2}\|\nabla u\|^{2\gamma+2} \\ &\quad - \frac{1}{p}\int\limits_\Omega u^p \ln|u|\, dx + \frac{1}{p^2}\|u\|_p^p. \end{aligned} \tag{2.1}$$

Lemma 2.1. *$E(t)$ is non-increasing function for $t \ge 0$ and*

$$E'(t) = -\|\nabla u_t\|^2 \le 0. \tag{2.2}$$

Proof. Multiplying the equation in (1.1) by u_t and integrating on Ω, we have

$$\langle u_{tt}, u_t\rangle - \left\langle M\left(\|\nabla u\|^2\right)\Delta u, u_t\right\rangle + \left\langle \Delta^2 u, u_t\right\rangle - \langle \Delta u_t, u_t\rangle = \left\langle |u|^{p-2} u \ln|u|, u_t\right\rangle,$$

$$\begin{aligned} &\int_\Omega u_{tt}u_t dx + \int_\Omega M\left(\|\nabla u\|^2\right)\nabla u \nabla u_t dx + \int_\Omega \Delta u \Delta u_t dx + \int_\Omega \nabla u_t \nabla u_t dx \\ = &\int_\Omega |u|^{p-2} u \ln|u| u_t dx, \end{aligned}$$

$$\begin{aligned} &\frac{d}{dt}\left(\frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\Delta u\|^2 + \frac{1}{2}\|\nabla u\|^2 + \frac{1}{2\gamma+2}\|\nabla u\|^{2\gamma+2}\right) \\ &+\frac{d}{dt}\left(-\frac{1}{p}\int_\Omega u^p \ln|u| dx + \frac{1}{p^2}\|u\|_p^p\right) \\ = &-\|\nabla u_t\|^2 \end{aligned}$$

and

$$E'(t) = -\|\nabla u_t\|^2 \le 0, \tag{2.3}$$

$$\begin{aligned} E(t) + \int_0^t \int_\Omega |\nabla u_t|^2 dx dt &= E(0), \\ E(t) &\le E(0). \end{aligned} \tag{2.4}$$

where

$$\begin{aligned} E(0) = &\frac{1}{2}\|u_1\|^2 + \frac{1}{2}\|\Delta u_0\|^2 + \frac{1}{2}\|\nabla u_0\|^2 + \frac{1}{2\gamma+2}\|\nabla u_0\|^{2\gamma+2} \\ &-\frac{1}{p}\int_\Omega u_0^p \ln|u_0| dx + \frac{1}{p^2}\|u_0\|_p^p. \end{aligned} \tag{2.5}$$

□

3. Global existence

In this part, we prove the global existence of solutions for problem (1.1).

First, we define the following functionals

$$\begin{aligned} J(u) = &\tfrac{1}{2}\|\Delta u\|^2 + \tfrac{1}{2}\|\nabla u\|^2 + \tfrac{1}{2\gamma+2}\|\nabla u\|^{2\gamma+2} \\ &-\tfrac{1}{p}\int_\Omega u^p \ln|u| dx + \tfrac{1}{p^2}\|u\|_p^p \end{aligned} \tag{3.1}$$

and

$$I(u) = \|\Delta u\|^2 + \|\nabla u\|^2 + \|\nabla u\|^{2\gamma+2} - \int_\Omega u^p \ln|u| dx. \tag{3.2}$$

Clearly, we have

$$\begin{aligned} J(u) &= \frac{1}{p} I(u) + \left(\frac{p-2}{2p}\right) \|\Delta u\|^2 + \left(\frac{p-2}{2p}\right) \|\nabla u\|^2 \\ &\quad + \left(\frac{p-2\gamma-2}{p(2\gamma+2)}\right) \|\nabla u\|^{2\gamma+2} + \frac{1}{p^2} \|u\|_p^p \\ &> \left(\frac{p-2}{2p}\right) \|\nabla u\|^2 \end{aligned} \tag{3.3}$$

and

$$E(t) = \frac{1}{2} \|u_t\|^2 + J(u). \tag{3.4}$$

The depth of the potential well is defined by

$$W = \left\{ u \in H_0^2(\Omega) \mid J(u) < d,\ I(u) > 0 \right\} \cup \{0\} \tag{3.5}$$

and

$$V = \left\{ u \in H_0^2(\Omega) \mid J(u) < d,\ I(u) < 0 \right\}. \tag{3.6}$$

Now, we state some properties of $I(u)$ and $J(u)$.

Lemma 3.1. *For any $u \in H_0^2(\Omega)$, $\|\nabla u\| \neq 0$ and let $g(\lambda) = J(\lambda u)$. Then, we get*

i) $\lim_{\lambda \to 0} g(\lambda) = 0,\ \lim_{\lambda \to \infty} g(\lambda) = -\infty,$

ii) there exists a unique λ_1 such that $g'(\lambda) = 0$,

iii) $g(\lambda)$ is strictly decreasing on $\lambda_1 < \lambda$, strictly increasing on $0 \le \lambda \le \lambda_1$ and takes the maximum at $\lambda = \lambda_1$; $I(\lambda u) = \lambda g'(\lambda)$ and $I(\lambda u) = \lambda \frac{d}{d\lambda} J(\lambda u)$.

$$I(\lambda u) \begin{cases} > 0, & 0 \le \lambda \le \lambda_1, \\ = 0, & \lambda = \lambda_1, \\ < 0, & \lambda_1 \le \lambda. \end{cases}$$

Proof. i) By the definition of $J(u)$, we get

$$\begin{aligned} g(\lambda) &= J(\lambda u) \\ &= \frac{1}{2} \|\lambda \Delta u\|^2 + \frac{1}{2} \|\lambda \nabla u\|^2 + \frac{1}{2\gamma+2} \|\lambda \nabla u\|^{2\gamma+2} \\ &\quad - \frac{1}{p} \int_\Omega (\lambda u)^p \ln|\lambda u|\, dx + \frac{1}{p^2} \int_\Omega |\lambda u|^p\, dx \\ &= \frac{1}{2} \lambda^2 \|\Delta u\|^2 + \frac{1}{2} \lambda^2 \|\nabla u\|^2 + \frac{1}{2\gamma+2} \lambda^{2\gamma+2} \|\nabla u\|^{2\gamma+2} \\ &\quad - \frac{1}{p} \lambda^p \int_\Omega u^p \ln|u|\, dx - \frac{1}{p} \lambda^p \ln \lambda \int_\Omega u^p dx + \frac{1}{p^2} \lambda^p \|u\|_p^p, \end{aligned} \tag{3.7}$$

which means $\lim\limits_{\lambda\to 0} g(\lambda)=0$, $\lim\limits_{\lambda\to\infty} g(\lambda)=-\infty$.

