

## ON NORMAL ALMOST PARACONTACT METRIC MANIFOLDS OF DIMENSION 3

I. Küpeli Erken

**Abstract.** In this study, we make the first contribution to investigating conditions under which normal almost paracontact metric manifold of dimension 3 has cyclic parallel Ricci tensor,  $\eta$ -parallel Ricci tensor, Ricci-semisymmetry and locally  $\tilde{\varphi}$ -symmetry. In the end an example of a 3-dimensional normal almost paracontact metric manifold which is locally  $\tilde{\varphi}$ -symmetric and has cyclic parallel Ricci tensor is presented.

**Keywords:** Normal almost paracontact metric manifolds, curvature tensor, cyclic parallel Ricci tensor, locally  $\varphi$ -symmetric,  $\eta$ -parallel Ricci tensor.

### 1. Introduction

Paracontact metric structures were introduced in [13], as a natural odd-dimensional counterpart to paraHermitian structures, and like contact metric structures correspond to the Hermitian ones. Paracontact metric manifolds  $(M^{2n+1}, \tilde{\varphi}, \xi, \eta, \tilde{g})$  have been studied by many authors in the recent years, particularly since the appearance of [22]. The curvature identities for different classes of almost paracontact metric manifolds were obtained e.g. in [8], [21], [22]. The importance of paracontact geometry, and in particular of para-Sasakian geometry, has been pointed out especially in the last years by several papers highlighting the interplays with the theory of para-Kähler manifolds and its role in pseudo-Riemannian geometry and mathematical physics (cf. e.g. [1],[2],[5],[6],[7],[14]).

Z. Olszak studied  $(M, \varphi, \xi, \eta, g)$  normal almost contact metric manifolds of dimension 3 [17]. He derives certain necessary and sufficient conditions for an almost contact metric structure on manifold to be normal and curvature properties of such structures and normal almost contact metric structures on a manifold of constant curvature are studied. Recently, J. Welyczko studied curvature and torsion of Frenet Legendre curves in 3-dimensional normal almost paracontact metric manifolds [20]. 3-dimensional normal almost contact metric manifolds which are satisfying certain curvature conditions were studied in [9], [10]. Later, C. L. Bejan

and M. Crasmareanu considered second order parallel tensors and Ricci solitons in 3-dimensional normal paracontact geometry [3].

To our knowledge, no effort has been made to investigate conditions under which normal almost paracontact metric manifolds of dimension 3 have cyclic parallel Ricci tensor,  $\eta$ -parallel Ricci tensor, Ricci-semisymmetry and locally  $\tilde{\varphi}$ -symmetry. In the present work we aim to investigate these curvature conditions for 3-dimensional normal almost paracontact metric manifolds.

Our work is organized as follows: Section 1 is the introductory section. Section 2 and Section 3, respectively, contain some basic and preliminary results related to the almost paracontact metric manifold and 3-dimensional normal almost paracontact manifolds. Section 4 is devoted to Ricci-semisymmetry in 3-dimensional normal almost paracontact manifolds and we give some equivalent conditions. In Section 5, we prove that 3-dimensional normal almost paracontact metric manifold which is not  $\beta$ -para-Sasakian manifold satisfying cyclic parallel Ricci tensor is a manifold of negative constant scalar curvature. In Section 6, we show that a 3-dimensional normal almost paracontact metric manifold is locally  $\tilde{\varphi}$ -symmetric if and only if the scalar curvature is constant along leaves of the canonical foliation. In Section 7, we obtain that a 3-dimensional normal almost paracontact metric manifold with  $\eta$ -parallel Ricci tensor is locally  $\tilde{\varphi}$ -symmetric. In the last section, we construct an example to illustrate the results obtained in previous sections.

## 2. Preliminaries

In this section we collect the formulas and results we need on paracontact metric manifolds. All manifolds in this note are assumed to be connected and smooth. We may refer to [13], [22] and references therein for more information about paracontact metric geometry.

An  $(2n+1)$ -dimensional smooth manifold  $M$  is said to have an *almost paracontact structure* if it admits a  $(1, 1)$ -tensor field  $\tilde{\varphi}$ , a vector field  $\xi$  and a 1-form  $\eta$  satisfying the following conditions:

- (i)  $\eta(\xi) = 1$ ,  $\tilde{\varphi}^2 = I - \eta \otimes \xi$ ,
- (ii) the tensor field  $\tilde{\varphi}$  induces an almost paracomplex structure on each fibre of  $\mathcal{D} = \ker(\eta)$ , i.e. the  $\pm 1$ -eigendistributions,  $\mathcal{D}^\pm := \mathcal{D}_{\tilde{\varphi}(\pm 1)}$  of  $\tilde{\varphi}$  have equal dimension  $n$ .

