FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 38, No 3 (2023), 535 - 558
https://doi.org/10.22190/FUMI221202035R
Original Scientific Paper

ON IMPULSIVE IMPLICIT RIESZ-CAPUTO FRACTIONAL DIFFERENTIAL EQUATIONS WITH RETARDATION AND ANTICIPATION IN BANACH SPACES

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Abstract. In this paper, we investigate the existence and Ulam stability results for a class of boundary value problems for implicit Riesz-Caputo fractional differential equations with non-instantaneous impulses involving both retarded and advanced arguments. The result are based on Mönch fixed point theorem associated with the technique of measure of noncompactness. An illustrative example is given to validate our main results.
Keywords: Riesz-Caputo fractional derivative, existence, measure of noncompactness, fixed point, Ulam stability, non-instantaneous impulses, retarded arguments, delay, implicit, anticipation.

1. Introduction

Because of its importance in the modeling and scientific understanding of natural processes, fractional calculus has long been an essential study topic in functional space theory. Several applications in viscoelasticity and electrochemistry have been studied. Non-integer derivatives of fractional order have been utilized successfully to generalize fundamental natural principles. For more details, we recommend [1, 2, 3, 4, 9, 10, 11, 24].

Received December 02, 2022. accepted May 11, 2023.
Communicated by Praveen Agarwal
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2010 *Mathematics Subject Classification.* Primary 26A33; Secondary 34B37, 34A08

There are numerous fractional derivatives, each with its own set of characteristics and uses. The Riemann-Liouville fractional derivative, introduced in 1847, and the Caputo derivative, created later in 1967, are two notable examples. Among the other notable derivatives are the Hilfer derivative (2000), the Hadamard derivative (1892), and the Caputo-Fabrizio derivative (2015). In many instances, the current condition of a process is determined by its past and future evolution. Stock price options, for example, depend on forecasting future market patterns. Similarly, fractional derivatives are used to describe the concentration of diffusion on a specific route in the anomalous diffusion problem. The Riesz derivative, a two-sided fractional operator, is especially helpful in this situation because it can capture both past and future memory effects. This property is particularly useful when describing fractional processes on a finite area. The Riemann-Liouville and Caputo fractional derivatives, which are one-sided fractional operators that only reflect past or future memory effects, are currently the center of much work on fractional differential equations. The flexibility of the Riesz derivative, on the other hand, has attracted notice and is garnering favor in the field. For further information, interested readers may refer to the works cited in [9, 10, 11].

In many cases, determining the exact solution of differential equations is difficult, if not unattainable. It is usual in such situations to investigate approximate solutions. It is essential to observe, however, that only steady approximations are accepted. As a consequence, different stability analysis techniques are used. S. M. Ulam, a mathematician, first raised the stability issue in functional equations in a 1940 lecture at Wisconsin University. In his presentation, Ulam posed the following challenge: "Under what conditions does an additive mapping exist near an approximately additive mapping?" [30]. The following year, Hyers provided an answer to Ulam's problem for additive functions defined on Banach spaces [13]. In 1978, Rassias further expanded upon Hyers' work, demonstrating the existence of unique linear mappings near approximate additive mappings [22]. Since then, numerous research articles in the literature have addressed the stabilities of various types of differential and integral equations. Interested readers may refer to [17, 26, 28, 15, 31, 8, 21] and their respective references for further details.

E. Hernandez and D. O'Regan [12], studied the existence of solutions to the novel class of abstract differential equations with noninstantaneous impulses. The papes [6, 32, 34, 25, 26, 27, 20] can be consulted for fundamental results and recent developments on differential equations with instantaneous and non-instantaneous impulses.

The authors of [9] studied the existence of solution for the following boundary value problem:

$$\begin{cases} {}_0^{RC}D_{\varkappa}^{\alpha} y(\theta) = g(\theta, y(\theta)), \quad \theta \in \Theta := [0, \varkappa], \\ y(0) = y_0, \quad y(\varkappa) = y_{\varkappa}, \end{cases}$$

where ${}_0^{RC}D_{\varkappa}^{\alpha}$ is a Riesz-Caputo derivative of order $0 < \alpha \leq 1$, $g : \Theta \times \mathbb{R} \to \mathbb{R}$ a continuous function and $y_0 \in \mathbb{R}$. Their arguments are based on Leray-Schauder fixed point theorem, and Schauder fixed point theorem.

In [16], Li and Wang discussed the following fractional problem:

$$\begin{gathered} {}_0^{RC}D_1^{\gamma}y(t) = f(t, y(t)), \quad t \in [0,1], \quad 0 < \gamma \leq 1, \\ y(0) = a, \quad y(1) = by(\eta), \end{gathered}$$

where ${}_0^{RC}D_1^{\gamma}$ is the Riesz Caputo derivative, $f \in C([0,1]\times\ [0,+\infty), [0,+\infty)), 0 < \eta < 1, a > 0, 0 < b < 2$. They found the positive solutions by applying the technique of monotone iterative.

Naas *et al.* [19] investigated the existence and uniqueness results of the following fractional differential equation with the Riesz-Caputo derivative:

$$\begin{cases} {}_0^{RC}D_T^{\vartheta}\varkappa(t) + \mathfrak{F}\left(t, \varkappa(t), {}_0^{RC}D_T^{\varsigma}\varkappa(t)\right) = 0, t \in \mathcal{J} := [0,T], \\ \varkappa(0) + \varkappa(T) = 0, \quad \mu\varkappa'(0) + \sigma\varkappa'(T) = 0, \end{cases}$$

where $1 < \vartheta \leq 2$ and $, 0 < \varsigma \leq 1, {}_0^{RC}D_T^{\kappa}$ is the Riesz-Caputo fractional derivative of order $\kappa \in \{\vartheta, \varsigma\}$, $\mathfrak{F} : \mathcal{J} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, is a continuous function, and μ, σ are nonnegative constants with $\mu > \sigma$. The existence and uniqueness of solutions for their problem are demonstrated with the Riesz-Caputo derivatives via Banach's, Schaefer's, and Krasnoselskii's fixed point theorems.

The authors of [25] established existence and stability results, with relevant fixed point theorems, to the boundary value problem:

$$\begin{cases} \left({}^{\zeta_1}\mathbb{D}_{\varkappa_i^+}^{\zeta_1,\zeta_2}x\right)(\vartheta) = f\left(\vartheta, x(\vartheta), \left({}^{\zeta_1}\mathbb{D}_{\varkappa_i^+}^{\zeta_1,\zeta_2}x\right)(\vartheta)\right); \ \vartheta \in \Omega_i, \ i = 0, \ldots, m, \\ \\ x(\vartheta) = \Psi_i(\vartheta, x(\vartheta)); \ \vartheta \in \tilde{\Omega}_i, \ i = 1, \ldots, m, \\ \\ \phi_1\left({}^{\zeta_1}\mathbb{J}_{a^+}^{1-\zeta_3}x\right)(a^+) + \phi_2\left({}^{\zeta_1}\mathbb{J}_{m^+}^{1-\zeta_3}x\right)(b) = \phi_3, \end{cases}$$

where ${}^{\zeta_1}\mathbb{D}_{\varkappa_i^+}^{\zeta_1,\zeta_2}, {}^{\zeta_1}\mathbb{J}_{a^+}^{1-\zeta_3}$ are the generalized Hilfer fractional derivative of order $\zeta_1 \in (0,1)$ and type $\zeta_2 \in [0,1]$ and generalized fractional integral of order $1-\zeta_3$ respectively, $\phi_1, \phi_2, \phi_3 \in \mathbb{R}$, $\phi_1 \neq 0$, $\Omega_i := (\varkappa_i, \vartheta_{i+1}]; i = 0, \ldots, m$, $\tilde{\Omega}_i := (\vartheta_i, \varkappa_i]; i = 1, \ldots, m$, $a = \vartheta_0 = \varkappa_0 < \vartheta_1 \leq \varkappa_1 < \vartheta_2 \leq \varkappa_2 < \ldots \leq \varkappa_{m-1} < \vartheta_m \leq \varkappa_m < \vartheta_{m+1} = b < \infty$, $x(\vartheta_i^+) = \lim_{\epsilon\to0^+} x(\vartheta_i + \epsilon)$ and $x(\vartheta_i^-) = \lim_{\epsilon\to0^-} x(\vartheta_i + \epsilon)$, $f : (a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function and $\Psi_i : \tilde{\Omega}_i \times \mathbb{R} \to \mathbb{R}; i = 1, \ldots, m$ are given continuous functions.

Motivated by the above-mentioned papers, first, we present some existence, uniqueness and Ulam stability results for the following fractional problem:

$$\left({}_{\varkappa_\jmath}^{RC}D_{\vartheta_{\jmath+1}}^{\zeta}y\right)(\vartheta) = \varphi\left(\vartheta, y^{\vartheta}(\cdot), \left({}_{\varkappa_\jmath}^{RC}D_{\vartheta_{\jmath+1}}^{\zeta}y\right)(\vartheta)\right); \ \vartheta \in \Omega_\jmath, \ \jmath = 0, \ldots, m, \tag{1.1}$$

$$y(\vartheta) = \Psi_\jmath(\vartheta, y(\vartheta_\jmath^-)); \ \vartheta \in \tilde{\Omega}_\jmath, \ \jmath = 1, \dots, m, \tag{1.2}$$

$$\delta_1 y(0) + \delta_2 y(\varkappa) = \delta_3, \tag{1.3}$$

$$y(\vartheta) = \hbar_1(\vartheta), \quad \vartheta \in [-\varpi, 0], \ \varpi > 0, \tag{1.4}$$

$$y(\vartheta) = \hbar_2(\vartheta), \quad \vartheta \in [\varkappa, \varkappa + \tilde{\varpi}], \ \tilde{\varpi} > 0, \tag{1.5}$$

where ${}^{RC}_{\varkappa_\jmath}D^{\zeta}_{\vartheta_{\jmath+1}}$ represent the Riesz-Caputo derivative of order $0 < \zeta \leq 1$, $\Theta := [0, \varkappa]$, $\delta_1, \delta_2 \in \mathbb{R}$, $\delta_3 \in \Xi$, where $\delta_1 \neq 0$, $\Omega_0 := [0, \vartheta_1]$, $\Omega_\jmath := (\varkappa_\jmath, \vartheta_{\jmath+1}]; \jmath = 1, \dots, m$, $\tilde{\Omega}_\jmath := (\vartheta_\jmath, \varkappa_\jmath]; \jmath = 1, \dots, m$, $0 = \vartheta_0 = \varkappa_0 < \vartheta_1 \leq \varkappa_1 < \vartheta_2 \leq \varkappa_2 < \ldots \leq \varkappa_{m-1} < \vartheta_m \leq \varkappa_m < \vartheta_{m+1} = \varkappa < \infty$, $y(\vartheta_\jmath^+) = \lim_{\epsilon \to 0^+} y(\vartheta_\jmath + \epsilon)$ and $y(\vartheta_\jmath^-) = \lim_{\epsilon \to 0^-} y(\vartheta_\jmath + \epsilon)$ represent the right and left hand limits of $y(\vartheta)$ at $\vartheta = \vartheta_\jmath$, $(\Xi, \|\cdot\|)$ is a Banach space, $\varphi : \Theta \times PC([-\varpi, \tilde{\varpi}], \Xi) \times \Xi \to \Xi$ is a given function, $\hbar_1 \in C([-\varpi, 0], \Xi)$, $\hbar_2 \in C([\varkappa, \varkappa + \tilde{\varpi}], \Xi)$, and $\Psi_\jmath : \tilde{\Omega}_\jmath \times \Xi \to \Xi$; $\jmath = 1, \dots, m$ are given continuous functions. For y defined on $[-\varpi, \varkappa + \tilde{\varpi}]$ and for any $\vartheta \in [0, \varkappa]$, y^ϑ is given by

$$y^\vartheta(\varrho) = y(\vartheta + \varrho), \quad \varrho \in [-\varpi, \tilde{\varpi}].$$

The following part refers to how the current paper is arranged. In Section 2, we present certain notations and review some preliminary information on the Riesz-Caputo fractional derivative and auxiliary results. Section 3 presents an existence result to the problem (1.1)-(1.5) based on Mönch's fixed point theorem associated with the technique of measure of noncompactness. The Ulam-Hyers-Rassias Stability for our problem is discussed in Section 4. Finally, in the final part, we provide an example to demonstrate the application of our study results.