ii) Now, differentiating $g(\lambda)$ with respect to λ,
we have

$$\begin{aligned}
& \frac{d}{d\lambda}J(\lambda u) \\
= \;& g'(\lambda)=\lambda\|\Delta u\|^2+\lambda\|\nabla u\|^2+\lambda^{2\gamma+1}\|\nabla u\|^{2\gamma+2} \\
& -\lambda^{p-1}\int_\Omega u^p\ln|u|\,dx-\lambda^{p-1}\ln\lambda\|u\|_p^p \\
= \;& \lambda\Big(\|\Delta u\|^2+\|\nabla u\|^2+\lambda^{2\gamma}\|\nabla u\|^{2\gamma+2} \\
& -\lambda^{p-2}\int_\Omega u^p\ln|u|\,dx-\lambda^{p-2}\ln\lambda\|u\|_p^p\Big) \\
= \;& \lambda\Big(\|\Delta u\|^2+\|\nabla u\|^2+\psi(\lambda)\Big),
\end{aligned} \tag{3.8}$$

where

$$\begin{aligned}
\psi(\lambda) &= \lambda^{2\gamma}\|\nabla u\|^{2\gamma+2}-\lambda^{p-2}\int_\Omega u^p\ln|u|\,dx-\lambda^{p-2}\ln\lambda\|u\|_p^p \\
&= \lambda^{p-2}\left(k\lambda^{2\gamma-p+2}-m-n\ln|\lambda|\right),
\end{aligned}$$

where $k=\|\nabla u\|^{2\gamma+2}\geq 0$, $m=\int\limits_\Omega u^p\ln|u|\,dx$, $n=\|u\|_p^p$. We observe from $2\gamma\leq p-2$ and $\gamma>0$ that $\lim\limits_{\lambda\to\infty}\psi(\lambda)=-\infty$, $\lim\limits_{\lambda\to 0}\psi(\lambda)=0$.

$$\begin{aligned}
\psi'(\lambda) &= (p-2)\lambda^{p-3}\left(k\lambda^{2\gamma+2-p}-m-n\ln|\lambda|\right) \\
&\quad +\lambda^{p-2}\left(k(2\gamma+2-p)\lambda^{2\gamma+1-p}-\frac{n}{\lambda}\right) \\
&= (p-2)\lambda^{p-3}\left(k\lambda^{2\gamma+2-p}-m-n\ln|\lambda|\right) \\
&\quad +\lambda^{p-3}\left(k(2\gamma+2-p)\lambda^{2\gamma+2-p}-n\right) \\
&= \lambda^{p-3}\left(2k\gamma\lambda^{2\gamma+2-p}-(p-2)m-(p-2)n\ln|\lambda|-n\right).
\end{aligned}$$

Where $g(\lambda)=2k\gamma\lambda^{2\gamma+2-p}-(p-2)m-(p-2)n\ln|\lambda|-n$.

$$\lim_{\lambda\to\infty} g(\lambda)=-\infty,\ \lim_{\lambda\to 0} g(\lambda)=0$$

and

$$g'(\lambda)=\frac{2k\gamma(2\gamma+2-p)\lambda^{2\gamma+2-p}-(p-2)m-(p-2)n}{\lambda}<0.$$

When $\lambda=\lambda^*$ and there exists a unique λ^* such that $g(\lambda^*)=0$. Consequently

$$\psi'(\lambda)\begin{cases} >0, & 0\le\lambda<\lambda_1, \\ =0, & \lambda=\lambda_1, \\ <0, & \lambda_1<\lambda. \end{cases}$$

Then we can see $\psi(\lambda)$ is monotone decreasing when $\lambda>\lambda^*$ and there exists a unique λ^* such that $\psi(\lambda^*)=0$. Then we have there is a $\lambda_1>\lambda^*$ such that $\lambda\left[\|\Delta u\|^2+\|\nabla u\|^2+\psi(\lambda)\right]=0$, which means $g'(\lambda_1)=0$.

iii) The result (ii) and from the definition of $I(u)$,

$$\begin{aligned} I(\lambda u) &= \|\lambda\Delta u\|^2+\|\lambda\nabla u\|^2+\|\lambda\nabla u\|^{2\gamma+2}-\int_\Omega|\lambda u|^p\ln|\lambda u|\,dx \\ &= \lambda^2\|\Delta u\|^2+\lambda^2\|\nabla u\|^2+\lambda^{2\gamma+2}\|\nabla u\|^{2\gamma+2} \\ &\quad -\lambda^p\int_\Omega u^p\ln|u|\,dx-\lambda^p\ln|\lambda|\int_\Omega u^p dx \\ &= \lambda\left(\lambda\|\Delta u\|^2+\lambda\|\nabla u\|^2+\lambda^{2\gamma+1}\|\nabla u\|^{2\gamma+2}\right) \\ &\quad +\left(-\lambda^{p-1}\int_\Omega u^p\ln|u|\,dx-\lambda^{p-1}\ln|\lambda|\,\|u\|_p^p\right) \\ &= \lambda\frac{dJ(\lambda u)}{d\lambda} \end{aligned} \tag{3.9}$$

$\square$

Lemma 3.2. *i) The definition the depth of potential well*

$$d=\inf_{u\in N}J(u),\, N=\left\{u\in H_0^2(\Omega)\setminus\{0\}:I(u)=0\right\}, \tag{3.10}$$

is equivalent to

$$d=\inf\left\{\sup_{\lambda\ge0}J(\lambda u)\mid u\in H_0^2(\Omega),\ \|\Delta u\|^2\ne0\right\}. \tag{3.11}$$

ii) d is defined as

$$d=\left(\frac{p-2}{2p}\right)\left(\frac{1}{C_*^{p+1}}\right)^{\frac{2}{p-1}}.$$

Proof. i) On one hand from (iii) of Lemma 3.1 it means that for any $u\in H_0^2(\Omega)$, there exist a λ_1 such that $I(\lambda_1 u)=0$, that is $\lambda_1 u\in N$. By the definition of d we obtain

$$J(\lambda_1 u)\ge d\ foranyu\in H_0^2(\Omega)\setminus\{0\}, \tag{3.12}$$

and because of Lemma 3.1 of property (iii), this λ_1 is also the maximizer of $J(\lambda u)$ such that

$$\sup_{\lambda\geq 0} J(\lambda u) = J(\lambda_1 u),$$

which by virtue of (3.12) means

$$\inf_{u\in H_0^2(\Omega)} \sup_{\lambda\geq 0} J(\lambda u) = \inf_{u\in H_0^2(\Omega)} J(\lambda_1 u) \geq d. \tag{3.13}$$

As $u \in H_0^2(\Omega)\setminus\{0\}$, we obtain d is not equivalent to 0, which gives (3.11). But then, from (3.11) it means that there exists λ^* such that

$$\sup_{\lambda\geq 0} J(\lambda u) = J(\lambda^* u).$$

Then from Lemma 3.1 we can deduce $\lambda^* = \lambda_1$. Again from Lemma 3.1 of property (iii) it shows that

$$I(\lambda^* u) = I(\lambda_1 u) = 0,$$

which means $\lambda^* u \in N$. By the definition of d we get

$$d = \inf_{\lambda^* u\in N} J(\lambda^* u),$$

that is

$$d = \inf_{u\in N} J(u). \tag{3.14}$$

This complete our proof for (i).