From the definition it follows that  $\tilde{\varphi}\xi = 0$ ,  $\eta \circ \tilde{\varphi} = 0$  and the endomorphism  $\tilde{\varphi}$  has rank  $2n$ . When the tensor field  $N_{\tilde{\varphi}} := [\tilde{\varphi}, \tilde{\varphi}] - 2d\eta \otimes \xi$  vanishes identically the almost paracontact manifold is said to be *normal*. If an almost paracontact manifold admits a pseudo-Riemannian metric  $\tilde{g}$  such that

$$(2.1) \quad \tilde{g}(\tilde{\varphi}X, \tilde{\varphi}Y) = -\tilde{g}(X, Y) + \eta(X)\eta(Y),$$

for all  $X, Y \in \Gamma(TM)$ , then we say that  $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$  is an *almost paracontact metric manifold*. Notice that any such a pseudo-Riemannian metric is necessarily of signature  $(n + 1, n)$ . For an almost paracontact metric manifold, there always exists an orthogonal basis  $\{X_1, \dots, X_n, Y_1, \dots, Y_n, \xi\}$  such that  $\tilde{g}(X_i, X_j) = \delta_{ij}$ ,  $\tilde{g}(Y_i, Y_j) = -\delta_{ij}$  and  $Y_i = \tilde{\varphi}X_i$ , for any  $i, j \in \{1, \dots, n\}$ . Such a basis is called a  $\tilde{\varphi}$ -basis.

If in addition  $F(X, Y) = d\eta(X, Y) = \tilde{g}(X, \tilde{\varphi}Y)$  for all vector fields  $X, Y$  on  $M$ ,  $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$  is said to be a *paracontact metric manifold* where  $F$  is the fundamental form. In a paracontact metric manifold one defines a symmetric, trace-free operator  $\tilde{h} := \frac{1}{2}\mathcal{L}_\xi\tilde{\varphi}$ . It is known [22] that  $\tilde{h}$  anti-commutes with  $\tilde{\varphi}$  and satisfies  $h\xi = 0$ ,  $\text{tr}\tilde{h} = \text{tr}\tilde{h}\tilde{\varphi} = 0$  and

$$(2.2) \quad \tilde{\nabla}\xi = -\tilde{\varphi} + \tilde{\varphi}\tilde{h},$$

where  $\tilde{\nabla}$  is the Levi-Civita connection of the pseudo-Riemannian manifold  $(M, \tilde{g})$ .

Moreover  $\tilde{h} \equiv 0$  if and only if  $\xi$  is a Killing vector field and in this case  $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$  is said to be a *K-paracontact manifold*. A normal paracontact metric manifold is called a *para-Sasakian manifold*. Also in this context the para-Sasakian condition implies the *K-paracontact* condition and the converse holds only in dimension 3. We also recall that any para-Sasakian manifold satisfies

$$(2.3) \quad \tilde{R}(X, Y)\xi = -(\eta(Y)X - \eta(X)Y).$$

Similarly to the class of almost contact metric manifolds [4], a normal almost paracontact metric manifold will be called para-Sasakian if  $F = d\eta$  [11] and quasi-para-Sasakian if  $dF = 0$ . Obviously, the class of para-Sasakian manifolds is contained in the class of quasi-para-Sasakian manifolds. The converse does not hold in general. A paracontact metric manifold will be called paracosymplectic if  $dF = 0, d\eta = 0$  [8].

### 3. Normal almost paracontact metric manifolds

**Proposition 3.1.** [20] *A  $(2n + 1)$ -dimensional almost paracontact metric manifold is normal if and only if*

$$(3.1) \quad \tilde{\varphi}(\tilde{\nabla}_X\tilde{\varphi})Y - (\tilde{\nabla}_{\tilde{\varphi}X}\tilde{\varphi})Y + (\tilde{\nabla}_X\eta)(Y)\xi = 0,$$

$\tilde{\nabla}$  being the Levi-Civita connection.

**Proposition 3.2.** [20] *For a 3-dimensional almost paracontact metric manifold  $M$  it holds*

$$(3.2) \quad (\tilde{\nabla}_X\tilde{\varphi})Y = \tilde{g}(\tilde{\varphi}\tilde{\nabla}_X\xi, Y)\xi - \eta(Y)\tilde{\varphi}\tilde{\nabla}_X\xi.$$

**Proposition 3.3.** [20] *For a 3-dimensional almost paracontact metric manifold  $M$  the following three conditions are mutually equivalent*

- (a)  $M$  is normal,  
 (b) there exist functions  $\alpha, \beta$  on  $M$  such that

$$(3.3) \quad (\tilde{\nabla}_X \tilde{\varphi})Y = \beta(\tilde{g}(X, Y)\xi - \eta(Y)X) + \alpha(\tilde{g}(\tilde{\varphi}X, Y)\xi - \eta(Y)\tilde{\varphi}X),$$