2. Preliminaries

In this section, we introduce some notations, definitions, and preliminary facts which are used throughout this paper.

We denote by $C(\Theta, \Xi)$ the Banach space of all continuous functions from Θ to Ξ, with the norm

$$\|\xi\|_\infty = \sup\{\|\xi(\vartheta)\| : \vartheta \in \Theta\}.$$

Let $\mathcal{X} = C([-\varpi, 0], \Xi)$ and $\tilde{\mathcal{X}} = C([\varkappa, \varkappa + \tilde{\varpi}], \Xi)$ be the spaces endowed, respectively, with the norms

$$\|\xi\|_{\mathcal{X}} = \sup\{\|\xi(\theta)\| : \theta \in [-\varpi, 0]\},$$

and

$$\|\xi\|_{\tilde{\mathcal{X}}} = \sup\{\|\xi(\theta)\| : \theta \in [\varkappa, \varkappa + \tilde{\varpi}]\}.$$

Consider the Banach space

$$PC(\Theta,\Xi) = \Big\{ y:\Theta\to\Xi : y|_{\tilde{\Omega}_\jmath} = \Psi_\jmath; \jmath = 1,\ldots,m,\ \ y|_{\Omega_\jmath} \in C(\Omega_\jmath,\Xi); \jmath = 0,\ldots,m,$$
$$\text{and there exist } u(\vartheta_\jmath^-),\ y(\vartheta_\jmath^+), y(\varkappa_\jmath^-),\ \text{and } y(\varkappa_\jmath^+)$$
$$\text{with } y(\vartheta_\jmath^-) = y(\vartheta_\jmath) \Big\},$$

with the norm

$$\|y\|_{PC} = \sup_{\vartheta\in\Theta} \|y(\vartheta)\|$$

Consider the weighted Banach space

$$PC\left([-\varpi,\tilde{\varpi}],\Xi\right) = \Big\{ y:[-\varpi,\tilde{\varpi}]\to\Xi : y|_{[\tau_\jmath,\tilde{\tau}_{\jmath+1}]} \in C([\tau_\jmath,\tilde{\tau}_{\jmath+1}],\Xi); \jmath = 0,\ldots,m,$$
$$\text{for each } \vartheta\in\Omega_\jmath,\ \text{and}\ \ y|_{[\tilde{\tau}_\jmath,\tau_\jmath]} \in C([\tilde{\tau}_\jmath,\tau_\jmath],\Xi); \jmath = 1,\ldots,m,$$
$$\text{for each } \vartheta\in\tilde{\Omega}_\jmath,\ \text{where } \tau_\jmath = \varkappa_\jmath - \vartheta \text{ and } \tilde{\tau}_\jmath = \vartheta_\jmath - \vartheta \text{ and}$$
$$\text{there exist } y(\tau_\jmath^-), y(\tilde{\tau}_\jmath^+), y(\tilde{\tau}_\jmath^-) \text{ and } y(\tau_\jmath^+); \jmath = 1,\ldots,m,$$
$$\text{with } y(\tau_\jmath^-) = y(\tau_\jmath) \Big\},$$

with the norm

$$\|y\|_{[-\varpi,\tilde{\varpi}]} = \sup_{\tau\in[-\varpi,\tilde{\varpi}]} \left\|y^\vartheta(\tau)\right\|$$

Next, we consider the Banach space

$$\mathbb{F} = \Big\{ y:[-\varpi,\varkappa+\tilde{\varpi}]\to\Xi : y|_{[-\varpi,0]} \in \mathcal{X}, y|_{[\varkappa,\varkappa+\tilde{\varpi}]} \in \tilde{\mathcal{X}} \text{ and }\ y|_{[0,\varkappa]} \in PC(\Theta,\Xi) \Big\}$$

with the norm

$$\|y\|_{\mathbb{F}} = \max\left\{\|y\|_{\mathcal{X}}, \|y\|_{\tilde{\mathcal{X}}}, \|y\|_{PC}\right\}.$$

Definition 2.1. ([14]) Let $\zeta > 0$. The left and right Riemann-Liouville fractional integrals of a function $\varphi \in C(\Theta,\Xi)$ of order ζ are given respectively by

$${}_0I_\vartheta^\zeta \varphi(\vartheta) = \frac{1}{\Gamma(\zeta)} \int_0^\vartheta (\vartheta-\varrho)^{\zeta-1}\varphi(\varrho)d\varrho,$$

and

$${}_\vartheta I_\varkappa^\zeta \varphi(\vartheta) = \frac{1}{\Gamma(\zeta)} \int_\vartheta^\varkappa (\varrho-\vartheta)^{\zeta-1}\varphi(\varrho)d\varrho.$$

Definition 2.2. ([14]) Let $\zeta > 0$. The Riesz fractional integral of a function $\varphi \in C(\Theta, \Xi)$ of order ζ is defined by

$$\begin{aligned} {}_0I^{\zeta}_{\varkappa}\varphi(\vartheta) &= \frac{1}{\Gamma(\zeta)}\int_0^{\varkappa} |\vartheta-\varrho|^{\zeta-1}\varphi(\varrho)d\varrho \\ &= {}_0I^{\zeta}_{\vartheta}\varphi(\vartheta) + {}_{\vartheta}I^{\zeta}_{\varkappa}\varphi(\vartheta), \end{aligned}$$

where ${}_0I^{\zeta}_{\vartheta}$ and ${}_{\vartheta}I^{\zeta}_{\varkappa}$ are the left and right fractional integrals of Riemann-Liouville.

Definition 2.3. ([14]) Let $\zeta \in (n, n+1]$, $n \in \mathbb{N}$. The left and right Caputo fractional derivatives of a function $\varphi \in C^{n+1}(\Theta, \Xi)$ of order ζ are given respectively by

$${}^C_0D^{\zeta}_{\vartheta}\varphi(\vartheta) = \frac{1}{\Gamma(n+1-\zeta)}\int_0^{\vartheta} (\vartheta-\varrho)^{n-\zeta}\varphi^{(n+1)}(\varrho)d\varrho,$$

and

$${}^C_{\vartheta}D^{\zeta}_{\varkappa}\varphi(\vartheta) = \frac{(-1)^{n+1}}{\Gamma(n+1-\zeta)}\int_{\vartheta}^{\varkappa} (\varrho-\vartheta)^{n-\zeta}\varphi^{(n+1)}(\varrho)d\varrho.$$

Definition 2.4. ([14]) Let $\zeta \in (n, n+1]$, $n \in \mathbb{N}$. The Riesz-Caputo fractional derivative of a function $\varphi \in C^{n+1}(\Theta, \Xi)$ of order ζ is given by

$$\begin{aligned} {}^{RC}_{0}D^{\zeta}_{\varkappa}\varphi(\vartheta) &= \frac{1}{\Gamma(n+1-\zeta)}\int_0^{\varkappa} |\vartheta-\varrho|^{n-\zeta}\varphi^{(n+1)}(\varrho)d\varrho \\ &= \frac{1}{2}({}^C_0D^{\zeta}_{\vartheta}\varphi(\vartheta) + (-1)^{n+1}{}^C_{\vartheta}D^{\zeta}_{\varkappa}\varphi(\vartheta)), \end{aligned}$$

where ${}^C_0D^{\zeta}_{\vartheta}$ is the left Caputo derivative and ${}^C_{\vartheta}D^{\zeta}_{\varkappa}$ is the right one. If we take $0 < \zeta \le 1$ and $\varphi \in C(\Theta, \Xi)$, we obtain

$${}^{RC}_{0}D^{\zeta}_{\varkappa}\varphi(\vartheta) = \frac{1}{2}({}^C_0D^{\zeta}_{\vartheta}\varphi(\vartheta) - {}^C_{\vartheta}D^{\zeta}_{\varkappa}\varphi(\vartheta)).$$

Lemma 2.1. *([14]) If $\xi \in C^{n+1}(\Theta, \Xi)$ and $\zeta \in (n, n+1]$, then we have*

$${}_0I^{\zeta}_{\vartheta}\,{}^C_0D^{\zeta}_{\vartheta}\xi(\vartheta) = \xi(\vartheta) - \sum_{\jmath=0}^{n}\frac{\xi^{(\jmath)}(0)}{\jmath!}\vartheta^{\jmath},$$

and

$${}_{\vartheta}I^{\zeta}_{\varkappa}\,{}^C_{\vartheta}D^{\zeta}_{\varkappa}\xi(\vartheta) = (-1)^{n+1}\left[\xi(\vartheta) - \sum_{\jmath=0}^{n}\frac{(-1)^{\jmath}\xi^{(\jmath)}(\varkappa)}{\jmath!}(\varkappa-\vartheta)^{\jmath}\right].$$

Consequently, we may have

$${}_0I^{\zeta}_{\varkappa}\,{}^{RC}_{0}D^{\zeta}_{\varkappa}\xi(\vartheta) = \frac{1}{2}({}_0I^{\zeta}_{\vartheta}\,{}^C_0D^{\zeta}_{\vartheta}\xi(\vartheta) + (-1)^{n+1}\,{}_{\vartheta}I^{\zeta}_{\varkappa}\,{}^C_{\vartheta}D^{\zeta}_{\varkappa}\xi(\vartheta)).$$