ii) By virtue of $I(u) = 0$, definition of $I(u)$ and embedding theorems, we obtain

$$\begin{aligned} \|\nabla u\|^2 &\leq \|\Delta u\|^2 + \|\nabla u\|^2 + \|\nabla u\|^{2\gamma+2} \\ &= \int_\Omega u^p \ln|u|\, dx \\ &\leq \|u\|_{p+1}^{p+1} \\ &\leq C_*^{p+1} \|\nabla u\|^{p+1}, \end{aligned} \tag{3.15}$$

$$\|\nabla u\| \geq \left(\frac{1}{C_*^{p+1}}\right)^{\frac{1}{p-1}}. \tag{3.16}$$

From the definition of d, we have $u \in N$. By (3.16) and $I(u) = 0$, we have

$$\begin{aligned} J(u) &= \frac{1}{p} I(u) + \left(\frac{p-2}{2p}\right) \|\Delta u\|^2 + \left(\frac{p-2}{2p}\right) \|\nabla u\|^2 \\ &\quad + \left(\frac{p-2\gamma-2}{p(2\gamma+2)}\right) \|\nabla u\|^{2\gamma+2} + \frac{1}{p^2} \|u\|_p^p \\ &\geq \left(\frac{p-2}{2p}\right) \|\Delta u\|^2 \\ &\geq \left(\frac{p-2}{2p}\right) \left(\frac{1}{C_*^{p+1}}\right)^{\frac{2}{p-1}} \\ &\geq d. \end{aligned}$$

we take $2\gamma \leq p-2$. Combining of (3.14) and (3.16), we can see clearly that

$$d = \left(\frac{p-2}{2p}\right) \left(\frac{1}{C_*^{p+1}}\right)^{\frac{2}{p-1}}$$

□

Definition 3.1. A function $u(x,t)$ is called a weak solution to problem (1.1) on $\Omega \times [0,T)$, if

$$u \in C\left((0,T); H_0^2(\Omega)\right) \cap C^1\left((0,T); H_0^1(\Omega)\right),$$

satisfy

$$\begin{cases} \int\limits_\Omega u_{tt}(x,t) w(x) \, dx + \int\limits_\Omega \Delta u \Delta w(x) \, dx + \int\limits_\Omega M\left(\|\nabla u\|^2\right) \nabla u \nabla w(x) \, dx \\ + \int\limits_\Omega \nabla u_t \nabla w(x) \, dx \;\; = \int\limits_\Omega u(x,t) \ln u(x,t) |u|^{p-2}(x,t) w(x) \, dx. \end{cases}$$

Where $u \in H_0^2(\Omega)$.

Lemma 3.3. *Let $u(t)$ be a weak solution problem of (1.1) and $u_0 \in H_0^2(\Omega)$, $u_1 \in H_0^1(\Omega)$. Suppose that $0 < E(0) < d$.*

i) If $I(u_0) > 0$, $u \in W$,

ii) If $I(u_0) < 0$, $u \in V$.

Proof. i) If $u(t)$ is a weak solution problem of (1.1) satisfying $0 < E(0) < d$, and for $t \in [0,T)$

$$\frac{1}{2} \|u_t\|^2 + J(u) = \frac{1}{2} \|u_1\|^2 + J(u_0) < d$$

under the conditions $u(t)$, $E(0) < d$, $u_1 \in H_0^1(\Omega)$ then by (2.4) says that

$$E(u(t)) < E(0) < d.$$

We shall prove $I(u(t)) > 0$ for $0 < t < T$. We will use conradiction and we suppose that; there is a $t_1 \in (0, T)$ such that $I(u(t_1)) < 0$. Observe by the continuity of $I(u(t))$ in t that there exists a $t^* \in (0, T)$ such that $I(u(t^*)) = 0$. Then by (3.10), we get

$$d > E(0) \geq E(u(t^*)) \geq J(u(t^*)) \geq d,$$

which is a contradiction.

ii) The proof of case (ii) is similar. □

Lemma 3.4. *Under the conditions of Lemma 3.3 in (i), we obtain*

$$E(0) \geq E(u) \geq J(u) > \left(\frac{p-2}{2p}\right) \|\nabla u\|^2 .$$

Proof. By definition of $J(u)$, $I(u)$ and $I(u) > 0$, we get

$$\begin{aligned} J(u) &= \frac{1}{p} I(u) + \left(\frac{p-2}{2p}\right) \|\Delta u\|^2 + \left(\frac{p-2}{2p}\right) \|\nabla u\|^2 \\ &\quad + \left(\frac{p-2\gamma-2}{p(2\gamma+2)}\right) \|\nabla u\|^{2\gamma+2} + \frac{1}{p^2} \|u\|_p^p \\ &> \left(\frac{p-2}{2p}\right) \|\Delta u\|^2 + \left(\frac{p-2}{2p}\right) \|\nabla u\|^2 \\ &\quad + \left(\frac{p-2\gamma-2}{p(2\gamma+2)}\right) \|\nabla u\|^{2\gamma+2} + \frac{1}{p^2} \|u\|_p^p \\ &> \left(\frac{p-2}{2p}\right) \|\nabla u\|^2 . \end{aligned}$$

Because of (3.4) and (2.4) we can see clearly that

$$E(0) \geq E(u) \geq J(u) > \left(\frac{p-2}{2p}\right) \|\nabla u\|^2 .$$

□

Theorem 3.1. *Let $u_0 \in H_0^2(\Omega), u_1 \in H_0^1(\Omega)$. If $I(u_0) > 0$ and $E(0) < d$ or $\|\nabla u_0\| = 0$, then problem (1.1) admits a global weak solution $u(t) \in L^\infty\left(0, \infty; H_0^2(\Omega)\right)$, $u_t(t) \in L^\infty\left(0, \infty; H_0^1(\Omega)\right)$.*

Proof. Let $\{w_j\}_{j=1}^\infty$ be a basis in space $H_0^2(\Omega)$.

$$\begin{aligned} u_0^m(x, 0) &= \sum_{j=1}^m a_j^m w_j(x) \to u_0 \text{ in } H_0^2(\Omega), \\ u_1^m(x) &= \sum_{j=1}^m b_j^m w_j(x) \to u_1 \text{ in } H_0^1(\Omega), \end{aligned}$$

for $j = 1, 2, ..., m$.