- (c) there exist functions  $\alpha, \beta$  on  $M$  such that

$$(3.4) \quad \tilde{\nabla}_X \xi = \alpha(X - \eta(X)\xi) + \beta\tilde{\varphi}X.$$

**Corollary 3.1.** [20] *The functions  $\alpha, \beta$  realizing (3.3) as well as (3.4) are given by*

$$(3.5) \quad 2\alpha = \text{Trace}\{X \longrightarrow \tilde{\nabla}_X \xi\}, \quad 2\beta = \text{Trace}\{X \longrightarrow \tilde{\varphi}\tilde{\nabla}_X \xi\}.$$

**Proposition 3.4.** [20] *For a 3-dimensional almost paracontact metric manifold  $M$ , the following three conditions are mutually equivalent*

- (a)  $M$  is quasi-para-Sasakian,  
 (b) there exists a function  $\beta$  on  $M$  such that

$$(3.6) \quad (\tilde{\nabla}_X \tilde{\varphi})Y = \beta(\tilde{g}(X, Y)\xi - \eta(Y)X),$$

- (c) there exists a function  $\beta$  on  $M$  such that

$$(3.7) \quad \tilde{\nabla}_X \xi = \beta\tilde{\varphi}X.$$

A 3-dimensional normal almost paracontact metric manifold is said to be

- paracosymplectic if  $\alpha = \beta = 0$  [8],
- quasi-para-Sasakian if and only if  $\alpha = 0$  and  $\beta \neq 0$  [11], [20],
- $\beta$ -para-Sasakian if and only if  $\alpha = 0$  and  $\beta \neq 0$  and  $\beta$  is constant, in particular, para-Sasakian if  $\beta = -1$  [20], [22],
- $\alpha$ -para-Kenmotsu if  $\alpha \neq 0$  and  $\alpha$  is constant and  $\beta = 0$ .

Now, after differentiating (3.4) covariantly and using (3.3) we find

$$\begin{aligned} \tilde{\nabla}_X \tilde{\nabla}_Y \xi &= \alpha(\tilde{\nabla}_X Y - \eta(\tilde{\nabla}_X Y)\xi) + (\alpha^2 - \beta^2)\tilde{g}(\tilde{\varphi}X, \tilde{\varphi}Y)\xi + \beta\tilde{\varphi}\tilde{\nabla}_X Y + X(\alpha)\tilde{\varphi}^2 Y \\ &\quad + X(\beta)\tilde{\varphi}Y - (\alpha^2 + \beta^2)\eta(Y)\tilde{\varphi}^2 X - 2\alpha\beta\eta(Y)\tilde{\varphi}X \end{aligned}$$

Therefore, for the curvature transformation  $R_{XY} = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]}$  we obtain

$$(3.8) \quad \begin{aligned} \tilde{R}(X, Y)\xi &= -\{Y(\alpha) + (\alpha^2 + \beta^2)\eta(Y)\}\tilde{\varphi}^2 X + \{X(\alpha) + (\alpha^2 + \beta^2)\eta(X)\}\tilde{\varphi}^2 Y \\ &\quad - \{Y(\beta) + 2\alpha\beta\eta(Y)\}\tilde{\varphi}X + \{X(\beta) + 2\alpha\beta\eta(X)\}\tilde{\varphi}Y \end{aligned}$$

and

$$(3.9) \quad \tilde{S}(Y, \xi) = -Y(\alpha) + \tilde{\varphi}Y(\beta) - \{\xi(\alpha) + 2(\alpha^2 + \beta^2)\}\eta(Y),$$

where  $\tilde{R}$  denotes the curvature tensor and  $\tilde{S}$  is the Ricci tensor.

From (3.8), we obtain

$$\tilde{R}(\xi, Y, Z, \xi) = (\xi(\alpha) + \alpha^2 + \beta^2)\tilde{g}(\tilde{\varphi}Y, \tilde{\varphi}Z) - (\xi(\beta) + 2\alpha\beta)\tilde{g}(\tilde{\varphi}Y, Z),$$

where  $\tilde{R}(X, Y, Z, W) = \tilde{g}(\tilde{R}(X, Y)Z, W)$ . By Bianchi identity the last equation follows that

$$(3.10) \quad \tilde{R}(\xi, Y, Z, \xi) = (\xi(\alpha) + \alpha^2 + \beta^2)\tilde{g}(\tilde{\varphi}Y, \tilde{\varphi}Z),$$

$$(3.11) \quad \xi(\beta) + 2\alpha\beta = 0.$$

Next, we recall that the curvature tensor of a 3-dimensional pseudo-Riemannian manifold satisfies

$$(3.12) \quad \begin{aligned} \tilde{R}(X, Y, Z, W) &= \tilde{g}(X, W)\tilde{S}(Y, Z) - \tilde{g}(X, Z)\tilde{S}(Y, W) + \tilde{g}(Y, Z)\tilde{S}(X, W) \\ &\quad - \tilde{g}(Y, W)\tilde{S}(X, Z) - \frac{1}{2}\tau\{\tilde{g}(X, W)\tilde{g}(Y, Z) - \tilde{g}(X, Z)\tilde{g}(Y, W)\}, \end{aligned}$$

where  $\tau$  is the scalar curvature.