In particular, if $0 < \zeta \leq 1$*, then we obtain*

$$ {}_0I^{\zeta}_{\varkappa} \, {}^{RC}_{0}D^{\zeta}_{\varkappa}\xi(\vartheta) = \xi(\vartheta) - \frac{1}{2}(\xi(0) + \xi(\varkappa)). $$

Lemma 2.2. *Let* $\sigma \in C(\Theta, \Xi)$ *and* $0 < \zeta \leq 1$*. Then* $y \in C(\Theta, \Xi)$ *is a solution of*

$$ {}^{RC}_{0}D^{\zeta}_{\varkappa}y(\vartheta) = \sigma(\vartheta), \quad \vartheta \in \Theta, \tag{2.1} $$

if and only if y *verifies the following integral equation:*

$$ y(\vartheta) \;\; = \;\; y(0) - \frac{1}{\Gamma(\zeta)}\int_0^{\varkappa} \varrho^{\zeta-1}\sigma(\varrho)d\varrho + \frac{1}{\Gamma(\zeta)}\int_0^{\varkappa} |\vartheta-\varrho|^{\zeta-1}\sigma(\varrho)d\varrho. \tag{2.2} $$

Proof. From Definition 2.2, Definition 2.4, and Lemma 2.1, we have

$$ {}_0I^{\zeta}_{\varkappa} \, {}^{RC}_{0}D^{\zeta}_{\varkappa}y(\vartheta) = y(\vartheta) - \frac{1}{2}(y(0) + y(\varkappa)), $$

which implies that

$$ \begin{aligned} y(\vartheta) &= \frac{1}{2}(y(0) + y(\varkappa)) + \; {}_0I^{\zeta}_{\varkappa}\sigma(\vartheta), \\ &= \frac{1}{2}(y(0) + y(\varkappa)) + \frac{1}{\Gamma(\zeta)}\int_0^{\varkappa} |\vartheta-\varrho|^{\zeta-1}\sigma(\varrho)d\varrho \\ &= \frac{1}{2}(y(0) + y(\varkappa)) + \frac{1}{\Gamma(\zeta)}\int_0^{\vartheta} (\vartheta-\varrho)^{\zeta-1}\sigma(\varrho)d\varrho + \frac{1}{\Gamma(\zeta)}\int_{\vartheta}^{\varkappa} (\varrho-\vartheta)^{\zeta-1}\sigma(\varrho)d\varrho. \end{aligned} $$

For $\vartheta = 0$, we have

$$ y(\varkappa) = y(0) - \frac{2}{\Gamma(\zeta)}\int_0^{\varkappa} \varrho^{\zeta-1}\sigma(\varrho)d\varrho. $$

Then, the final solution is given by:

$$ y(\vartheta) \;\; = \;\; y(0) - \frac{1}{\Gamma(\zeta)}\int_0^{\varkappa} \varrho^{\zeta-1}\sigma(\varrho)d\varrho + \frac{1}{\Gamma(\zeta)}\int_0^{\varkappa} |\vartheta-\varrho|^{\zeta-1}\sigma(\varrho)d\varrho. $$

Conversely, we can easily show by Lemma 2.1 that if ξ verifies equation (2.2), then it satisfied the equation (2.1). $\square$

Definition 2.5. ([7]) Let X be a Banach space and let Υ_X be the family of bounded subsets of X. The Kuratowski measure of noncompactness is the map $\alpha : \Upsilon_X \longrightarrow [0, \infty)$ defined by

$$ \alpha(\chi) = inf\left\{ \varepsilon > 0 : \chi \subset \bigcup_{j=1}^{m} \chi_j, diam(\chi_j) \leq \varepsilon \right\}, $$

where $\chi \in \Upsilon_X$. The map α satisfies the following properties:

- $\alpha(\chi) = 0 \Leftrightarrow \overline{\chi}$ is compact (χ is relatively compact).
- $\alpha(\chi) = \alpha(\overline{\chi})$.
- $\chi_1 \subset \chi_2 \Rightarrow \alpha(\chi_1) \leq \alpha(\chi_2)$.
- $\alpha(\chi_1 + \chi_2) \leq \alpha(\Omega_1) + \alpha(\Omega_2)$.
- $\alpha(c\chi) = |c|\alpha(\chi)$, $c \in X$.
- $\alpha(conv\chi) = \alpha(\chi)$.

Theorem 2.1. Mönch's fixed point theorem [18] *Let D be a non-empty, closed, bounded and convex subset of a Banach space X such that $0 \in D$ and let $\mathcal{H} : D \longrightarrow D$ be a continuous mapping. If the implication*

$$\Omega = \overline{conv}\mathcal{H}(\Omega) \text{ or } \Omega = \mathcal{H}(\Omega) \cup \{0\} \Rightarrow \alpha(\Omega) = 0, \tag{2.3}$$

holds for every subset Ω of D, then $\mathcal{H}$ has at least one fixed point.

3. Main Results

We study the fractional differential equation that follows:

$$\left({}^{RC}_{\varkappa_\jmath}D^{\zeta}_{\vartheta_{\jmath+1}}y\right)(\vartheta) = \sigma(\vartheta);\ \vartheta \in \Omega_\jmath,\ \jmath = 0, \ldots, m, \tag{3.1}$$

where $0 < \zeta \leq 1$, with the conditions

$$y(\vartheta) = \Psi_\jmath(\vartheta, y(\vartheta_\jmath^-));\ \vartheta \in \tilde{\Omega}_\jmath,\ \jmath = 1, \ldots, m, \tag{3.2}$$

$$\delta_1 y(0) + \delta_2 y(\varkappa) = \delta_3, \tag{3.3}$$

$$y(\vartheta) = \hbar_1(\vartheta), \quad \vartheta \in [-\varpi, 0],\ \varpi > 0, \tag{3.4}$$

$$y(\vartheta) = \hbar_2(\vartheta), \quad \vartheta \in [\varkappa, \varkappa + \tilde{\varpi}],\ \tilde{\varpi} > 0, \tag{3.5}$$

where $\delta_1, \delta_2 \in \mathbb{R}$, $\delta_3 \in \Xi$, $\delta_1 \neq 0$, $\sigma(\cdot) \in C(\Theta, \Xi)$, $\hbar_1 \in C([-\varpi, 0], \Xi)$, $\hbar_2 \in C([\varkappa, \varkappa + \tilde{\varpi}], \Xi)$, and $\Psi_\jmath : \tilde{\Omega}_\jmath \times \Xi \to \Xi$; $\jmath = 1, \ldots, m$ are given continuous functions.

Theorem 3.1. *The function $y(\cdot)$ verifies* (3.1)-(3.3) *if and only if it verifies*

$$y(\vartheta)=\begin{cases}\dfrac{\delta_3}{\delta_1}-\dfrac{\delta_2\Psi_m(\varkappa_m,y(t_m^-))}{\delta_1}+\dfrac{\delta_2}{\Gamma(\zeta)\delta_1}\displaystyle\int_{\varkappa_m}^{\varkappa}\varrho^{\zeta-1}\sigma(\varrho)\,d\varrho\\ -\dfrac{\delta_2}{\Gamma(\zeta)\delta_1}\displaystyle\int_{\varkappa_m}^{\varkappa}|\varkappa-\varrho|^{\zeta-1}\sigma(\varrho)\,d\varrho-\dfrac{1}{\Gamma(\zeta)}\displaystyle\int_0^{\vartheta_1}\varrho^{\zeta-1}\sigma(\varrho)\,d\varrho\\ +\dfrac{1}{\Gamma(\zeta)}\displaystyle\int_0^{\vartheta_1}|\vartheta-\varrho|^{\zeta-1}\sigma(\varrho)\,d\varrho,\quad \vartheta\in\Omega_0,\\ \\ \Psi_\jmath(\varkappa_\jmath,y(\vartheta_\jmath^-))-\dfrac{1}{\Gamma(\zeta)}\displaystyle\int_{\varkappa_\jmath}^{\vartheta_{\jmath+1}}\varrho^{\zeta-1}\sigma(\varrho)\,d\varrho\\ +\dfrac{1}{\Gamma(\zeta)}\displaystyle\int_{\varkappa_\jmath}^{\vartheta_{\jmath+1}}|\vartheta-\varrho|^{\zeta-1}\sigma(\varrho)\,d\varrho,\quad \vartheta\in\Omega_\jmath;\jmath=1,\ldots,m,\\ \\ \Psi_\jmath(\vartheta,y(\vartheta_\jmath^-)),\ \vartheta\in\tilde{\Omega}_\jmath;\jmath=1,\ldots,m,\\ \\ y(\vartheta)=\hbar_1(\vartheta),\quad \vartheta\in[-\varpi,0],\\ \\ y(\vartheta)=\hbar_2(\vartheta),\quad \vartheta\in[\varkappa,\varkappa+\tilde{\varpi}].\end{cases}$$

Proof. Assume y satisfies (3.1)-(3.3). If $\vartheta\in\Omega_0$, then

$${}_0^{RC}D_{\vartheta_1}^{\zeta}y(\vartheta)=\sigma(\vartheta).$$

By Lemma 2.2, we get

$$y(\vartheta)\ =\ y(0)-\frac{1}{\Gamma(\zeta)}\int_0^{\vartheta_1}\varrho^{\zeta-1}\sigma(\varrho)\,d\varrho+\frac{1}{\Gamma(\zeta)}\int_0^{\vartheta_1}|\vartheta-\varrho|^{\zeta-1}\sigma(\varrho)\,d\varrho.$$

If $\vartheta\in\tilde{\Omega}_1$, the we have $y(\vartheta)=\Psi_1(\vartheta,y(\vartheta_1^-))$.
If $\vartheta\in\Omega_1$, then Lemma 2.2 implies

$$\begin{aligned}y(\vartheta)&=y(\varkappa_1)-\frac{1}{\Gamma(\zeta)}\int_{\varkappa_1}^{\vartheta_2}\varrho^{\zeta-1}\sigma(\varrho)\,d\varrho+\frac{1}{\Gamma(\zeta)}\int_{\varkappa_1}^{\vartheta_2}|\vartheta-\varrho|^{\zeta-1}\sigma(\varrho)\,d\varrho\\&=\Psi_1(\varkappa_1,y(\vartheta_1^-))-\frac{1}{\Gamma(\zeta)}\int_{\varkappa_1}^{\vartheta_2}\varrho^{\zeta-1}\sigma(\varrho)\,d\varrho+\frac{1}{\Gamma(\zeta)}\int_{\varkappa_1}^{\vartheta_2}|\vartheta-\varrho|^{\zeta-1}\sigma(\varrho)\,d\varrho.\end{aligned}$$