We look for the approximate solutions

$$u^m(x,t) = \sum_{j=1}^{m} h_j^m(t)\, w_j(x),$$

$m = 1, 2, ...$

$$\begin{cases} \int_\Omega \left(u_{tt}^m w_k + \Delta u^m \Delta w_k + M\left(\|\nabla u\|^2\right) \nabla u^m \nabla w_k + \nabla u_t^m \nabla w_k\right) dx \\ \qquad = \int_\Omega u^m |u^m|^{p-2} \ln |u^m|\, w_k dx, \qquad k = 1, 2, ..., m. \end{cases} \tag{3.17}$$

(3.17) multiplying $h_j^m(t)$ and if gathers for k. According to the standard existence theory for ordinary differantial equation, one can obtain functions

$$h_j : [0, t_m) \to R, \quad j = 1, 2, ..., m,$$

which satisfy (3.17) in a maximal interval $[0, t_m)$, $0 < t_m \leq T$. Now, we show that $t_m = T$ and that the local solution is uniformly bounded independent of m and t. For this purpose, let us replace w by u_t^m in (3.17) and integrate by parts, we have

$$\frac{d}{dt} E^m(t) = -\|\nabla u_t^m\|^2 \leq 0, \tag{3.18}$$

where

$$\begin{aligned} & \frac{d}{dt}\left(\frac{1}{2}\|u_t^m\|^2 + \frac{1}{2}\|\Delta u^m\|^2 + \frac{1}{2}\|\nabla u^m\|^2 + \frac{1}{2\gamma+2}\|\nabla u^m\|^{2\gamma+2}\right) \\ & + \frac{d}{dt}\left(-\frac{1}{p}\int_\Omega |u^m|^p \ln|u^m|\, dx + \frac{1}{p^2}\|u^m\|_p^p\right) \\ = \; & -\|\nabla u_t^m\|^2 \end{aligned} \tag{3.19}$$

Integrating (3.18) from 0 to t, and using of (3.4), we obtain

$$\frac{1}{2}\|u_t^m\|^2 + J(u^m) + \int_0^t \|\nabla u_s^m\|^2 ds = E^m(0). \tag{3.20}$$

By virtue problem of (3.17) initial data, while $m \to \infty$ we obtain $E^m(0) \to E(0)$. By choosing of large m we have

$$\frac{1}{2}\|u_t^m\|^2 + J(u^m) + \int_0^t \|\nabla u_s^m\|^2 ds < d. \tag{3.21}$$

From Lemma 3.4

$$\begin{aligned} J(u^m) \; = \; & \frac{1}{p} I(u^m) + \left(\frac{p-2}{2p}\right)\|\Delta u^m\|^2 + \left(\frac{p-2}{2p}\right)\|\nabla u^m\|^2 \\ & + \left(\frac{p-2\gamma-2}{p(2\gamma+2)}\right)\|\nabla u^m\|^{2\gamma+2} + \frac{1}{p^2}\|u^m\|_p^p. \end{aligned}$$

Then, we have

$$\frac{1}{2}\|u_t^m(0)\|^2 + J(u^m(0)) = E(0)$$

and initial data, for choosing large m and $0 \le t < \infty$, we get $u^m(0) \in W$. By (3.21) and an argument similar to Lemma 3.3, by choosing large m and $0 \le t < \infty$, we have $u^m(t) \in W$. Therefore, by virtue of (3.21) and (3.1) we get

$$\begin{aligned}
&\frac{1}{2}\|u_t^m\|^2 + \frac{1}{2}\|\Delta u^m\|^2 + \frac{1}{2}\|\nabla u^m\|^2 \\
&+\frac{1}{2\gamma+2}\|\nabla u^m\|^{2\gamma+2} - \frac{1}{p}\int_\Omega |u^m|^p \ln|u^m|\,dx \\
&+\frac{1}{p^2}\|u^m\|_p^p + \int_0^t \|\nabla u_s^m\|^2\,ds \\
&< d \qquad (3.22)
\end{aligned}$$

where $0 \le t < \infty$ and $p \ge 2\gamma + 2$. For a sufficiently large m and $0 \le t < \infty$, (3.22) gives

$$\|u_t^m\|^2 < 2d,$$

$$\|\Delta u^m\|^2 < \frac{2p}{p-2}d,$$

$$\|\nabla u^m\|^2 < \frac{2p}{p-2}d,$$

$$\|\nabla u^m\|^{2\gamma+2} < \frac{p(2\gamma+2)}{p-2\gamma-2}d,$$

$$\|u^m\|_p^p < p^2 d,$$

and

$$\int_0^t \|\nabla u_s^m\|^2\,ds < d.$$

Then, we obtain

$$\begin{cases}
u^m, \text{ is uniformly bounded in } L^\infty\left(0,\infty; H_0^2(\Omega)\right), \\
u_t^m, \text{ is uniformly bounded in } L^\infty\left(0,\infty; L^2(\Omega)\right), \\
|u^m|^p, \text{ is uniformly bounded in } L^\infty\left(0,\infty; L^p(\Omega)\right), \\
|u_t^m|^{p+1}, \text{ is uniformly bounded in } L^{p+1}\left(0,\infty; L^{p+1}(\Omega)\right).
\end{cases}$$

By using the Sobolev embedding inequality, (3.21) and (3.22), we get

$$\begin{aligned}\int_\Omega |u^m|^p \ln |u^m|\, dx &\leq \|u^m\|_{p+1}^{p+1} \\ &\leq C_p \|\nabla u^m\|_2^{p+1} \\ &< \left(\frac{2pd}{C_1(p-2)}\right)^{\frac{p+1}{2}},\end{aligned}$$

so that we obtain

$$|u^m|^{p+1}, \text{ is uniformly bounded in } L^\infty\left(0,\infty; L^{p+1}(\Omega)\right).$$

Then integrating (3.17) with respect to t , for $0 \leq t < \infty$, we have

$$\begin{aligned}&(u_t^m, w_k) + (\nabla u^m, \nabla w_k) + \int_0^t (\triangle u^m, \Delta w_k)\, dk \\ &- \int_0^t M\left(\|\nabla u^m\|^2\right)(\nabla u^m, \nabla w_k)\, dk \\ = \; &(u_1, w_k) + (\triangle u_0, \Delta w_k) + \int_0^t \left(\ln |u^m| |u^m|^{p-1}, w_k\right) dk.\end{aligned} \tag{3.23}$$

Therefore, up to a subsequence, we may pass to the limit in (3.23), and get a weak solution (u) to problem (1.1) with the above regularity. On the other hand, initial data conditions in (3.17) we may conclude $(u(x,0)) = (u_0)$ in $H_0^2(\Omega)$ and $(u_t(x,0)) = (u_1)$ in $H_0^1(\Omega)$. □

4. Decay Estimates

In this part, we study the decay estimates for the solutions of problem (1.1).

Theorem 4.1. *Let $u_0(x) \in H_0^2(\Omega)$, $u_1(x) \in H_0^1(\Omega)$. Suppose that $E(0) < d$, $I(u_0) > 0$ or $\|\nabla u_0\| = 0$. Then*

$$E(t) \leq Ne^{-nt}, t \geq 0,$$

where N and n are positive constants.