In order to compute  $\tilde{S}(Y, Z)$  we will use (3.9), (3.10) and (3.12).

$$(3.13) \quad \begin{aligned} \tilde{S}(Y, Z) &= -(\xi(\alpha) + \alpha^2 + \beta^2 + \frac{1}{2}\tau)\tilde{g}(\tilde{\varphi}Y, \tilde{\varphi}Z) + \eta(Z)(\tilde{\varphi}Y(\beta) - Y(\alpha)) \\ &\quad + \eta(Y)(\tilde{\varphi}Z(\beta) - Z(\alpha)) - 2(\alpha^2 + \beta^2)\eta(Y)\eta(Z). \end{aligned}$$

Combining the above formula with (3.12), we obtain

$$(3.14) \quad \begin{aligned} \tilde{R}(X, Y)Z &= (2(\alpha^2 + \beta^2) + \frac{1}{2}\tau)(\tilde{g}(Y, Z)X - \tilde{g}(X, Z)Y) \\ &\quad - (3(\alpha^2 + \beta^2) + \frac{1}{2}\tau)(\tilde{g}(Y, Z)\eta(X)\xi - \tilde{g}(X, Z)\eta(Y)\xi) \\ &\quad + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y, \end{aligned}$$

for  $\alpha = \text{constant}$  and  $\beta = \text{constant}$ .

From (3.14), we can give following:

**Theorem 3.1.** *Let  $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$  be a 3-dimensional normal almost paracontact metric manifold of constant curvature  $\tau$  then  $\tau = -6(\alpha^2 + \beta^2)$ .*

From now on, we accept  $\alpha, \beta$  as constants.

#### 4. Ricci-semisymmetry in 3-dimensional normal almost paracontact manifolds

A semi-Riemannian manifold  $(M, \tilde{g})$ ,  $n \geq 3$ , is said to be *Ricci-semisymmetric* if on  $M$  we have

$$(4.1) \quad \tilde{R}(X, Y).\tilde{S} = 0,$$

for any  $X, Y \in \chi(M)$ ,  $\tilde{R}$  denotes the curvature tensor and  $\tilde{S}$  is the Ricci tensor of type  $(0, 2)$  of the manifold.

From (4.1) we get

$$\tilde{S}(\tilde{R}(X, Y)U, V) + \tilde{S}(U, \tilde{R}(X, Y)V) = 0.$$

Replacing  $X$  and  $U$  by  $\xi$  in the last equation, we obtain

$$(4.2) \quad \tilde{S}(\tilde{R}(\xi, Y)\xi, V) + \tilde{S}(\xi, \tilde{R}(\xi, Y)V) = 0.$$

By virtue of (3.8) and (3.9), we have

$$(4.3) \quad \tilde{S}(\xi, \tilde{R}(\xi, Y)V) = 2(\alpha^2 + \beta^2)((\alpha^2 + \beta^2)\tilde{g}(\tilde{\varphi}^2 Y, V) + 2\alpha\beta\tilde{g}(\tilde{\varphi} Y, V)).$$

Using (3.13), one can compute

$$(4.4) \quad \begin{aligned} \tilde{S}(\tilde{R}(\xi, Y)\xi, V) &= (\alpha^2 + \beta^2)(-\alpha^2 + \beta^2 + \frac{\tau}{2})\tilde{g}(\tilde{\varphi} Y, \tilde{\varphi} V) \\ &\quad + 2\alpha\beta(-\alpha^2 + \beta^2 + \frac{\tau}{2})\tilde{g}(Y, \tilde{\varphi} V). \end{aligned}$$

Applying these equations in (4.2), we get

$$(4.5) \quad (3(\alpha^2 + \beta^2) + \frac{\tau}{2})(-\alpha^2 + \beta^2)\tilde{g}(\tilde{\varphi} Y, \tilde{\varphi} V) + 2\alpha\beta\tilde{g}(\tilde{\varphi} Y, V) = 0.$$

Let us assume that  $M$  is non-paracosymplectic. From (4.5), precisely following cases occurs.