If $\vartheta\in\tilde{\Omega}_2$, the we have $y(\vartheta)=\Psi_2(\vartheta,y(\vartheta_2^-))$.
If $\vartheta\in\Omega_2$, then Lemma 2.2 implies

$$\begin{aligned}y(\vartheta)&=y(\varkappa_2)-\frac{1}{\Gamma(\zeta)}\int_{\varkappa_2}^{\vartheta_3}\varrho^{\zeta-1}\sigma(\varrho)\,d\varrho+\frac{1}{\Gamma(\zeta)}\int_{\varkappa_2}^{\vartheta_3}|\vartheta-\varrho|^{\zeta-1}\sigma(\varrho)\,d\varrho\\&=\Psi_2(\varkappa_2,y(\vartheta_2^-))-\frac{1}{\Gamma(\zeta)}\int_{\varkappa_2}^{\vartheta_3}\varrho^{\zeta-1}\sigma(\varrho)\,d\varrho+\frac{1}{\Gamma(\zeta)}\int_{\varkappa_2}^{\vartheta_3}|\vartheta-\varrho|^{\zeta-1}\sigma(\varrho)\,d\varrho.\end{aligned}$$

Repeating the process in this way, for $\vartheta \in \Theta$ we can obtain

$$y(\vartheta)=\begin{cases} y(0)-\dfrac{1}{\Gamma(\zeta)}\displaystyle\int_0^{\vartheta_1}\varrho^{\zeta-1}\sigma(\varrho)\,d\varrho \\ +\dfrac{1}{\Gamma(\zeta)}\displaystyle\int_0^{\vartheta_1}|\vartheta-\varrho|^{\zeta-1}\sigma(\varrho)\,d\varrho, \quad \vartheta\in\Omega_0, \\ \\ \Psi_\jmath(\varkappa_\jmath, y(\vartheta_\jmath^-))-\dfrac{1}{\Gamma(\zeta)}\displaystyle\int_{\varkappa_\jmath}^{\vartheta_{\jmath+1}}\varrho^{\zeta-1}\sigma(\varrho)\,d\varrho \\ +\dfrac{1}{\Gamma(\zeta)}\displaystyle\int_{\varkappa_\jmath}^{\vartheta_{\jmath+1}}|\vartheta-\varrho|^{\zeta-1}\sigma(\varrho)\,d\varrho, \quad \vartheta\in\Omega_\jmath; \jmath=1,\ldots,m, \\ \\ \Psi_\jmath(\vartheta, y(\vartheta_\jmath^-)),\ \vartheta\in\tilde{\Omega}_\jmath; \jmath=1,\ldots,m. \end{cases}$$

Taking $\vartheta=\varkappa$ in (3.), we obtain

$$y(\varkappa)=\Psi_m(\varkappa_m, y(t_m^-))-\frac{1}{\Gamma(\zeta)}\int_{\varkappa_m}^{\varkappa}\varrho^{\zeta-1}\sigma(\varrho)\,d\varrho+\frac{1}{\Gamma(\zeta)}\int_{\varkappa_m}^{\varkappa}|\varkappa-\varrho|^{\zeta-1}\sigma(\varrho)\,d\varrho.$$

Using the condition (3.3), we get

$$\begin{aligned} y(0)=&\frac{\delta_3}{\delta_1}-\frac{\delta_2\Psi_m(\varkappa_m, y(t_m^-))}{\delta_1}+\frac{\delta_2}{\Gamma(\zeta)\delta_1}\int_{\varkappa_m}^{\varkappa}\varrho^{\zeta-1}\sigma(\varrho)\,d\varrho \\ &-\frac{\delta_2}{\Gamma(\zeta)\delta_1}\int_{\varkappa_m}^{\varkappa}|\varkappa-\varrho|^{\zeta-1}\sigma(\varrho)\,d\varrho. \end{aligned}$$

Substituting the value of $y(0)$ in (3.), we obtain (3.1).

Reciprocally, for $\vartheta\in\Omega_0$, taking $\vartheta=0$, we get

$$\begin{aligned} y(0)=&\frac{\delta_3}{\delta_1}-\frac{\delta_2\Psi_m(\varkappa_m, y(t_m^-))}{\delta_1}+\frac{\delta_2}{\Gamma(\zeta)\delta_1}\int_{\varkappa_m}^{\varkappa}\varrho^{\zeta-1}\sigma(\varrho)\,d\varrho \\ &-\frac{\delta_2}{\Gamma(\zeta)\delta_1}\int_{\varkappa_m}^{\varkappa}|\varkappa-\varrho|^{\zeta-1}\sigma(\varrho)\,d\varrho, \end{aligned}$$

and for $\vartheta\in\Omega_m$, taking $\vartheta=\varkappa$, we get

$$\begin{aligned} y(\varkappa)=&\Psi_m(\varkappa_m, y(t_m^-))-\frac{1}{\Gamma(\zeta)}\int_{\varkappa_m}^{\varkappa}\varrho^{\zeta-1}\sigma(\varrho)\,d\varrho \\ &+\frac{1}{\Gamma(\zeta)}\int_{\varkappa_m}^{\varkappa}|\varkappa-\varrho|^{\zeta-1}\sigma(\varrho)\,d\varrho. \end{aligned}$$

Thus, we can obtain $\delta_1 y(0)+\delta_2 y(\varkappa)=\delta_3$, which implies that (3.3) is verified. Next, apply ${}^{RC}_{\varkappa_\jmath}D^{\zeta}_{\vartheta_{\jmath+1}}(\cdot)$ on both sides of (3.1), where $\jmath=0,\ldots,m$. Then, by Lemma 2.2 we get the equation (3.1). Also, it is clear that y verifies (3.2), (3.4) and (3.5). $\square$

Lemma 3.1. *Let* $0 < \zeta \leq 1$, $\varphi : \Theta \times PC\left([-\varpi, \tilde{\varpi}], \Xi\right) \times \Xi \to \Xi$ *is a given function,* $\hbar_1(\cdot) \in \mathcal{X}$ *and* $\hbar_2(\cdot) \in \tilde{\mathcal{X}}$, *then* $y \in \mathbb{F}$ *verifies* (1.1)-(1.5) *if and only if* y *is the fixed point of the operator* $\aleph : \mathbb{F} \to \mathbb{F}$ *defined by*

$$\aleph y(\vartheta) = \begin{cases} \dfrac{\delta_3}{\delta_1} - \dfrac{\delta_2 \Psi_m(\varkappa_m, y(t_m^-))}{\delta_1} + \dfrac{\delta_2}{\Gamma(\zeta)\delta_1} \displaystyle\int_{\varkappa_m}^{\varkappa} \varrho^{\zeta-1} \sigma(\varrho)\, d\varrho \\ -\dfrac{\delta_2}{\Gamma(\zeta)\delta_1} \displaystyle\int_{\varkappa_m}^{\varkappa} |\varkappa - \varrho|^{\zeta-1} \sigma(\varrho)\, d\varrho - \dfrac{1}{\Gamma(\zeta)} \displaystyle\int_0^{\vartheta_1} \varrho^{\zeta-1} \sigma(\varrho)\, d\varrho \\ +\dfrac{1}{\Gamma(\zeta)} \displaystyle\int_0^{\vartheta_1} |\vartheta - \varrho|^{\zeta-1} \sigma(\varrho)\, d\varrho, \quad \vartheta \in \Omega_0, \\ \\ \Psi_\jmath(\varkappa_\jmath, y(\vartheta_\jmath^-)) - \dfrac{1}{\Gamma(\zeta)} \displaystyle\int_{\varkappa_\jmath}^{\vartheta_{\jmath+1}} \varrho^{\zeta-1} \sigma(\varrho)\, d\varrho \\ +\dfrac{1}{\Gamma(\zeta)} \displaystyle\int_{\varkappa_\jmath}^{\vartheta_{\jmath+1}} |\vartheta - \varrho|^{\zeta-1} \sigma(\varrho)\, d\varrho, \quad \vartheta \in \Omega_\jmath; \jmath = 1, \dots, m, \\ \\ \Psi_\jmath(\vartheta, y(\vartheta_\jmath^-)), \ \vartheta \in \tilde{\Omega}_\jmath; \jmath = 1, \dots, m, \\ \\ y(\vartheta) = \hbar_1(\vartheta), \quad \vartheta \in [-\varpi, 0], \\ \\ y(\vartheta) = \hbar_2(\vartheta), \quad \vartheta \in [\varkappa, \varkappa + \tilde{\varpi}]. \end{cases}$$

where σ *be a function satisfying the functional equation*

$$\sigma(\vartheta) = \varphi\left(\vartheta, y^\vartheta(\cdot), \sigma(\vartheta)\right).$$

Obviously, the fixed points of the operator $\aleph$ *are solutions of the problem* (1.1)-(1.5).

Let us assume the following assumptions:

(*Ax1*) The function $\varphi : \Theta \times PC\left([-\varpi, \tilde{\varpi}], \Xi\right) \times \Xi \to \Xi$ is continuous.

(*Ax2*) There exist constants $\psi_1, \wp_\jmath > 0$ and $0 < \psi_2 < 1$ such that

$$\|\varphi(\vartheta, \xi, \gamma) - \varphi(\vartheta, \bar{\xi}, \bar{\gamma})\| \leq \psi_1 \|\xi - \bar{\xi}\|_{[-\varpi, \tilde{\varpi}]} + \psi_2 \|\gamma - \bar{\gamma}\|$$

and

$$\|\Psi_\jmath(\vartheta, \gamma) - \Psi_\jmath(\vartheta, \bar{\gamma})\| \leq \wp_\jmath \|\gamma - \bar{\gamma}\|$$

for any $\xi, \bar{\xi} \in PC\left([-\varpi, \tilde{\varpi}], \Xi\right)$, $\gamma, \bar{\gamma} \in \Xi$ and $\vartheta \in \Omega_\jmath$; $\jmath = 0, \dots, m$, where $\wp^* = \max_{\jmath = 1, \dots, m} \{\wp_\jmath\}$.

(*Ax3*) For each bounded sets $\beta_1 \in PC\left([-\varpi, \tilde{\varpi}], \Xi\right)$ and $\beta_2 \in \Xi$ and for each $\vartheta \in \Omega_\jmath$; $\jmath = 0, \dots, m$, we have

$$\alpha\left(\varphi(\vartheta, \beta_1, \beta_2)\right) \leq \psi_1 \sup_{s \in [-\varpi, \tilde{\varpi}]} \alpha(\beta_1(s)) + \psi_2 \alpha(\beta_2)$$

and

$$\alpha\left(\Psi_\jmath(\vartheta, \beta_2)\right) \leq \wp_\jmath \alpha(\beta_2).$$

Remark 3.1. ([5]) It is worth noting that the hypotheses (*Ax2*) and (*Ax3*) are equivalent.

We are now in a position to prove the existence result of the problem (1.1)-(1.5) based on the Mönch fixed point theorem.