Proof. Small enough for $\epsilon > 0$. Let

$$L(t) = E(t) + \varepsilon \int_\Omega u u_t + \frac{\varepsilon}{2}\|\nabla u\|^2, \tag{4.1}$$

then we observe for sufficient small ε that there exist positive constants λ_1, λ_2, such that

$$\lambda_1 E(t) \leq L(t) \leq \lambda_2 E(t), \tag{4.2}$$

and $L(t) > 0$ for any $t \geq 0$.

By multiplying the (1.1) by u and integrating on Ω, we obtain

$$\begin{aligned} \int_\Omega u_{tt} u dx &= \int_\Omega u^2 |u|^{p-2} \ln |u| \, dx \\ &- \int_\Omega M\left(\|\nabla u\|^2\right) \nabla u \nabla u dx - \int_\Omega \Delta u \Delta u dx - \int_\Omega \nabla u_t \nabla u dx. \end{aligned} \tag{4.3}$$

By derivative of (4.1) and using of (2.3) and (2.1) we obtain

$$\begin{aligned} L'(t) &= E'(t) + \varepsilon \left(\|u_t\|^2 + \int_\Omega u u_{tt} dx + \int_\Omega \nabla u_t \nabla u dx \right) \\ &\leq -\|\nabla u_t\|^2 + \varepsilon \|u_t\|^2 + \varepsilon \int_\Omega |u|^p \ln |u| \, dx \\ &\quad -\varepsilon \|\nabla u\|^2 - \varepsilon \|\nabla u\|^{2\gamma+2} - \varepsilon \|\Delta u\|^2 \\ &\leq -\|\nabla u_t\|^2 + \varepsilon \|u_t\|^2 + \varepsilon \|u\|_{p+1}^{p+1} \\ &\quad -\varepsilon \|\nabla u\|^2 - \varepsilon \|\nabla u\|^{2\gamma+2} - \varepsilon \|\Delta u\|^2 \end{aligned} \tag{4.4}$$

Now, our aim is to estimate every term of (4.4) severally.

$$\|\nabla u\|^2 < \frac{2p}{p-2} J(u) \leq \frac{2p}{p-2} E(t) \leq \frac{2p}{p-2} E(0).$$

Thanks to Sobolev embedding inequality and Lemma 3.4, we conclude

$$\begin{aligned} \int_\Omega |u|^p \ln |u| \, dx &\leq \|u\|_{p+1}^{p+1} \\ &\leq C_*^{p+1} \|\nabla u\|^{p+1} \\ &\leq C_*^{p+1} \left(\frac{2p}{p-2} E(0) \right)^{\frac{p-1}{2}} \|\nabla u\|^2 \\ &= \alpha \|\nabla u\|^2, \end{aligned} \tag{4.5}$$

where $\alpha = C_*^{p+1} \left(\frac{2p}{p-2} E(0) \right)^{\frac{p-1}{2}}$. $E(0) < d$ and from Lemma 3.2

$$d = \left(\frac{p-2}{2p} \right) \left(\frac{1}{C_*^{p+1}} \right)^{\frac{2}{p-1}} > E(0), \tag{4.6}$$

$$\alpha = C_*^{p+1} \left(\frac{2p}{p-2} E(0) \right)^{\frac{p-1}{2}} < 1. \tag{4.7}$$

From (2.5), (4.5) and taking $0 \leq k \leq 1$, we get

$$\begin{aligned}
\|u\|_{p+1}^{p+1} &= (1-k)\|u\|_{p+1}^{p+1} + k\|u\|_{p+1}^{p+1} \\
&< (1-k)\,\alpha\|\nabla u\|^{2} + k\|u\|_{p+1}^{p+1} \\
&\leq (1-k)\,\alpha\|\nabla u\|^{2} - kpE(t) \\
&\quad + kp\left(\frac{1}{2}\|u_t\|^{2} + \frac{1}{2}\|\nabla u\|^{2} + \frac{1}{2\gamma+2}\|\nabla u\|^{2\gamma+2}\right) \\
&\quad + kp\left(\frac{1}{2}\|\Delta u\|^{2} + \frac{1}{p^{2}}\|u\|_{p}^{p}\right) \qquad (4.8)
\end{aligned}$$

substituting (4.8) for (4.4),

$$\begin{aligned}
L'(t) &\leq \left(\frac{k\varepsilon p}{2} + \varepsilon\right)\|u_t\|^{2} - \varepsilon\left(1 - \frac{kp}{2} - (1-k)\,\alpha\right)\|\nabla u\|^{2} \\
&\quad - \varepsilon\left(1 - \frac{kp}{2\gamma+2}\right)\|\nabla u\|^{2\gamma+2} - \varepsilon\left(1 - \frac{kp}{2}\right)\|\Delta u\|^{2} \\
&\quad - \|\nabla u_t\|^{2} + \frac{k\varepsilon}{p}\|u\|_{p}^{p} - kp\epsilon E(t) \qquad (4.9)
\end{aligned}$$

From the Sobolev embedding we have

$$\begin{aligned}
\|u\|_{p}^{p} &\leq C_1^{p}\|\nabla u\|_{2}^{p} \\
&\leq C_1^{p}\left(\frac{2p}{p-2}E(0)\right)^{\frac{p-2}{2}} \frac{2p}{p-2}E(t), \qquad (4.10)
\end{aligned}$$

$$\|u_t\|^{2} \leq C_2^{2}\|\nabla u_t\|^{2}, \qquad (4.11)$$

$$\begin{aligned}
\|u\|_{p+1}^{p+1} &\leq C_3^{p+1}\|\nabla u\|^{p+1} \\
&< C_3^{p+1}\left(\frac{2p}{p-2}E(0)\right)^{\frac{p-2}{2}} \frac{2p}{p-2}E(t). \qquad (4.12)
\end{aligned}$$

If choosing $k < \frac{2}{p}, 1 - \frac{kp}{2} - (1-k)\,\alpha < 0$ and using together (4.7), (4.10), (4.11) and (4.12),