Case 1:  $(3(\alpha^2 + \beta^2) + \frac{\tau}{2}) = 0,$

Case 2:  $(-\alpha^2 + \beta^2)\tilde{g}(\tilde{\varphi} Y, \tilde{\varphi} V) + 2\alpha\beta\tilde{g}(\tilde{\varphi} Y, V) = 0,$

Case 3:  $(3(\alpha^2 + \beta^2) + \frac{\tau}{2}) = 0$  and  $(-\alpha^2 + \beta^2)\tilde{g}(\tilde{\varphi} Y, \tilde{\varphi} V) + 2\alpha\beta\tilde{g}(\tilde{\varphi} Y, V) = 0.$

We now check, case by case. From Case 1, we get  $\tau = -6(\alpha^2 + \beta^2)$ . Using (2.1) and (3.13), we obtain  $\tilde{S}(Y, Z) = -2(\alpha^2 + \beta^2)\tilde{g}(Y, Z)$  which implies that  $M$  is an Einstein manifold with constant negative curvature  $-6(\alpha^2 + \beta^2)$ . Case 2 and Case 3 hold if and only if  $\alpha = \beta = 0$ . But this contradicts with the type of manifold. So we omit Case 2 and Case 3.

Hence we can give the following theorem.

**Theorem 4.1.** *Let  $M$  be a 3-dimensional non-paracosymplectic normal almost paracontact metric manifold. Then the following conditions are equivalent.*

- (i)  $M$  is an Einstein manifold,
- (ii) The Ricci tensor  $\tilde{S}$  of  $M$  is parallel, i.e.,  $\tilde{\nabla}\tilde{S} = 0$
- (iii)  $M$  is Ricci-semisymmetric.

**5. Cyclic parallel Ricci tensors in 3-dimensional normal almost paracontact manifolds**

The Ricci tensor  $\tilde{S}$  of a semi-Riemannian manifold  $M$  is said to have *cyclic-parallel* Ricci tensor if  $C\tilde{S} = 0$ , namely

$$(5.1) \quad (\tilde{\nabla}_X \tilde{S})(Y, Z) + (\tilde{\nabla}_Y \tilde{S})(Z, X) + (\tilde{\nabla}_Z \tilde{S})(X, Y) = 0,$$

for all vector fields  $X, Y, Z$  [12].

From (5.1), we get  $\tau$  is constant. By (2.1) and (3.13), we get

$$(5.2) \quad (\tilde{\nabla}_X \tilde{S})(Y, Z) = -(3(\alpha^2 + \beta^2) + \frac{1}{2}\tau)(\eta(Y)(\tilde{\nabla}_X \eta)Z + \eta(Z)(\tilde{\nabla}_X \eta)Y).$$

Combining (5.1) and (5.2), it follows that

$$(5.3) \quad - (3(\alpha^2 + \beta^2) + \frac{1}{2}\tau) \begin{pmatrix} \eta(Y)(\tilde{\nabla}_X \eta)Z + \eta(Z)(\tilde{\nabla}_X \eta)Y \\ + \eta(Z)(\tilde{\nabla}_Y \eta)X + \eta(X)(\tilde{\nabla}_Y \eta)Z \\ + \eta(X)(\tilde{\nabla}_Z \eta)Y + \eta(Y)(\tilde{\nabla}_Z \eta)X \end{pmatrix} = 0.$$

We take into account  $\tilde{\varphi}$ -basis and put  $Y = Z = e_i$  in (5.3) and by the help of (3.4), we obtain

$$(3(\alpha^2 + \beta^2) + \frac{1}{2}\tau)\alpha\eta(X) = 0.$$

Then the following cases occurs:

(i)  $M$  is  $\beta$ -para-Sasakian if  $\alpha = 0$ . In particular, for  $\beta = -1$  the manifold is para-Sasakian.

(ii)  $M$  has negative scalar curvature  $(-6(\alpha^2 + \beta^2))$  if  $(3(\alpha^2 + \beta^2) + \frac{1}{2}\tau)\eta(X) = 0$ .

Conversely, if  $\tau = -6(\alpha^2 + \beta^2)$ , by virtue of (5.2) we obtain  $(\tilde{\nabla}_X \tilde{S})(Y, Z) = 0$ , namely  $M$  has cyclic-parallel Ricci tensor.

So, we can give the following lemma:

**Lemma 5.1.** *A 3-dimensional normal almost paracontact metric manifold which is not a  $\beta$ -para-Sasakian satisfies cyclic parallel Ricci tensor if and only if  $\tau = -6(\alpha^2 + \beta^2)$ .*

From (3.14), we have

**Theorem 5.1.** *If a 3-dimensional normal almost paracontact metric manifold which is not a  $\beta$ -para-Sasakian satisfying cyclic parallel Ricci tensor, then the manifold is a manifold of constant curvature  $\tau = -6(\alpha^2 + \beta^2)$ .*

### 6. Locally $\tilde{\varphi}$ -symmetry in 3-dimensional normal almost paracontact manifolds

In  $K$ -contact and Sasakian geometry, local symmetry is a very strong condition: a locally symmetric  $K$ -contact manifold necessarily has a constant curvature equal to ([16], [19]). In [18], Takahashi introduced a weaker condition for a Sasakian manifold. He calls a Sasakian space *locally  $\tilde{\varphi}$ -symmetric* if its Riemannian curvature tensor  $\tilde{R}$  satisfies the condition

$$\varphi^2((\nabla_X R)(Y, Z, W)) = 0,$$

for all vector fields  $X, Y, Z$  and  $W$  orthogonal to the characteristic vector field  $\xi$ , where  $\nabla$  denotes the Levi-Civita connection.