Theorem 3.2. *Assume that the assumptions* $(Ax1)$*-*$(Ax2)$ *hold. If*

$$\beta := (|\delta_1| + |\delta_2|)\left[\frac{\wp^*}{|\delta_1|} + \frac{\varkappa^\zeta \psi_1}{\Gamma(\zeta+1)|\delta_1|(1-\psi_2)}\right] < 1, \tag{3.6}$$

then the implicit fractional problem (1.1)-(1.5) *has a solution on* Θ.

Proof. The proof will be given in several steps.

Step 1: We show that the operator $\aleph$ defined in (3.1), transforms the ball $B_\omega = B(0,\omega) = \{y \in \mathbb{F} : \|y\|_{\mathbb{F}} \le \omega\}$ into itself.
Let ω a positive constant such that

$$\omega \ge \max\left\{ \frac{\dfrac{\|\delta_3\| + (|\delta_1|+|\delta_2|)\tilde{L}}{|\delta_1|} + \dfrac{(|\delta_1|+|\delta_2|)\varkappa^\zeta \varphi^*}{\Gamma(\zeta+1)|\delta_1|(1-\psi_2)}}{1 - (|\delta_1|+|\delta_2|)\left[\dfrac{\wp^*}{|\delta_1|} + \dfrac{\varkappa^\zeta\psi_1}{\Gamma(\zeta+1)|\delta_1|(1-\psi_2)}\right]}, \|\hbar_1\|_{\mathcal{X}}, \|\hbar_2\|_{\tilde{\mathcal{X}}} \right\},$$

such that

$$\tilde{L} = \max_{1\le \jmath \le m}\{\sup\{\|\Psi_\jmath(\vartheta,0)\|,\ \vartheta \in \tilde{\Omega}_\jmath\}\},$$

and

$$\varphi^* = \sup_{\vartheta\in\Theta} \|\varphi(\vartheta,0,0)\|.$$

For each $\vartheta \in [-\varpi, 0]$, we have

$$\begin{aligned}\|\aleph y(\vartheta)\| &\le \|\hbar_1\|_{\mathcal{X}}\\ &\le \omega,\end{aligned}$$

and for each $\vartheta \in [\varkappa, \varkappa + \tilde{\varpi}]$, we have

$$\begin{aligned}\|\aleph y(\vartheta)\| &\le \|\hbar_2\|_{\tilde{\mathcal{X}}}\\ &\le \omega.\end{aligned}$$

By the hypothesis (*Ax2*), for $\vartheta \in \Theta$, we have

$$\begin{aligned}\|\sigma(\vartheta)\| &= \|\varphi\left(\vartheta, y^\vartheta(\cdot), \sigma(\vartheta)\right) - \varphi(\vartheta,0,0)\| + \|\varphi(\vartheta,0,0)\|\\ &\le \psi_1\|y^\vartheta\|_{[-\varpi,\tilde{\varpi}]} + \psi_2\|\sigma(\vartheta)\| + \varphi^*\end{aligned}$$

and

$$\begin{aligned}\|\Psi_\jmath(\vartheta, y(\vartheta))\| &= \|\Psi_\jmath(\vartheta,y(\vartheta)) - \Psi_\jmath(\vartheta,0)\| + \|\Psi_\jmath(\vartheta,0)\|\\ &\le \wp^*\|y\|_{\mathbb{F}} + \tilde{L},\end{aligned}$$

which implies that

$$\|\sigma(\vartheta)\| \le \psi_1\omega + \psi_2\|\sigma(\vartheta)\| + \varphi^*,$$

then

$$\|\sigma(\vartheta)\| \le \frac{\varphi^* + \psi_1\omega}{1-\psi_2},$$

and

$$\|\Psi_\jmath(\vartheta, y(\vartheta))\| \le \wp^*\omega + \tilde{L}.$$

Thus, for $\vartheta \in \Omega_0$ and by (3.1) and hypothesis ($Ax2$), we obtain

$$\begin{aligned}
\|(\aleph y)(\vartheta)\| &\le \frac{\|\delta_3\|}{|\delta_1|} + \frac{|\delta_2|\|\Psi_m(\varkappa_m, y(t_m^-))\|}{|\delta_1|} + \frac{|\delta_2|}{\Gamma(\zeta)|\delta_1|}\int_{\varkappa_m}^{\varkappa} \varrho^{\zeta-1}\|\sigma(\varrho)\|d\varrho \\
&\quad + \frac{|\delta_2|}{\Gamma(\zeta)|\delta_1|}\int_{\varkappa_m}^{\varkappa} |\varkappa-\varrho|^{\zeta-1}\|\sigma(\varrho)\|d\varrho \\
&\quad + \frac{1}{\Gamma(\zeta)}\int_0^{\vartheta_1} \varrho^{\zeta-1}\|\sigma(\varrho)\|d\varrho \\
&\quad + \frac{1}{\Gamma(\zeta)}\int_0^{\vartheta_1} |\vartheta-\varrho|^{\zeta-1}\|\sigma(\varrho)\|d\varrho \\
&\le \frac{\|\delta_3\|}{|\delta_1|} + \frac{|\delta_2|(\wp^*\omega+\tilde{L})}{|\delta_1|} + \frac{2|\delta_2|\varkappa^\zeta(\varphi^*+\psi_1\omega)}{\Gamma(\zeta+1)|\delta_1|(1-\psi_2)} + \frac{2\varkappa^\zeta(\varphi^*+\psi_1\omega)}{\Gamma(\zeta+1)(1-\psi_2)} \\
&\le \left[\frac{\|\delta_3\| + |\delta_2|\tilde{L}}{|\delta_1|} + \frac{2(|\delta_1|+|\delta_2|)\varkappa^\zeta\varphi^*}{\Gamma(\zeta+1)|\delta_1|(1-\psi_2)}\right] \\
&\quad + \omega\left[\frac{|\delta_2|\wp^*}{|\delta_1|} + \frac{2(|\delta_1|+|\delta_2|)\varkappa^\zeta\psi_1}{\Gamma(\zeta+1)|\delta_1|(1-\psi_2)}\right].
\end{aligned}$$

For $\vartheta \in \Omega_\jmath; \jmath = 1, \ldots, m$, we obtain

$$\begin{aligned}
\|(\aleph y)(\vartheta)\| &\le \|\Psi_\jmath(\varkappa_\jmath, y(\vartheta_\jmath^-))\| + \frac{1}{\Gamma(\zeta)}\int_{\varkappa_\jmath}^{\vartheta_{\jmath+1}} \varrho^{\zeta-1}\|\sigma(\varrho)\|d\varrho \\
&\quad + \frac{1}{\Gamma(\zeta)}\int_{\varkappa_\jmath}^{\vartheta_{\jmath+1}} |\vartheta-\varrho|^{\zeta-1}\|\sigma(\varrho)\|d\varrho \\
&\le (\wp^*\omega + \tilde{L}) + \frac{2\varkappa^\zeta(\varphi^*+\psi_1\omega)}{\Gamma(\zeta+1)(1-\psi_2)},
\end{aligned}$$

and for $\vartheta \in \tilde{\Omega}_\jmath; \jmath = 1, \ldots, m$, we have

$$\begin{aligned}
\|(\aleph y)(\vartheta)\| &\le \left\|\Psi_\jmath(\vartheta, y(\vartheta_\jmath^-))\right\| \\
&\le \wp^*\omega + \tilde{L}.
\end{aligned}$$

Then for each $\vartheta \in \Theta$ we get

$$\begin{aligned}\|\aleph y\|_{PC} \leq & \left[\frac{\|\delta_3\| + (|\delta_1| + |\delta_2|)\tilde{L}}{|\delta_1|} + \frac{2(|\delta_1| + |\delta_2|)\varkappa^{\zeta}\varphi^*}{\Gamma(\zeta+1)|\delta_1|(1-\psi_2)}\right] \\ & + \omega(|\delta_1| + |\delta_2|)\left[\frac{\wp^*}{|\delta_1|} + \frac{2\varkappa^{\zeta}\psi_1}{\Gamma(\zeta+1)|\delta_1|(1-\psi_2)}\right] \\ \leq & \ \omega.\end{aligned}$$

Then, for each $\vartheta \in [-\varpi, \varkappa + \tilde{\varpi}]$ we obtain

$$\|\aleph y\|_{\mathbb{F}} \leq \omega.$$

Step 2: $\aleph : B_\omega \to B_\omega$ is continuous.
Let $\{y_n\}$ be a sequence such that $y_n \longrightarrow y$ in $\mathbb{F}$. For each $\vartheta \in [-\varpi, 0] \cup [\varkappa, \varkappa + \tilde{\varpi}]$, we have

$$\|\aleph y_n(\vartheta) - \aleph y(\vartheta)\| = 0.$$

And for $\vartheta \in \Omega_0$ we have

$$\begin{aligned}\|\aleph y(\vartheta) - \aleph y_n(\vartheta)\| \leq & \frac{|\delta_2|}{|\delta_1|}\left\|\Psi_m(\varkappa_m, y(t_m^-)) - \Psi_m(\varkappa_m, y_n(t_m^-))\right\| \\ & + \frac{\delta_2}{\Gamma(\zeta)\delta_1}\int_{\varkappa_m}^{\varkappa} \varrho^{\zeta-1}\|\sigma(\varrho) - \sigma_n(\varrho)\|d\varrho \\ & + \frac{\delta_2}{\Gamma(\zeta)\delta_1}\int_{\varkappa_m}^{\varkappa} |\varkappa - \varrho|^{\zeta-1}\|\sigma(\varrho) - \sigma_n(\varrho)\|d\varrho \\ & + \frac{1}{\Gamma(\zeta)}\int_0^{\vartheta_1} \varrho^{\zeta-1}\|\sigma(\varrho) - \sigma_n(\varrho)\|d\varrho \\ & + \frac{1}{\Gamma(\zeta)}\int_0^{\vartheta_1} |\vartheta - \varrho|^{\zeta-1}\|\sigma(\varrho) - \sigma_n(\varrho)\|d\varrho,\end{aligned}$$

and for $\vartheta \in \Omega_\jmath; \jmath = 1, \ldots, m$, we have

$$\begin{aligned}\|\aleph y(\vartheta) - \aleph y_n(\vartheta)\| \leq & \left\|\Psi_\jmath(\varkappa_\jmath, y(\vartheta_\jmath^-)) - \Psi_\jmath(\varkappa_\jmath, y_n(\vartheta_\jmath^-))\right\| \\ & + \frac{1}{\Gamma(\zeta)}\int_{\varkappa_\jmath}^{\vartheta_{\jmath+1}} \varrho^{\zeta-1}\|\sigma(\varrho) - \sigma_n(\varrho)\|d\varrho \\ & + \frac{1}{\Gamma(\zeta)}\int_{\varkappa_\jmath}^{\vartheta_{\jmath+1}} |\vartheta - \varrho|^{\zeta-1}\|\sigma(\varrho) - \sigma_n(\varrho)\|d\varrho,\end{aligned}$$

where σ and σ_n be functions satisfying the functional equations

$$\begin{aligned}\sigma(\vartheta) &= \varphi(\vartheta, y^{\vartheta}(\cdot), \sigma(\vartheta)), \\ \sigma_n(\vartheta) &= \varphi(\vartheta, y_n^{\vartheta}(\cdot), \sigma_n(\vartheta)).\end{aligned}$$

