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COMMON FIXED POINT RESULTS USING $(E.A)$ AND CLR-PROPERTIES IN S-METRIC SPACES

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Abstract. In this paper, we prove some common fixed point results for two pairs of weakly compatible mappings satisfying $(E.A)$ property and CLR-property in the framework of S-metric spaces and provide some examples to support the outcomes. We also prove well-posedness of a fixed point problem. Our findings generalize and extend a number of previously published findings.
Keywords: Common fixed point, S-metric space, $(E : A)$-property, CLR-property, weakly compatible condition.

1. Introduction

Banach fixed point theorem ([4]) (or in short BCP) in a complete metric space has been generalized in many spaces. This famous result is stated as follows.

Theorem 1.1. *Let (X, d) be a complete metric space and let $\mathcal{R}: X \to X$ be a self-mapping. If there exists $k \in [0, 1)$ such that*

$$d(\mathcal{R}(x), \mathcal{R}(y)) \leq k\, d(x, y), \tag{1.1}$$

for all $x, y \in X$, then $\mathcal{R}$ has a unique fixed point $z \in X$.

Moreover, for any $u_0 \in X$, the sequence $\{u_n\} \subset X$ defined by $u_{n+1} = \mathcal{R}u_n$, $n \in \mathbb{N}$, is convergent to the fixed point $z \in X$. Inequality (1.1) also implies the continuity of $\mathcal{R}$.

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Over last few decades, a number of generalizations of metric spaces, such as 2-metric spaces, D^*-metric spaces, b-metric spaces and partial metric spaces, have thus appeared in numerous papers. These generalizations were then used to extend the study of fixed point theory. For more discussions of such generalizations, we refer to [3, 5, 6].

In 2006, Mustafa and Sims [13] introduced G-metric spaces as a generalization of metric spaces and proved the existence of fixed points under different contractive conditions.

In 2012, Sedghi et al. [16] generalized the notion of metric by introducing the following concept:

Definition 1.1. ([16]) Let X be a nonempty set and let $S\colon X^3 \to [0,\infty)$ be a function satisfying the following conditions for all u, v, z, $t \in X$:

($S1$) $S(u,v,z) = 0$ if and only if $u = v = z$;

($S2$) $S(u,v,z) \leq S(u,u,t) + S(v,v,t) + S(z,z,t)$.

Then the function S is called an S-metric on X and the pair (X,S) is called an S-metric space (in short SMS).

Some examples of such S-metric spaces are as follows.

Example 1.1. ([16])

(1) Let $X = \mathbb{R}^n$ and $\|\cdot\|$ a norm on X, then $S(u,v,z) = \|v + z - 2u\| + \|v - z\|$ is an S-metric on X.

(2) Let $X = \mathbb{R}^n$ and $\|\cdot\|$ a norm on X, then $S(u,v,z) = \|u - z\| + \|v - z\|$ is an S-metric on X.

Example 1.2. ([17]) Let $X = \mathbb{R}$ be the real line. Then $S(u,v,z) = |u - z| + |v - z|$ for all $u,v,z \in \mathbb{R}$ is an S-metric on X. This S-metric on X is called the usual S-metric on X.

Example 1.3. ([12]) Let X be a non-empty set and d be an ordinary metric on X. Then $S(u,v,z) = d(u,z) + d(v,z)$ for all $u,v,z \in \mathbb{R}$ is an S-metric on X.

Example 1.4. ([19]) Let X be a non-empty set and d_1, d_2 be two ordinary metrics on X. Then $S(u,v,z) = d_1(u,z) + d_2(v,z)$ for all $u,v,z \in X$ is an S-metric on X.

Recently, Sedghi et al. [18] have proved some existence results of the unique common fixed point for a pair of weakly compatible self mappings satisfying some Φ-type contractive conditions in the setting of S-metric spaces.

2. Basic Properties and Auxiliary Results of an S-Metric Space

We need the following definitions and lemmas in the sequel.