The definition of locally  $\tilde{\varphi}$ -symmetry can be adapted to the definition given [18] for almost paracontact metric manifolds.

**Theorem 6.1.** *A 3-dimensional normal almost paracontact metric manifold is locally  $\tilde{\varphi}$ -symmetric if and only if scalar curvature  $\tau$  is constant along the leaves of the canonical foliation.*

*Proof.* Let us consider a 3-dimensional normal almost paracontact metric manifold. If we differentiate (3.14) covariantly with respect to  $W$ , we obtain

$$\begin{aligned} (\tilde{\nabla}_W \tilde{R})(X, Y)Z &= \frac{d\tau(W)}{2}(\tilde{g}(Y, Z)X - \tilde{g}(X, Z)Y) \\ &+ \tilde{g}(X, Z) \left[ \frac{d\tau(W)}{2} \eta(Y)\xi + (3(\alpha^2 + \beta^2) + \frac{1}{2}\tau)((\tilde{\nabla}_W \eta)(Y)\xi + \eta(Y)\tilde{\nabla}_W \xi) \right] \\ (6.1) \quad &- \tilde{g}(Y, Z) \left[ \frac{d\tau(W)}{2} \eta(X)\xi + (3(\alpha^2 + \beta^2) + \frac{1}{2}\tau)((\tilde{\nabla}_W \eta)(X)\xi + \eta(X)\tilde{\nabla}_W \xi) \right] \\ &- \left[ \frac{d\tau(W)}{2} \eta(Y)\eta(Z)X + (3(\alpha^2 + \beta^2) + \frac{1}{2}\tau)((\tilde{\nabla}_W \eta)(Y)\eta(Z)X \right. \\ &\quad \left. + \eta(Y)(\tilde{\nabla}_W \eta)(Z)X + \eta(Y)\eta(Z)\tilde{\nabla}_W X \right] \\ &+ \left[ \frac{d\tau(W)}{2} \eta(X)\eta(Z)Y + (3(\alpha^2 + \beta^2) + \frac{1}{2}\tau)((\tilde{\nabla}_W \eta)(X)\eta(Z)Y \right. \\ &\quad \left. + \eta(X)(\tilde{\nabla}_W \eta)(Z)Y + \eta(X)\eta(Z)\tilde{\nabla}_W Y \right]. \end{aligned}$$

For the locally  $\tilde{\varphi}$ -symmetry condition, we take  $X, Y, Z$  and  $W$  orthogonal to  $\xi$ . Then (6.1) returns to

$$(6.2) \quad \tilde{\varphi}^2((\tilde{\nabla}_W \tilde{R})(X, Y)Z) = \frac{d\tau(W)}{2}(\tilde{g}(Y, Z)X - \tilde{g}(X, Z)Y).$$

So, (6.2) completes the proof.  $\square$

Theorem 6.1 leads to following corollary:

**Corollary 6.1.** *A 3-dimensional para-Sasakian, para-Kenmotsu or paracosymplectic manifold is locally  $\tilde{\varphi}$ -symmetric if and only if scalar curvature  $\tau$  is constant along leaves of the canonical foliation.*

### 7. $\eta$ -parallel Ricci tensors in 3-dimensional normal almost paracontact manifolds

The Ricci tensor  $\tilde{S}$  of an almost paracontact metric manifold is said to have  $\eta$ -parallel Ricci tensor if

$$(7.1) \quad (\tilde{\nabla}_X \tilde{S})(\tilde{\varphi} Y, \tilde{\varphi} Z) = 0,$$

for all vector fields  $X, Y, Z$  [15].

**Theorem 7.1.** *A 3-dimensional normal almost paracontact metric manifold with  $\eta$ -parallel Ricci tensor has constant scalar curvature.*

*Proof.* Let us consider a 3-dimensional normal almost paracontact metric manifold. If we put  $Y = \tilde{\varphi} Y, Z = \tilde{\varphi} Z$  in (3.13), we obtain

$$(7.2) \quad \tilde{S}(\tilde{\varphi} Y, \tilde{\varphi} Z) = -(\alpha^2 + \beta^2 + \frac{1}{2}\tau)(\tilde{g}(Y, Z) - \eta(Y)\eta(Z)).$$

After differentiating (7.2) covariantly, we get

$$\begin{aligned} (\tilde{\nabla}_X \tilde{S})(\tilde{\varphi} Y, \tilde{\varphi} Z) &= -\frac{d\tau(X)}{2}(\tilde{g}(Y, Z) - \eta(Y)\eta(Z)) \\ &\quad + (\alpha^2 + \beta^2 + \frac{1}{2}\tau)((\tilde{\nabla}_X \eta)(Y)\eta(Z) + \eta(Y)(\tilde{\nabla}_X \eta)(Z)). \end{aligned}$$