Since $y_n \to y$ and hypothesis $(Ax2)$, we have that $\sigma_n(\vartheta) \to \sigma(\vartheta)$ as $n \to \infty$ for each $\vartheta \in \Theta$.
For $\vartheta \in \tilde{\Omega}_\jmath; \jmath = 1, \ldots, m$, we have

$$\begin{aligned}\|\aleph y(\vartheta) - \aleph y_n(\vartheta)\| &\leq \left\|\Psi_\jmath(\vartheta, y(\vartheta_\jmath^-)) - \Psi_\jmath(\vartheta, y_n(\vartheta_\jmath^-))\right\| \\ &\leq \wp^* \|y - y_n\|_{PC}.\end{aligned}$$

Since φ and $\Psi_\jmath; i = 1, \ldots, m$ are continuous, then we have

$$\|\aleph y_n - \aleph y\|_{\mathbb{F}} \to 0 \text{ as } n \to \infty.$$

Step 3: $\aleph(B_\omega)$ is bounded and equicontinuous.
Since $\aleph(B_\omega) \subset B_\omega$ and B_ω is bounded, then $\aleph(B_\omega)$ is bounded.
For $\nu_1, \nu_2 \in \Omega_\jmath; \jmath = 0, \ldots, m$, we have

$$\begin{aligned}&\|(\aleph y)(\nu_1) - (\aleph y)(\nu_2)\| \\ &\leq \left\| \frac{1}{\Gamma(\zeta)} \int_{\varkappa_\jmath}^{\vartheta_{\jmath+1}} |\nu_1 - \varrho|^{\zeta-1} \sigma(\varrho)\, d\varrho - \frac{1}{\Gamma(\zeta)} \int_{\varkappa_\jmath}^{\vartheta_{\jmath+1}} |\nu_2 - \varrho|^{\zeta-1} \sigma(\varrho)\, d\varrho \right\| \\ &\leq \frac{1}{\Gamma(\zeta)} \int_{\varkappa_\jmath}^{\vartheta_{\jmath+1}} \left| |\nu_1 - \varrho|^{\zeta-1} - |\nu_2 - \varrho|^{\zeta-1} \right| \|\sigma(\varrho)\| d\varrho \\ &\leq \frac{\varphi^* + \psi_1 \omega}{\Gamma(\zeta)(1-\psi_2)} \int_{\varkappa_\jmath}^{\vartheta_{\jmath+1}} \left| |\nu_1 - \varrho|^{\zeta-1} - |\nu_2 - \varrho|^{\zeta-1} \right| d\varrho.\end{aligned}$$

Note that

$$\|(\aleph y)(\nu_1) - (\aleph y)(\nu_2)\| \to 0 \quad \text{as} \quad \nu_1 \to \nu_2.$$

And for $\nu_1, \nu_2 \in \tilde{\Omega}_\jmath; \jmath = 1, \ldots, m$,

$$\|(\aleph y)(\nu_1) - (\aleph y)(\nu_2)\| \leq \left\|\Psi_\jmath(\nu_1, y(\vartheta_\jmath^-)) - \Psi_\jmath(\nu_2, y(\vartheta_\jmath^-))\right\|,$$

and since $\Psi_\jmath$ are continuous that

$$\|(\aleph y)(\nu_1) - (\aleph y)(\nu_2)\| \to 0 \quad \text{as} \quad \nu_1 \to \nu_2.$$

Hence, $\aleph(B_\omega)$ is bounded and equicontinuous.

Step 4: The implication (2.3) of Theorem 2.1 holds.
Now let $\mathfrak{V}$ be an equicontinuous subset of B_ω such that $\mathfrak{V} \subset \overline{\aleph(\mathfrak{V})} \cup \{0\}$, therefore the function $\vartheta \longrightarrow d(\vartheta) = \alpha(\mathfrak{V}(\vartheta))$ is continuous on $[-\varpi, \varkappa + \tilde{\varpi}]$. By $(Ax3)$ and the properties of the measure α, for each $\vartheta \in \Omega_\jmath; \jmath = 0, \ldots, m$, we have

$$\begin{aligned}d(\vartheta) &\leq \alpha\left((\aleph\mathfrak{V})(\vartheta) \cup \{0\}\right) \\ &\leq \alpha\left((\aleph\mathfrak{V})(\vartheta)\right) \\ &\leq \frac{|\delta_2|\wp^*\|d\|_{PC}}{|\delta_1|} + \frac{|\delta_2|\varkappa^\zeta\psi_1\|d\|_{[-\varpi,\tilde{\varpi}]}}{\Gamma(\zeta+1)|\delta_1|(1-\psi_2)} + \frac{\varkappa^\zeta\psi_1\|d\|_{[-\varpi,\tilde{\varpi}]}}{\Gamma(\zeta+1)(1-\psi_2)} \\ &\leq \|d\|_{\mathbb{F}} \left[\frac{|\delta_2|\wp^*}{|\delta_1|} + \frac{(|\delta_1| + |\delta_2|)\varkappa^\zeta\psi_1}{\Gamma(\zeta+1)|\delta_1|(1-\psi_2)}\right].\end{aligned}$$

For $\vartheta \in \tilde{\Omega}_\jmath; \jmath = 1, \ldots, m$, we have

$$d(\vartheta) \leq \wp^* \|d\|_{\mathbb{F}}.$$

For $\vartheta \in [-\varpi, 0] \cup [\varkappa, \varkappa + \tilde{\varpi}]$, we have

$$d(\vartheta) = \alpha(\hbar_1(\vartheta)) = \alpha(\hbar_2(\vartheta)) = 0.$$

Thus

$$\|d\|_{\mathbb{F}} \leq \beta \|d\|_{\mathbb{F}}.$$

From (3.6), we get $\|d\|_{\mathbb{F}} = 0$, that is $d(\vartheta) = \alpha(\mathfrak{V}(\vartheta)) = 0$, for each $\vartheta \in [-\varpi, \varkappa + \tilde{\varpi}]$, and then $\mathfrak{V}(\vartheta)$ is relatively compact in Ξ. In view of the Ascoli-Arzela Theorem, $\mathfrak{V}$ is relatively compact in B_ω. By Theorem 2.1, we deduce that $\aleph$ has a fixed point, which is a solution to (1.1)-(1.5). □

4. Ulam-Hyers-Rassias Stability

Now, we consider the Ulam stability for problem (1.1)-(1.5). For this, we take inspiration from the following papers [23, 33, 26, 35] and the references therein. Let $y \in \mathbb{F}$, $\epsilon > 0$, $\Delta_1, \Delta_2 > 0$, $\lambda > 0$, and $\mathrm{Im} : \Theta \longrightarrow [0, \infty)$ be a continuous function. We consider the following inequalities:

$$\begin{cases} \left\| \left({}^{RC}_{\varkappa_\jmath} D^{\zeta}_{\vartheta_{\jmath+1}} y \right)(\vartheta) - \varphi \left(\vartheta, y^{\vartheta}(\cdot), \left({}^{RC}_{\varkappa_\jmath} D^{\zeta}_{\vartheta_{\jmath+1}} y \right)(\vartheta) \right) \right\| \leq \epsilon, \vartheta \in \Omega_\jmath, \jmath = 0, \ldots, m, \\ \left\| y(\vartheta) - \Psi_\jmath(\vartheta, y(\vartheta_\jmath^-)) \right\| \leq \epsilon, \quad \vartheta \in \tilde{\Omega}_\jmath, \jmath = 1, \ldots, m, \\ \|y(\vartheta) - \hbar_1(\vartheta)\| \leq \epsilon, \quad \vartheta \in [-\varpi, 0], \\ \|y(\vartheta) - \hbar_2(\vartheta)\| \leq \epsilon, \quad \vartheta \in [\varkappa, \varkappa + \tilde{\varpi}], \end{cases} \tag{4.1}$$

$$\begin{cases} \left\| \left({}^{RC}_{\varkappa_\jmath} D^{\zeta}_{\vartheta_{\jmath+1}} y \right)(\vartheta) - \varphi \left(\vartheta, y^{\vartheta}(\cdot), \left({}^{RC}_{\varkappa_\jmath} D^{\zeta}_{\vartheta_{\jmath+1}} y \right)(\vartheta) \right) \right\| \leq \mathrm{Im}(\vartheta), \vartheta \in \Omega_\jmath, \jmath = 0, \ldots, m, \\ \left\| y(\vartheta) - \Psi_\jmath(\vartheta, y(\vartheta_\jmath^-)) \right\| \leq \lambda, \quad \vartheta \in \tilde{\Omega}_\jmath, \jmath = 1, \ldots, m, \\ \|y(\vartheta) - \hbar_1(\vartheta)\| \leq \Delta_1, \quad \vartheta \in [-\varpi, 0], \\ \|y(\vartheta) - \hbar_2(\vartheta)\| \leq \Delta_2, \quad \vartheta \in [\varkappa, \varkappa + \tilde{\varpi}], \end{cases} \tag{4.2}$$

and

$$\begin{cases} \left\| \left({}^{RC}_{\varkappa_\jmath} D^{\zeta}_{\vartheta_{\jmath+1}} y \right)(\vartheta) - \varphi\left(\vartheta, y^{\vartheta}(\cdot), \left({}^{RC}_{\varkappa_\jmath} D^{\zeta}_{\vartheta_{\jmath+1}} y \right)(\vartheta)\right) \right\| \leq \epsilon \mathrm{Im}(\vartheta), \vartheta \in \Omega_\jmath, \jmath = 0, \dots, m, \\ \|y(\vartheta) - \Psi_\jmath(\vartheta, y(\vartheta_\jmath^-))\| \leq \epsilon\lambda, \quad \vartheta \in \tilde{\Omega}_\jmath, \jmath = 1, \dots, m, \\ \|y(\vartheta) - \hbar_1(\vartheta)\| \leq \epsilon\Delta_1, \quad \vartheta \in [-\varpi, 0], \\ \|y(\vartheta) - \hbar_2(\vartheta)\| \leq \epsilon\Delta_2, \quad \vartheta \in [\varkappa, \varkappa + \tilde{\varpi}]. \end{cases} \tag{4.3}$$

Definition 4.1. ([33, 26, 35]) Problem (1.1)-(1.5) is Ulam-Hyers (U-H) stable if there exists a real number $a_\varphi > 0$ such that for each $\epsilon > 0$ and for each solution $x \in \mathbb{F}$ of inequality (4.1) there exists a solution $y \in \mathbb{F}$ of (1.1)-(1.5) with

$$\|x(\vartheta) - y(\vartheta)\| \leq \epsilon a_\varphi, \qquad \vartheta \in \Theta.$$