Definition 2.1. Let (X, S) be an S-metric space. For $r > 0$ and $x \in X$ we define the open ball $B_S(x, r)$ and closed ball $B_S[x, r]$ with center x and radius r as follows, respectively:

$$B_S(x, r) = \{y \in X : S(y, y, x) < r\},$$

$$B_S[x, r] = \{y \in X : S(y, y, x) \leq r\}.$$

Example 2.1. ([17]) Let $X = \mathbb{R}$. Denote by $S(x, y, z) = |y + z - 2x| + |y - z|$ for all $x, y, z \in \mathbb{R}$. Then

$$\begin{aligned} B_S(1, 2) &= \{y \in \mathbb{R} : S(y, y, 1) < 2\} = \{y \in \mathbb{R} : |y - 1| < 1\} \\ &= \{y \in \mathbb{R} : 0 < y < 2\} = (0, 2), \end{aligned}$$

and

$$\begin{aligned} B_S[2, 4] &= \{y \in \mathbb{R} : S(y, y, 2) \leq 4\} = \{y \in \mathbb{R} : |y - 2| \leq 2\} \\ &= \{y \in \mathbb{R} : 0 \leq y \leq 4\} = [0, 4]. \end{aligned}$$

Definition 2.2. ([16], [17]) Let (X, S) be an S-metric space and $A \subset X$.

- The subset A is said to be an open subset of X, if for every $x \in A$ there exists $r > 0$ such that $B_S(x, r) \subset A$.
- A sequence $\{t_n\}$ in X converges to $t \in X$ if $S(t_n, t_n, t) \to 0$ as $n \to \infty$, that is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $S(t_n, t_n, t) < \varepsilon$. We denote this by $\lim_{n\to\infty} t_n = t$ or $t_n \to t$ as $n \to \infty$.
- A sequence $\{t_n\}$ in X is called a Cauchy sequence if $S(t_n, t_n, t_m) \to 0$ as $n, m \to \infty$, that is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$ we have $S(t_n, t_n, t_m) < \varepsilon$.
- The S-metric space (X, S) is called complete if every Cauchy sequence in X is convergent in X.
- Let τ be the set of all $A \subset X$ with the property that for each $x \in A$ and there exists $r > 0$ such that $B_S(x, r) \subset A$. Then τ is a topology on X (induced by the S-metric space).
- A nonempty subset A of X is S-closed if closure of A coincides with A.

Lemma 2.1. *([16], Lemma 2.5) Let (X, S) be an S-metric space. Then, we have $S(u, u, v) = S(v, v, u)$ for all $u, v \in X$.*

Lemma 2.2. *([16], Lemma 2.12) Let (X, S) be an S-metric space. If $t_n \to t$ and $u_n \to u$ as $n \to \infty$ then $S(t_n, t_n, u_n) \to S(t, t, u)$ as $n \to \infty$.*

Lemma 2.3. *([7], Lemma 8) Let (X, S) be an S-metric space and A is a nonempty subset of X. Then A is S-closed if and only if for any sequence $\{t_n\}$ in A such that $t_n \to t$ as $n \to \infty$, then $t \in A$.*

Lemma 2.4. *([16]) Let (X, S) be an S-metric space. If $r > 0$ and $x \in X$, then the ball $B_S(x, r)$ is an open subset of X.*

Lemma 2.5. *([17]) The limit of a sequence $\{t_n\}$ in an S-metric space (X, S) is unique.*

Lemma 2.6. *([16]) Let (X, S) be an S-metric space. Then any convergent sequence $\{t_n\}$ in X is Cauchy.*

In the following lemma we see the relationship between a metric and S-metric.