If the Ricci tensor is  $\eta$ -parallel, from the above equation we have

$$(7.3) \quad -\frac{d\tau(X)}{2}(\tilde{g}(Y, Z) - \eta(Y)\eta(Z)) + (\alpha^2 + \beta^2 + \frac{1}{2}\tau)((\tilde{\nabla}_X \eta)(Y)\eta(Z) + \eta(Y)(\tilde{\nabla}_X \eta)(Z))$$

We take into account  $\tilde{\varphi}$ -basis and put  $Y = Z = e_i$  in (7.3), we obtain

$$(7.4) \quad -d\tau(X) + 2(\alpha^2 + \beta^2 + \frac{1}{2}\tau)(\tilde{\nabla}_X \eta)(\xi) = 0.$$

Using (3.4) it can be easily seen that  $(\tilde{\nabla}_X \eta)(\xi) = 0$ . Hence (7.4) implies that

$$d\tau(X) = 0.$$

Thus the proof of the theorem is completed.  $\square$

The following result is a consequence from Theorem 6.1 and Theorem 7.1.

**Corollary 7.1.** *A 3-dimensional normal almost paracontact metric manifold with  $\eta$ -parallel Ricci tensor is locally  $\tilde{\varphi}$ -symmetric.*

### 8. Example

Now, we give an example of a 3-dimensional normal almost paracontact metric manifold.

**Example 8.1.** We consider the 3-dimensional manifold

$$M = \{(x, y, z) \in \mathbb{R}^3, z \neq 0\}$$

and the vector fields

$$X = \frac{\partial}{\partial x}, \quad \tilde{\varphi}X = \frac{\partial}{\partial y}, \quad \xi = (x+2y)\frac{\partial}{\partial x} + (2x+y)\frac{\partial}{\partial y} + \frac{\partial}{\partial z}.$$

The 1-form  $\eta = dz$  defines an almost paracontact structure on  $M$  with characteristic vector field  $\xi = (x+2y)\frac{\partial}{\partial x} + (2x+y)\frac{\partial}{\partial y} + \frac{\partial}{\partial z}$ . Let  $\tilde{g}$ ,  $\tilde{\varphi}$  be the pseudo-Riemannian metric and the  $(1, 1)$ -tensor field given by

$$\tilde{g} = \begin{pmatrix} 1 & 0 & -\frac{1}{2}(x+2y) \\ 0 & -1 & \frac{1}{2}(2x+y) \\ -\frac{1}{2}(x+2y) & \frac{1}{2}(2x+y) & 1 - (2x+y)^2 + (x+2y)^2 \end{pmatrix},$$

$$\tilde{\varphi} = \begin{pmatrix} 0 & 1 & -(2x+y) \\ 1 & 0 & -(x+2y) \\ 0 & 0 & 0 \end{pmatrix},$$

with respect to the basis  $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ .

Using (3.4) we have

$$\begin{aligned} \tilde{\nabla}_X X &= -\xi, & \tilde{\nabla}_{\tilde{\varphi}X} X &= 0, & \tilde{\nabla}_\xi X &= -2\tilde{\varphi}X, \\ \tilde{\nabla}_X \tilde{\varphi}X &= 0, & \tilde{\nabla}_{\tilde{\varphi}X} \tilde{\varphi}X &= \xi, & \tilde{\nabla}_\xi \tilde{\varphi}X &= -2X, \\ \tilde{\nabla}_X \xi &= X, & \tilde{\nabla}_{\tilde{\varphi}X} \xi &= \tilde{\varphi}X, & \tilde{\nabla}_\xi \xi &= 0. \end{aligned}$$

for  $\alpha = 1$  and  $\beta = 0$ . Hence the manifold is a para-Kenmotsu manifold. One can easily compute,

$$(8.1) \quad \begin{aligned} \tilde{R}(X, \tilde{\varphi}X)\xi &= 0, & \tilde{R}(\tilde{\varphi}X, \xi)\xi &= -\tilde{\varphi}X, & \tilde{R}(X, \xi)\xi &= -X, \\ \tilde{R}(X, \tilde{\varphi}X)\tilde{\varphi}X &= X, & \tilde{R}(\tilde{\varphi}X, \xi)\tilde{\varphi}X &= -\xi, & \tilde{R}(X, \xi)\tilde{\varphi}X &= 0, \\ \tilde{R}(X, \tilde{\varphi}X)X &= \tilde{\varphi}X, & \tilde{R}(\tilde{\varphi}X, \xi)X &= 0, & \tilde{R}(X, \xi)X &= \xi. \end{aligned}$$

We have the constant scalar curvature as follows,

$$\tau = S(X, X) - S(\tilde{\varphi}X, \tilde{\varphi}X) + S(\xi, \xi) = -6.$$

Hence,  $M$  satisfies cyclic parallel Ricci tensor. The manifold is not a  $\beta$ -para-Sasakian manifold because of  $\alpha \neq 0$ . Namely, Lemma 5.1 holds for  $\alpha = 1, \beta = 0$  ( $\tau = -6 = -6(\alpha^2 + \beta^2)$ ). By Theorem 6.1,  $M$  is also locally  $\tilde{\varphi}$ -symmetric.