$$
\begin{aligned}
L'(t) &\leq \epsilon\left(C_2^2\left(\frac{kp}{2}+1\right)-1\right)\|\nabla u_t\|^2+\varepsilon\left(-1+\frac{kp}{2}+(1-k)\alpha\right)\frac{2p}{p-2}E(t)\\
&\quad -\varepsilon\left(1-\frac{kp}{2\gamma+2}\right)\|\nabla u\|^{2\gamma+2}-\varepsilon\left(1-\frac{kp}{2}\right)\|\Delta u\|^2+\frac{\beta\varepsilon}{p^2}\|u\|_p^p\\
&\quad +\frac{k\varepsilon}{p}C_1^p\left(\frac{2p}{p-2}E(0)\right)^{\frac{p-2}{2}}\frac{2p}{p-2}E(t)-k\varepsilon pE(t)\\
&\quad +\varepsilon C_3^{p+1}\left(\frac{2p}{p-2}E(0)\right)^{\frac{p-2}{2}}\frac{2p}{p-2}E(t)\\
&\leq \epsilon\left(C_2^2\left(\frac{kp}{2}+1\right)-1\right)\|\nabla u_t\|^2-\varepsilon\left(1-\frac{kp}{2}\right)\|\Delta u\|^2\\
&\quad +\epsilon\left(-1+\frac{kp}{2}+(1-k)\alpha\right)\frac{2p}{p-2}E(t)\\
&\quad -\varepsilon\left(1-\frac{kp}{2\gamma+2}\right)\|\nabla u\|^{2\gamma+2}+\frac{k\varepsilon}{p}C_1^p\left(\frac{2p}{p-2}E(0)\right)^{\frac{p-2}{2}}\frac{2p}{p-2}E(t)\\
&\quad -k\varepsilon pE(t)+\varepsilon C_3^{p+1}\left(\frac{2p}{p-2}E(0)\right)^{\frac{p-2}{2}}\frac{2p}{p-2}E(t)\\
&= h_1\|\nabla u_t\|^2+h_2\|\nabla u\|^{2\gamma+2}+h_3\frac{2p}{p-2}E(t)+h_4\|\Delta u\|^2, \qquad (4.13)
\end{aligned}
$$

where

$$h_1=\epsilon\left(C_2^2\left(\frac{kp}{2}+1\right)-1\right),$$

$$h_2=-\varepsilon\left(1-\frac{kp}{2\gamma+2}\right),$$

$$
\begin{aligned}
h_3 &= \varepsilon\left(-1+\frac{kp}{2}+(1-k)\alpha\right)+\frac{k\varepsilon}{p}C_1^p\left(\frac{2p}{p-2}E(0)\right)^{\frac{p-2}{2}}\\
&\quad +\varepsilon C_3^{p+1}\left(\frac{2p}{p-2}E(0)\right)^{\frac{p-2}{2}}-k\epsilon\left(\frac{p-2}{2}\right),
\end{aligned}
$$

$$h_4=-\varepsilon\left(1-\frac{kp}{2}\right).$$

$k\to 0$ and we choose ϵ small enough so that, $2\gamma+2<p$, $0\leq k\leq\frac{2}{p}\leq\frac{2\gamma+2}{p}\leq 1$, using (4.7)

$$
\begin{aligned}
L'(t) &\leq h_3\frac{2p}{p-2}E(t)\\
&\leq h_3\frac{2p}{p-2}\frac{L(t)}{\lambda_1} \qquad (4.14)
\end{aligned}
$$

Finally, a simple integration of (4.14) over $(0,t)$ then yields

$$E\left(t\right) \leq Ne^{-nt}.$$

where $N = \frac{L(0)}{\lambda_1}$ and $n = -h_3\frac{2p}{\lambda_2(p-2)}$. This completed our proof. □

5. Blow up

In this section, we establish the upper and lower bounds for the blow-up time.

5.1. Upper bound for the blow up time

In this part, we prove an upper bound for the blow up time.

Lemma 5.1. *[13, 15]. Let $\Phi\left(t\right)$ be a positive C^2 function, which satisfies, for $t > 0$, inequality*

$$\Phi\left(t\right)\Phi''\left(t\right) - \left(1+\beta\right)\left[\Phi'\left(t\right)\right]^2 \geq 0, \tag{5.1}$$

with some $\beta > 0$. If $\Phi\left(0\right) > 0$ and $\Phi'\left(0\right) > 0$, then there exist a time $T^* \leq \frac{\Phi(0)}{\beta\Phi'(0)}$ such that

$$\lim_{t\to T^{*-}} \Phi\left(t\right) = \infty. \tag{5.2}$$

Theorem 5.1. *Assume that $u_0\left(x\right) \in V$, $u_1\left(x\right) \in H_0^1\left(\Omega\right)$. Suppose that $2 < p < \frac{2n}{n-4}$, then the solution u of problem (1.1) blow up in finite time; that is the maximum existence time T^* of u is finite and*

$$\lim_{t\to T^*}\left(\left\|u\right\|^2 + \int_\Omega \left\|\nabla u\right\|^2 d\tau\right) = +\infty. \tag{5.3}$$

Moreover, the upper bound for blow up time T^* is given by

$$T^* \leq \frac{2bT_0^2 + 2\left\|u_0\right\|^2}{\left(p-2\right)bT_0 + \left(p-2\right)\int_\Omega u_0u_1dx - 2\left\|\nabla u_0\right\|^2}, \tag{5.4}$$

where b and T_0 will be chosen in (5.14) and (5.15).

Proof. By contradiction, we assume that u is global, then $T^* = +\infty$. For any $T > 0$, we assume that $\Phi : [0,T] \to R^+$ defined by

$$\Phi\left(t\right) = \left\|u\right\|^2 + \int_0^t \left\|\nabla u\right\|^2 d\tau + \left(T-t\right)\left\|\nabla u_0\right\|^2 + b\left(T_0+t\right)^2, \tag{5.5}$$

where b and T_0 are positive fixed which will be specified later.

Firstly, we compute the first order differential and second order differential of $\Phi(t)$, respectively, as follows

$$\begin{aligned}\Phi'(t) &= 2\int_\Omega u_t u dx + \|\nabla u\|^2 - \|\nabla u_0\|^2 + 2b(T_0+t)\\ &= 2\int_\Omega u_t u dx + 2\int_0^t\int_\Omega \nabla u \nabla u_t dx d\tau + 2b(T_0+t),\end{aligned} \tag{5.6}$$

and

$$\begin{aligned}\Phi''(t) &= 2\int_\Omega |u_t|^2 dx + 2\int_\Omega u_{tt} u dx + 2\int_\Omega \nabla u \nabla u_t dx + 2b\\ &= 2\int_\Omega |u_t|^2 dx + 2\int_\Omega u_{tt} u dx - 2\int_\Omega u \Delta u_t dx + 2b\\ &= 2\int_\Omega |u_t|^2 dx + 2\int_\Omega u\left[u_{tt} - \Delta u_t\right] dx + 2b\\ &= 2\int_\Omega |u_t|^2 dx + 2\int_\Omega u\left[\left(1+\|\nabla u\|^{2\gamma}\right)\Delta u - \Delta^2 u + |u|^{p-2} u \ln|u|\right] dx + 2b\\ &= 2\int_\Omega |u_t|^2 dx + 2b\\ &\quad -2\left[\int_\Omega \left(1+\|\nabla u\|^{2\gamma}\right)|\nabla u|^2 dx + \int_\Omega |\Delta u|^2 dx - \int_\Omega |u|^p \ln|u| dx\right]\\ &= 2\int_\Omega |u_t|^2 dx - 2\left[\|\nabla u\|^2 + \|\nabla u\|^{2(\gamma+1)} + \|\Delta u\|^2 - \int_\Omega |u|^p \ln|u| dx\right] + 2b\\ &= 2\|u_t\|^2 - 2I(u) + 2b,\end{aligned} \tag{5.7}$$

where

$$I(u) = \|\nabla u\|^2 + \|\nabla u\|^{2(\gamma+1)} + \|\Delta u\|^2 - \int_\Omega |u|^p \ln|u| dx.$$