Definition 4.2. ([33, 26, 35]) Problem (1.1)-(1.5) is generalized Ulam-Hyers (G.U-H) stable if there exists $K_\varphi : C([0,\infty),[0,\infty))$ with $K_\varphi(0) = 0$ such that for each $\epsilon > 0$ and for each solution $x \in \mathbb{F}$ of inequality (4.1) there exists a solution $y \in \mathbb{F}$ of (1.1)-(1.5) with

$$|x(\vartheta) - y(\vartheta)| \leq K_\varphi(\epsilon), \qquad \vartheta \in \Theta.$$

Definition 4.3. ([33, 26, 35]) Problem (1.1)-(1.5) is Ulam-Hyers-Rassias (U-H-R) stable with respect to $(\mathrm{Im}, \lambda, \Delta_1, \Delta_2)$ if there exists a real number $a_{\varphi,\mathrm{Im}} > 0$ such that for each $\epsilon > 0$ and for each solution $x \in \mathbb{F}$ of inequality (4.3) there exists a solution $y \in \mathbb{F}$ of (1.1)-(1.5) with

$$\|x(\vartheta) - y(\vartheta)\| \leq \epsilon a_{\varphi,\mathrm{Im}}(\mathrm{Im}(\vartheta) + \lambda + \Delta_1 + \Delta_2), \qquad \vartheta \in \Theta.$$

Definition 4.4. ([33, 26, 35]) Problem (1.1)-(1.5) is generalized Ulam-Hyers-Rassias (G.U-H-R) stable with respect to $(\mathrm{Im}, \lambda, \Delta_1, \Delta_2)$ if there exists a real number $a_{\varphi,\mathrm{Im}} > 0$ such that for each solution $x \in \mathbb{F}$ of inequality (4.3) there exists a solution $y \in \mathbb{F}$ of (1.1)-(1.5) with

$$\|x(\vartheta) - y(\vartheta)\| \leq a_{\varphi,\mathrm{Im}}(\mathrm{Im}(\vartheta) + \lambda + \Delta_1 + \Delta_2), \qquad \vartheta \in \Theta.$$

Remark 4.1. It is clear that :

1. Definition 4.1 $\Longrightarrow$ Definition 4.2
2. Definition 4.3 $\Longrightarrow$ Definition 4.4
3. Definition 4.3 for $\mathrm{Im}(.) = \lambda = \Delta_1 = \Delta_2 = 1 \Longrightarrow$ Definition 4.1

Remark 4.2. A function $y \in \mathbb{F}$ is a solution of inequality (4.3) if and only if there exist $\upsilon \in \mathbb{F}$ and a sequence $\upsilon_\jmath, \jmath = 0, \dots, m+2$ such that

1. $\|\upsilon(\vartheta)\| \leq \epsilon \mathrm{Im}(\vartheta)$, $\vartheta \in \Omega_\jmath, \jmath = 0, \dots, m$; $\|\upsilon_\jmath\| \leq \epsilon\lambda$, $\vartheta \in \tilde{\Omega}_\jmath, \jmath = 1, \dots, m$, $\|\upsilon_{m+1}\| \leq \epsilon\Delta_1$ and $\|\upsilon_{m+2}\| \leq \epsilon\Delta_2$.

2. $\left({}^{RC}_{\varkappa_\jmath}D^{\zeta}_{\vartheta_{\jmath+1}}y\right)(\vartheta)=\varphi\left(\vartheta,y^{\vartheta}(\cdot),\left({}^{RC}_{\varkappa_\jmath}D^{\zeta}_{\vartheta_{\jmath+1}}y\right)(\vartheta)\right)+\upsilon(\vartheta)$, $\vartheta\in\Omega_\jmath, \jmath=0,\dots,m$,

3. $y(\vartheta)=\Psi_\jmath(\vartheta,y(\vartheta_\jmath^-))+\upsilon_\jmath$, $\vartheta\in\tilde{\Omega}_\jmath, \jmath=1,\dots,m$.

4. $y(\vartheta)=\hbar_1(\vartheta)+\upsilon_{m+1}, \quad \vartheta\in[-\varpi,0]$.

5. $y(\vartheta)=\hbar_2(\vartheta)+\upsilon_{m+2}, \quad \vartheta\in[\varkappa,\varkappa+\tilde{\varpi}]$.

Theorem 4.1. *Assume that in addition to* (Ax1)-(Ax3) *and (3.6), the following hypothesis holds.*

(Ax4) *There exist a nondecreasing function* $\mathrm{Im}:\Theta\longrightarrow[0,\infty)$ *and* $\ell_{\mathrm{Im}}>0$ *such that for each* $\vartheta\in\Omega_\jmath;\jmath=0,\dots,m$, *we have*

$$({}_0I^{\zeta}_{\varkappa}\mathrm{Im})(\vartheta)\le\ell_{\mathrm{Im}}\mathrm{Im}(\vartheta),$$

Then the problem (1.1)-(1.5) *is U-H-R stable with respect to* (Im,λ).

Proof. Let $x\in\mathbb{F}$ be a solution if inequality (4.3), and let us assume that y is the unique solution of the problem

$$\begin{cases}\left({}^{RC}_{\varkappa_\jmath}D^{\zeta}_{\vartheta_{\jmath+1}}y\right)(\vartheta)=\varphi\left(\vartheta,y^{\vartheta}(\cdot),\left({}^{RC}_{\varkappa_\jmath}D^{\zeta}_{\vartheta_{\jmath+1}}y\right)(\vartheta)\right);\ \vartheta\in\Omega_\jmath,\ \jmath=0,\dots,m,\\ y(\vartheta)=\Psi_\jmath(\vartheta,y(\vartheta_\jmath^-));\ \vartheta\in\tilde{\Omega}_\jmath,\ \jmath=1,\dots,m,\\ y(\vartheta)=\hbar_1(\vartheta),\quad\vartheta\in[-\varpi,0],\\ y(\vartheta)=\hbar_2(\vartheta),\quad\vartheta\in[\varkappa,\varkappa+\tilde{\varpi}],\\ \delta_1y(0)+\delta_2y(\varkappa)=\delta_3,\\ y(\varkappa_\jmath)=x(\varkappa_\jmath);\jmath=0,\dots,m,\\ y(\vartheta_\jmath)=x(\vartheta_\jmath);\jmath=1,\dots,m+1.\end{cases}$$

By Theorem 3.1, we obtain for each $\vartheta \in \Theta$

$$y(\vartheta) = \begin{cases} \dfrac{\delta_3}{\delta_1} - \dfrac{\delta_2 \Psi_m(\varkappa_m, y(t_m^-))}{\delta_1} + \dfrac{\delta_2}{\Gamma(\zeta)\delta_1} \displaystyle\int_{\varkappa_m}^{\varkappa} \varrho^{\zeta-1} \sigma(\varrho)\, d\varrho \\ -\dfrac{\delta_2}{\Gamma(\zeta)\delta_1} \displaystyle\int_{\varkappa_m}^{\varkappa} |\varkappa - \varrho|^{\zeta-1} \sigma(\varrho)\, d\varrho - \dfrac{1}{\Gamma(\zeta)} \displaystyle\int_0^{\vartheta_1} \varrho^{\zeta-1} \sigma(\varrho)\, d\varrho \\ +\dfrac{1}{\Gamma(\zeta)} \displaystyle\int_0^{\vartheta_1} |\vartheta - \varrho|^{\zeta-1} \sigma(\varrho)\, d\varrho, \quad \vartheta \in \Omega_0, \\ \\ \Psi_\jmath(\varkappa_\jmath, y(\vartheta_\jmath^-)) - \dfrac{1}{\Gamma(\zeta)} \displaystyle\int_{\varkappa_\jmath}^{\vartheta_{\jmath+1}} \varrho^{\zeta-1} \sigma(\varrho)\, d\varrho \\ +\dfrac{1}{\Gamma(\zeta)} \displaystyle\int_{\varkappa_\jmath}^{\vartheta_{\jmath+1}} |\vartheta - \varrho|^{\zeta-1} \sigma(\varrho)\, d\varrho, \quad \vartheta \in \Omega_\jmath; \jmath = 1, \ldots, m, \\ \\ \Psi_\jmath(\vartheta, y(\vartheta_\jmath^-)), \ \vartheta \in \tilde{\Omega}_\jmath; \jmath = 1, \ldots, m, \\ \\ y(\vartheta) = \hbar_1(\vartheta), \quad \vartheta \in [-\varpi, 0], \\ \\ y(\vartheta) = \hbar_2(\vartheta), \quad \vartheta \in [\varkappa, \varkappa + \tilde{\varpi}], \end{cases}$$

where σ be a function satisfying the functional equations

$$\sigma(\vartheta) = \varphi(\vartheta, y^\vartheta(\cdot), \sigma(\vartheta)).$$

Since x is a solution of the inequality (4.3), by Remark 4.2, we have

$$\begin{cases} \left({}^{RC}_{\varkappa_\jmath}D^{\zeta}_{\vartheta_{\jmath+1}} x\right)(\vartheta) = \varphi\left(\vartheta, x^\vartheta(\cdot), \left({}^{RC}_{\varkappa_\jmath}D^{\zeta}_{\vartheta_{\jmath+1}} x\right)(\vartheta)\right) + \upsilon(\vartheta), \vartheta \in \Omega_\jmath, \jmath = 0, \ldots, m, \\ \\ x(\vartheta) = \Psi_\jmath(\vartheta, x(\vartheta_\jmath^-)) + \upsilon_\jmath, \vartheta \in \tilde{\Omega}_\jmath, \jmath = 1, \ldots, m, \\ \\ x(\vartheta) = \hbar_1(\vartheta) + \upsilon_{m+1}, \quad \vartheta \in [-\varpi, 0], \\ \\ x(\vartheta) = \hbar_2(\vartheta) + \upsilon_{m+2}, \quad \vartheta \in [\varkappa, \varkappa + \tilde{\varpi}]. \end{cases} \tag{4.4}$$

Clearly, the solution of (4.4) is given by

$$x(\vartheta) = \begin{cases} x(\varkappa_\jmath) - \dfrac{1}{\Gamma(\zeta)} \displaystyle\int_{\varkappa_\jmath}^{\vartheta_{\jmath+1}} \varrho^{\zeta-1} (\sigma_x(\varrho) + \upsilon(\varrho)) d\varrho \\ +\dfrac{1}{\Gamma(\zeta)} \displaystyle\int_{\varkappa_\jmath}^{\vartheta_{\jmath+1}} |\vartheta - \varrho|^{\zeta-1} (\sigma_x(\varrho) + \upsilon(\varrho)) d\varrho, \quad if\ \vartheta \in \Omega_\jmath, \jmath = 1, \ldots, m, \\ \\ \Psi_\jmath(\vartheta, x(\vartheta_\jmath^-)) + \upsilon_\jmath, \qquad if\ \vartheta \in \tilde{\Omega}_\jmath, \jmath = 1, \ldots, m, \\ \\ y(\vartheta) = \hbar_1(\vartheta) + \upsilon_{m+1}, \quad \vartheta \in [-\varpi, 0], \\ \\ y(\vartheta) = \hbar_2(\vartheta) + \upsilon_{m+2}, \quad \vartheta \in [\varkappa, \varkappa + \tilde{\varpi}], \end{cases}$$

where σ_x be a function satisfying the functional equations

$$\sigma_x(\vartheta) = \varphi(\vartheta, x_n^{\vartheta}(\cdot), \sigma_x(\vartheta)).$$