## REFERENCES

1. D.V. ALEKSEEVSKI, V. CORTÉS, A.S. GALAEV, T. LEISTNER, *Cones over pseudo-Riemannian manifolds and their holonomy*, J. Reine Angew. Math. **635** (2009), 23-69.
2. D.V. ALEKSEEVSKI, C. MEDORI, A. TOMASSINI, *Maximally homogeneous para-CR manifolds*, Ann. Glob. Anal. Geom. **30** (2006), 1-27.
3. C. L. BEJAN, M. CRASMAREANU, *Second order parallel tensors and Ricci solitons in 3-dimensional normal paracontact geometry*, Ann. Global Anal. Geom. **46(2)** (2014), 117-127.
4. D. E. BLAIR, *Riemannian Geometry of Contact and Symplectic Manifolds*, Progress in Mathematics Vol. **203**, Birkhäuser, Boston, 2002.
5. B. CAPPELLETTI-MONTANO, I. KUPELI ERKEN, C. MURATHAN, *Nullity conditions in paracontact geometry*, Diff. Geom. Appl. **30** (2012), 665-693.
6. V. CORTÉS, C. MAYER, T. MOHAUPT, F. SAUERESSING, *Special geometry of Euclidean supersymmetry*, 1. Vector multiplets. J. High Energy Phys. (2004) 03:028: 73.
7. V. CORTÉS, M.A. LAWN, L. SCHÄFER, *Affine hyperspheres associated to special para-Kähler manifolds*, Int. J. Geom. Methods Mod. Phys. **3** (2006), 995-1009.
8. P. DACKO, *On almost para-cosymplectic manifolds*, Tsukuba J. Math. **28** (2004), 193-213.
9. U.C. DE, A. YILDIZ, A. F. YALINIZ, *Locally  $\phi$ -symmetric normal almost contact metric manifolds of dimension 3*, Appl. Math. Lett. **22** (2009), 723-727.
10. U.C. DE, A. K. MONDAL, *On 3-dimensional normal almost contact metric manifolds satisfying certain curvature conditions*, Commun. Korean Math. Soc. **24** (2009), 265-275.
11. S. ERDEM, *On almost (para)contact (hyperbolic) metric manifolds and harmonicity of  $(\varphi, \varphi')$ -holomorphic maps between them*, Houston J. Math. **28** (2002), 21-45.
12. A. GRAY, *Einstein-like manifolds which are not Einstein*, Geom. Dedicata **7 (3)** (1978), 259-280.
13. S. KANEYUKI, F. L. WILLIAMS, *Almost paracontact and parahodge structures on manifolds*, Nagoya Math. J. **99** (1985), 173-187.
14. I. KUPELI ERKEN, C. MURATHAN, *A Complete Study of Three-Dimensional Paracontact  $(\kappa, \mu, \nu)$ -spaces*, Submitted. Available in Arxiv:1305.1511 v3 [*math.DG*].
15. M. KON, *Invariant submanifolds in Sasakian manifolds*, Math. Ann. **219** (1976), 227-290.
16. M. OKUMURA, *Some remarks on spaces with a certain contact structure*, Tohoku Math. J. **14** (1962), 135-145.
17. Z. OLSZAK, *Normal almost contact metric manifolds of dimension three*, Ann. Polon. Math. XLVII (1986), 41-50.
18. T. TAKAHASHI, *Sasakian  $\phi$ -symmetric spaces*, Tohoku Math. J. **29** (1977), 91-113.
19. S. TANNO, *Locally symmetric K-contact Riemannian manifolds*, Proc. Japan Acad. **43** (1967), 581-583.
20. J. WELYCZKO, *On Legendre Curves in 3-Dimensional Normal Almost Paracontact Metric Manifolds*, Result. Math. **54** (2009), 377-387.
21. J. WELYCZKO, *On basic curvature identities for almost (para)contact metric manifolds*. Available in Arxiv: 1209.4731v1 [*math.DG*].
22. S. ZAMKOVY, *Canonical connections on paracontact manifolds*, Ann. Glob. Anal. Geom. **36** (2009), 37-60.

Irem Kpeli Erken  
Uludag University  
Faculty of Science  
Department of Mathematics  
16059, Bursa, Turkey  
iremkuveli@uludag.edu.tr