Through a direct calculation, we have

$$\begin{aligned}&\Phi(t)\Phi''(t) - \frac{p+2}{4}\left[\Phi'(t)\right]^2\\ =\ & 2\Phi(t)\left(\|u_t\|^2 - \|\nabla u\|^2 - \|\nabla u\|^{2(\gamma+1)} - \|\Delta u\|^2 + \int_\Omega |u|^p \ln|u| dx + b\right)\\ &+(p+2)\left[B(t) - \left(\Phi(t) - (T-t)\|\nabla u_0\|^2\right)\left(\|u_t\|^2 + \int_0^t \|\nabla u_t\|^2 d\tau + b\right)\right],\end{aligned} \tag{5.8}$$

where

$$\begin{aligned}B(t) &= \left(\|u\|^2 + \int_0^t \|\nabla u\|^2 d\tau + b(T_0+t)^2\right)\left(\|u_t\|^2 + \int_0^t \|\nabla u_t\|^2 d\tau + b\right)\\ &\quad - \left(\int_\Omega u_t u dx + \int_0^t\int_\Omega \nabla u \nabla u_t dx d\tau + 2b(T_0+t)\right)^2.\end{aligned} \tag{5.9}$$

Using Schwarz inequality and Young inequality, it is not difficult to verify that $B(t) \geq 0$ for any $t \in [0,T]$. As a consequence, from (5.8) we arrive that

$$\Phi(t)\Phi''(t) - \frac{p+2}{4}\left[\Phi'(t)\right]^2 \geq \Phi(t)\xi(t), \tag{5.10}$$

where $\xi(t) : [0,T] \to R$ is defined by

$$\begin{aligned} \xi(t) &= -p\|u_t\|^2 - 2\|\nabla u\|^2 - 2\|\Delta u\|^2 - \frac{2p^2-(p-2\gamma-2)}{2p(\gamma+1)}\|\nabla u\|^{2(\gamma+1)} \\ &\quad +2\int_\Omega |u|^p \ln|u|\,dx + (p-2)\int_0^t \|\nabla u_t\|^2\,d\tau - pb. \end{aligned} \tag{5.11}$$

Furthermore, by the definition of $E(t)$ and Lemma 3.3, it follows that

$$\begin{aligned} \xi(t) &= -2pE(t) + (p-2)\|\nabla u\|^2 + (p-2)\|\Delta u\|^2 + \frac{(p-2\gamma-2)}{2p(\gamma+1)}\|\nabla u\|^{2(\gamma+1)} \\ &\quad +\frac{2}{p}\|u\|_p^p + (p-2)\int_0^t \|\nabla u_t\|^2\,d\tau - pb \\ &\geq -2pd + (p-2)\|\nabla u\|^2 + (p-2)\|\Delta u\|^2 + \left(\frac{p-2\gamma-2}{p(2\gamma+2)}\right)\|\nabla u\|^{2(\gamma+1)} \\ &\quad +\frac{2}{p}\|u\|_p^p + (p-2)\int_0^t \|\nabla u_t\|^2\,d\tau - pb. \end{aligned} \tag{5.12}$$

From $u_0(x) \in V$, $u_1(x) \in H_0^1(\Omega)$ and Lemma 3.3, we obtain $u(x) \in V$, $u(x) \in H_0^1(\Omega)$ for all $t \geq 0$, which implies that $I(u) < 0$. Hence there exists a $\lambda_* \in (0,1)$ such that $I(\lambda_* u) = 0$. Thus by the definition of d and (3.3), we get that

$$\begin{aligned} & \frac{p-2}{2p}\|\nabla u\|^2 + \frac{p-2}{2p}\|\Delta u\|^2 \\ & +\left(\frac{p-2\gamma-2}{2p^2(2\gamma+2)}\right)\|\nabla u\|^{2\gamma+2} + \frac{1}{p^2}\|u\|_p^p \\ \geq\ & \frac{(p-2)\lambda_*^2}{2p}\|\nabla u\|^2 + \frac{(p-2)\lambda_*^2}{2p}\|\Delta u\|^2 \\ & +\left(\frac{p-2\gamma-2}{2p^2(2\gamma+2)}\right)\lambda_*^{2\gamma+2}\|\nabla u\|^{2\gamma+2} + \frac{\lambda_*^p}{p^2}\|u\|_p^p \\ \geq\ & J(\lambda_* u) \\ \geq\ & d. \end{aligned} \tag{5.13}$$

Choosing b small enough shuch that

$$\begin{aligned} 0 &< b \leq \frac{1}{p}\left[(p-2)\|\nabla u\|^2 + (p-2)\|\Delta u\|^2 + \frac{2}{p}\|u\|_p^p\right] \\ &\quad +\frac{1}{p}\left[\left(\frac{p-2\gamma-2}{p(2\gamma+2)}\right)\|\nabla u\|^{2(\gamma+1)} + (p-2)\int_0^t \|\nabla u_t\|^2\,d\tau - 2pd\right] \end{aligned} \tag{5.14}$$

The combination of (5.12)-(5.14) implies that $\xi(t) \geq 0$. Hence, by the above discussion, we have

$$\Phi(t)\,\Phi''(t) - \frac{p+2}{4}\left[\Phi'(t)\right]^2 \geq 0.$$

From the definition of $B(t)$, it is easy to know that $\Phi(0) = \|u_0\|^2 + T\|\nabla u_0\|^2 d\tau + bT_0^2 > 0$. We choose T_0 large enough shuch that

$$T_0 > \frac{(p-2)\left(\|u_0\|^2 + \|u_1\|^2\right) + 4\|\nabla u_0\|^2}{2(p-2)b}, \tag{5.15}$$

which fulfills the requirement of

$$\begin{aligned} \Phi'(0) &= 2\int_\Omega u_0 u_1 dx + 2bT_0 \\ &\geq 2bT_0 - \|u_0\|^2 - \|u_1\|^2 - \frac{4\|\nabla u_0\|^2}{p-2} \\ &\geq 0. \end{aligned} \tag{5.16}$$

Then, according to Lemma 5.1, we obtain that $\Phi(t)$ goes to ∞ as t tends to some T^* satisfying

$$T^* \leq \frac{4\Phi(0)}{(p-2)\Phi'(0)} = \frac{2bT_0^2 + 2\|u_0\|^2 + 2T\|\nabla u_0\|^2}{(p-2)bT_0 + (p-2)\int_\Omega u_0u_1dx},$$

which means that

$$T^* \leq \frac{4\left(bT_0^2 + \|u_0\|^2\right)}{(p-2)bT_0 + (p-2)\int_\Omega u_0u_1dx - 2\|\nabla u_0\|^2}. \tag{5.17}$$

Finally, for fixed T_0, choose T as

$$T > \frac{4bT_0^2 + 4\|u_0\|^2}{(p-2)bT_0 - 4\|\nabla u_0\|^2 - (p-2)\left(\|u_0\|^2 + \|u_1\|^2\right)}. \tag{5.18}$$

The combination of (5.17) and (5.18), we see that $T > T^*$. This contradicts to our assumption, which finished the proof. □

5.2. Lower bound for the blow up time

Our aim is to state a lower bound for the blow up time of problem (1.1).