Hence, for each $\vartheta \in \Omega_\jmath, \jmath = 0, \dots, m$, we have

$$\begin{aligned}\|x(\vartheta) - y(\vartheta)\| &\leq \frac{1}{\Gamma(\zeta)} \int_{\varkappa_\jmath}^{\vartheta_{\jmath+1}} |\vartheta - \varrho|^{\zeta-1} \|\sigma_x(\varrho) - \sigma(\varrho)\| d\varrho + \left({}_0I_{\varkappa}^{\zeta}\|\upsilon(\tau)\|\right) \\ &\leq \epsilon \ell_{\mathrm{Im}} \mathrm{Im}(\vartheta) + \frac{\psi_1}{\Gamma(\zeta)} \|x - y\|_{\mathbb{F}} \int_0^{\varkappa} |\vartheta - \varrho|^{\zeta-1} d\varrho \\ &\leq \epsilon \ell_{\mathrm{Im}} \mathrm{Im}(\vartheta) + \frac{\psi_1 \varkappa^{\zeta}}{\Gamma(\zeta)(1-\psi_2)} \|x - y\|_{\mathbb{F}}.\end{aligned}$$

And for each $\vartheta \in \tilde{\Omega}_\jmath, \jmath = 1, \dots, m$, we have

$$\begin{aligned}\|x(\vartheta) - y(\vartheta)\| &\leq \|\Psi_\jmath(\vartheta, x(\vartheta_\jmath^-)) - \Psi_\jmath(\vartheta, y(\vartheta_\jmath^-))\| + \|\upsilon_\jmath\| \\ &\leq \wp^* \|x(\vartheta) - y(\vartheta)\| + \epsilon\lambda \\ &\leq \wp^* \|x - y\|_{\mathbb{F}} + \epsilon\lambda.\end{aligned}$$

For each $\vartheta \in [-\varpi, 0]$, we have

$$\begin{aligned}\|x(\vartheta) - y(\vartheta)\| &\leq \|\upsilon_{m+1}\| \\ &\leq \epsilon\Delta_1.\end{aligned}$$

And for each $\vartheta \in [\varkappa, \varkappa + \tilde{\varpi}]$, we have

$$\begin{aligned}\|x(\vartheta) - y(\vartheta)\| &\leq \|\upsilon_{m+2}\| \\ &\leq \epsilon\Delta_2.\end{aligned}$$

Thus

$$\|x - y\|_{\mathbb{F}} \leq [\epsilon \ell_{\mathrm{Im}} \mathrm{Im}(\vartheta) + \epsilon\lambda + \epsilon\Delta_1 + \epsilon\Delta_2] + \left[\wp^* + \frac{\psi_1 \varkappa^{\zeta}}{\Gamma(\zeta)(1-\psi_2)}\right] \|x - y\|_{\mathbb{F}}.$$

Then for each $\vartheta \in \Theta$, we have

$$\|x - y\|_{\mathbb{F}} \leq a_{\varphi,\mathrm{Im}} \epsilon(\lambda + \mathrm{Im}(\vartheta) + \Delta_1 + \Delta_2),$$

where

$$a_{\varphi,\mathrm{Im}} = \frac{1 + \ell_{\mathrm{Im}}}{1 - \left[\wp^* + \frac{\psi_1 \varkappa^{\zeta}}{\Gamma(\zeta)(1-\psi_2)}\right]}.$$

Hence, the problem (1.1)-(1.5) is U-H-R stable with respect to $(\mathrm{Im}, \lambda, \Delta_1, \Delta_2)$. □

Remark 4.3. If the conditions $(Ax1)$-$(Ax2)$ and (3.6) are satisfied, then by Theorem 4.1 and Remark 4.1, it is clear that problem (1.1)-(1.5) is U-H-R stable and G.U-H-R stable. And if $\mathrm{Im}(.) = \lambda = \Delta_1 = \Delta_2 = 1$, then problem (1.1)-(1.5) is also G.U-H stable and U-H stable.

5. An Example

Let

$$\Xi = l^1 = \left\{ \xi = (\xi_1, \xi_2, \ldots, \xi_n, \ldots), \sum_{n=1}^{\infty} |\xi_n| < \infty \right\}$$

be the Banach space with the norm

$$\|\xi\| = \sum_{n=1}^{\infty} |\xi_n|.$$

Consider the following impulsive problem which is an example of our problem (1.1)-(1.5).

$$\left({}^{RC}_{\varkappa_\jmath} D^{\frac{1}{2}}_{\vartheta_{\jmath+1}} y \right)(\vartheta) = \varphi\left(\vartheta, y^{\vartheta}(\cdot), \left({}^{RC}_{\varkappa_\jmath} D^{\zeta}_{\vartheta_{\jmath+1}} y \right)(\vartheta) \right); \ \vartheta \in \Omega_0 \cup \Omega_1, \tag{5.1}$$

$$y(\vartheta) = \Psi_1(\vartheta, y(\vartheta_1^-)) \in \tilde{\Omega}_1, \tag{5.2}$$

$$y(0) + y(\varkappa) = 0, \tag{5.3}$$

$$y(\vartheta) = \hbar_1(\vartheta), \quad \vartheta \in [-\pi, 0], \ \varpi > 0, \tag{5.4}$$

$$y(\vartheta) = \hbar_2(\vartheta), \quad \vartheta \in [\pi, 2\pi], \ \tilde{\varpi} > 0, \tag{5.5}$$

where $\Omega_0 = (0, 2]$, $\Omega_1 = (3, \pi]$, $\tilde{\Omega}_1 = (2, 3]$, $\varkappa_0 = 0$, $\vartheta_1 = e$ and $\varkappa_1 = 3$, with $\zeta = \frac{1}{2}$, $\jmath \in \{0, 1\}$, $\delta_1 = \delta_2 = 1$, $\delta_3 = 0$ and $\varpi = \tilde{\varpi} = \pi$, and

$$y = (y_1, y_2, \ldots, y_n, \ldots),$$

$$\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n, \ldots),$$

$${}^{RC}_{\varkappa_\jmath} D^{\frac{1}{2}}_{\vartheta_{\jmath+1}} y = \left({}^{RC}_{\varkappa_\jmath} D^{\frac{1}{2}}_{\vartheta_{\jmath+1}} y_1, {}^{RC}_{\varkappa_\jmath} D^{\frac{1}{2}}_{\vartheta_{\jmath+1}} y_2, \ldots, {}^{RC}_{\varkappa_\jmath} D^{\frac{1}{2}}_{\vartheta_{\jmath+1}} y_n, \ldots \right).$$

Set

$$\varphi(\vartheta, y_1, y_2) = \frac{1 + 2|\cos(\vartheta)| + \|y_1\|_{[-\varpi, \tilde{\varpi}]} + \|y_2(\vartheta)\|}{215 + 323e^{\vartheta}}, \ \vartheta \in \Omega_0 \cup \Omega_1,$$

and

$$\Psi_1(\vartheta, y_2(\vartheta_1^-)) = \frac{|\sin(\vartheta)| + \|y_2(\vartheta)\|}{312e^{\vartheta}},$$

where $y_1 \in PC([-\pi, \pi], \Xi), \ y_2 \in \Xi$.

Clearly, the function φ is continuous. Hence the condition (*Ax1*) is satisfied.

For each $x_1, y_1 \in PC([-\pi, \pi], \Xi)$, $x_2, y_2 \in \Xi$ and $\vartheta \in \Theta$, we have

$$\begin{aligned} \|f(\vartheta, x_1, x_2) - f(\vartheta, y_1, y_2)\| &\le \frac{1}{215 + 323e^{\vartheta}} \left(\|x_1 - y_1\|_{[-\varpi, \tilde{\varpi}]} + \|x_2 - y_2\| \right) \\ &\le \frac{1}{538} \left(\|x_1 - y_1\|_{[-\varpi, \tilde{\varpi}]} + \|x_2 - y_2\| \right), \end{aligned}$$

and

$$\begin{aligned}\|\Psi_1(\vartheta, x_2(\vartheta_1^-)) - \Psi_1(\vartheta, y_2(\vartheta_1^-))\| &\le \frac{\|x_2(\vartheta) - y_2(\vartheta)\|}{312e^{\vartheta}}\\ &\le \frac{1}{312}\|x_2(\vartheta) - y_2(\vartheta)\|,\end{aligned}$$

Hence condition (*Ax2*) is satisfied with $\psi_1 = \psi_2 = \dfrac{1}{538}$ and $\wp^* = \dfrac{1}{312}$.
And, the condition (3.6) of Theorem 3.2 is verified, for

$$\begin{aligned}\beta &= (|\delta_1| + |\delta_2|)\left[\frac{\wp^*}{|\delta_1|} + \frac{\varkappa^{\zeta}\psi_1}{\Gamma(\zeta+1)|\delta_1|(1-\psi_2)}\right]\\ &= \frac{2}{312} + \frac{8}{537}\\ &\approx 0.02130783555\\ &< 1.\end{aligned}$$

Then the problem (5.1)-(5.3) has a unique solution in $PC([0,\pi], \Xi)$.
Hypothesis (*Ax4*) is satisfied with $\lambda = \Delta_1 = \Delta_2 = 1$, $\mathrm{Im}(\vartheta) = 3\sqrt{\pi}$ and $\ell_{\mathrm{Im}} = 4$. Indeed, for each $\vartheta \in \Omega_0 \cup \Omega_1$, we get

$$\begin{aligned}{}_0I_{\pi}^{\zeta}\sqrt{\pi} &= \frac{1}{\Gamma(\zeta)}\int_0^{\pi} |\vartheta - \varrho|^{\zeta-1}\sqrt{\pi}d\varrho\\ &\le \frac{\sqrt{\pi}}{\Gamma(\zeta)}\int_0^{\vartheta} (\vartheta - \varrho)^{\zeta-1}d\varrho + \frac{\sqrt{\pi}}{\Gamma(\zeta)}\int_{\vartheta}^{\pi} (\varrho - \vartheta)^{\zeta-1}d\varrho\\ &\le 16\sqrt{\pi}.\end{aligned}$$

Consequently, Theorem 4.1 implies that the problem (5.1)-(5.3) is U-H-R stable.

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