Theorem 5.2. *Under the conditions of Theorem 5.1 and* $2 < p < 2\left(1 + \frac{2}{n-4}\right)$, *then the solutions* u *of the problem* (1.1) *become unbounded at finite time* $t = T^*$ *with*

$$\lim_{t\to T^{*-}} \|u_t\|^2 + \|\nabla u\|^2 + \|\Delta u\|^2 + \frac{1}{\gamma+1}\|\nabla u\|^{2\gamma+2} = +\infty. \tag{5.19}$$

Moreover, the lower bound for blow up time T^* is given by

$$\int_{\Psi(0)}^{\infty} \frac{d\theta}{\theta + (e(p-1))^{-2}|\Omega| + (e\mu)^{-2} C_2^{2(p-1+\mu)} \theta^{p-1+\mu}} \leq T^*. \tag{5.20}$$

where $0 < \mu < \frac{2n-2}{n-2} - p$ *and*

$$\Psi(0) = \|u_1\|^2 + \|\nabla u_0\|^2 + \|\Delta u_0\|^2 + \frac{1}{\gamma+1}\|\nabla u_0\|^{2\gamma+2}.$$

Proof. Let us define the function

$$\Psi(t) = \|u_t\|^2 + \|\nabla u\|^2 + \|\Delta u\|^2 + \frac{1}{\gamma+1}\|\nabla u\|^{2\gamma+2}. \tag{5.21}$$

By differentiating $\Psi(t)$ with respect to t and using of the (1.1), we get

$$\begin{aligned}
\Psi'(t) &= 2\int_\Omega u_t u_{tt} dx + 2\int_\Omega \nabla u \nabla u_t dx + 2\int_\Omega \Delta u \Delta u_t dx + 2\|\nabla u\|^{2\gamma} \int_\Omega \nabla u \nabla u_t dx \\
&= 2\int_\Omega u_t u_{tt} dx - 2\int_\Omega u_t \Delta u dx + 2\int_\Omega u_t \Delta^2 u dx - 2\|\nabla u\|^{2\gamma}\int_\Omega u_t \Delta u dx \\
&= 2\int_\Omega u_t \left[u_{tt} - M\left(\|\nabla u\|^2\right)\Delta u + \Delta^2 u\right] dx \\
&= 2\int_\Omega u_t \left[\Delta u_t + |u|^{p-2} u \ln|u|\right] dx \\
&= -2\int_\Omega |\nabla u_t|^2 dx + 2\int_\Omega u_t |u|^{p-2} u \ln|u| dx \\
&= -2\|\nabla u_t\|^2 dx + 2\int_\Omega u_t |u|^{p-2} u \ln|u| dx, \qquad (5.22)
\end{aligned}$$

since $2 < p < \left(1 + \frac{n}{n-2}\right)$, we chose $\mu > 0$ small enough such that $(p-1+\mu) <$

$\frac{n}{n-2}$. Hence, by the Young's and Sobolev inequalities, we have from (5.21) that

$$\begin{aligned}
& 2\int_\Omega u_t u^{p-2} u \ln |u| \, dx \\
\leq \; & 2\left[\int_\Omega \frac{|u_t|^2}{2} dx + \int_\Omega \frac{\left|u^{p-2} u \ln |u|\right|^2}{2} dx\right] \\
= \; & \int_\Omega |u_t|^2 \, dx + \int_\Omega \left|u^{p-2} u \ln |u|\right|^2 dx \\
= \; & \|u_t\|^2 + \int_\Omega \left|u^{p-2} u \ln |u|\right|^2 dx \\
= \; & \|u_t\|^2 + \int_{x\in\Omega:|u|<1} \left|u^{p-2} u \ln |u|\right|^2 dx + \int_{x\in\Omega:|u|\geq 1} \left|u^{p-2} u \ln |u|\right|^2 dx \\
\leq \; & \|u_t\|^2 + (e(p-1))^{-2} |\Omega| + (e\mu)^{-2} \int\limits_{x\in\Omega:|u|\geq 1} |u|^{2(p-1+\mu)} \, dx \\
\leq \; & \|u_t\|^2 + (e(p-1))^{-2} |\Omega| + (e\mu)^{-2} C_2^{2(p-1+\mu)} \|\nabla u\|_2^{2(p-1+\mu)} \\
\leq \; & \Psi(t) + (e(p-1))^{-2} |\Omega| + (e\mu)^{-2} C_2^{2(p-1+\mu)} \Psi(t)^{p-1+\mu}, \qquad (5.23)
\end{aligned}$$

where we used $\left|x^{p-1} \log x\right| \leq (e(p-1))^{-1}$ for $0 < x < 1$, while $x^{-\mu} \log x \leq (e\mu)^{-1}$ for $x \geq 1, \mu > 0$, and C_2 is the Sobolev constant satisfying $\|u\|_{p-1+\mu} \leq C_2 \|\nabla u\|_2$. The combination of (5.22) and (5.23), it follows that

$$\Psi'(t) \leq \Psi(t) + (e(p-1))^{-2} |\Omega| + (e\mu)^{-2} C_2^{2(p-1+\mu)} \Psi(t)^{2(p-1+\mu)} \tag{5.24}$$

integrating the inequality (5.24) from 0 to t, we have

$$\int\limits_{\Psi(0)}^{\Psi(t)} \frac{d\theta}{\theta + (e(p-1))^{-2} |\Omega| + (e\mu)^{-2} C_2^{2(p-1+\mu)} \theta^{p-1+\mu}} \leq t,$$

where $0 < \mu < \frac{2n-2}{n-2} - p$.

From the results of Theorem 5.1. It is easy to see that there exists a finite time T^* such that the solutions u blow up with $\lim\limits_{t\to T^{*-}} \Psi(t) = +\infty$. Therefore, we obtain a lower bound for T^* given by

$$\int\limits_{\Psi(0)}^{\infty} \frac{d\theta}{\theta + (e(p-1))^{-2} |\Omega| + (e\mu)^{-2} C_2^{2(p-1+\mu)} \theta^{p-1+\mu}} \leq T^*.$$

Clearly, the integral is bound since exponent $p - 1 + \mu > 1$. This completes the proof of Theorem 5.2. $\square$

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