Lemma 2.7. *([8]) Let (X, d) be a metric space. Then the following properties are satisfied:*

(Ω_1) *$S_d(x, y, z) = d(x, z) + d(y, z)$ for all $x, y, z \in X$ is an S-metric on X.*

(Ω_2) *$x_n \to x$ in (X, d) if and only if $x_n \to x$ in (X, S_d).*

(Ω_3) *$\{x_n\}$ is Cauchy in (X, d) if and only if $\{x_n\}$ is Cauchy in (X, S_d).*

(Ω_4) *(X, d) is complete if and only if (X, S_d) is complete.*

We call the function S_d defined in Lemma 2.7 (1) as the S-metric generated by the metric d. An example of an S-metric which is not generated by any metric can be found in [8, 14].

Definition 2.3. ([16]) Let (X, S) be an S-metric space. A mapping $\mathcal{T}: X \to X$ is said to be a contraction if there exists a constant $0 \leq L < 1$ such that

$$S(\mathcal{T}u, \mathcal{T}v, \mathcal{T}z) \leq L\, S(u, v, z) \tag{2.1}$$

for all $u, v, z \in X$.

Remark 2.1. If the S-metric space (X, S) is complete then the mapping defined as above has a unique fixed point (see [16], Theorem 3.1).

Definition 2.4. ([16]) Let (X, S) and (Y, S') be two S-metric spaces. A function $g: X \to Y$ is said to be continuous at a point $t_0 \in X$ if for every sequence $\{t_n\}$ in X with $S(t_n, t_n, t_0) \to 0$, $S'(g(t_n), g(t_n), g(t_0)) \to 0$ as $n \to \infty$. We say that g is continuous on X if g is continuous at every point $t_0 \in X$.

Definition 2.5. Let X be a non-empty set and let $\mathcal{T}, g: X \to X$ be two self mappings of X. Then a point $x \in X$ is called a

(Γ_1) fixed point of operator $\mathcal{T}$ if $\mathcal{T}x = x$;

(Γ_2) common fixed point of $\mathcal{T}$ and g if $\mathcal{T}x = gx = x$.

Definition 2.6. ([1]) Let f and g be single valued self-mappings on a set X. If $w = fu = gu$ for some $u \in X$, then u is called a coincidence point of f and g, and w is called a point of coincidence of f and g.

Definition 2.7. ([10]) Let f and g be single valued self-mappings on a set X. Mappings f and g are said to be commuting if $fgu = gfu$ for all $u \in X$.

Definition 2.8. ([11]) Let f and g be single valued self-mappings on a set X. Mappings f and g are said to be weakly compatible if they commute at their coincidence points, i.e., if $fu = gu$ for some $u \in X$ implies $fgu = gfu$.

Definition 2.9. ([2]) Let (X, S) be an S-metric space and let $A, S: X \to X$ be two self mappings of X. The pair (A, S) is said to have the $(E.A)$-property if there exists a sequence $\{t_n\}$ in X such that $\lim_{n\to\infty} At_n = \lim_{n\to\infty} St_n = t$ for some $t \in X$.

Example 2.2. Let $X = [0, 1]$ and let $f, g: X \to X$ be defined by $f(x) = \frac{x}{2}$ and $g(x) = \frac{x}{4}$. Define the function $S: X^3 \to [0, \infty)$ by

$$S(x, y, z) = \begin{cases} 0, & \text{if } x{=}y{=}z, \\ \max\{x, y, z\}, & \text{if otherwise,} \end{cases}$$

for all $x, y, z \in X$, then S is an S-metric on X. Now, we show that the pair (f, g) satisfies the $(E.A)$ property. For this, consider the sequence $\{t_n\} = \{\frac{1}{2n+1}\}_{n\geq 1}$. Clearly $\{t_n\}$ is in X and note that $ft_n = \frac{t_n}{2} = \frac{1}{2(2n+1)}$ and $gt_n = \frac{t_n}{4} = \frac{1}{4(2n+1)}$ for all $n \in \mathbb{N}$. This implies that

$$S(ft_n, ft_n, 0) = S\left(\frac{1}{2(2n+1)}, \frac{1}{2(2n+1)}, 0\right) = \max\left\{\frac{1}{2(2n+1)}, \frac{1}{2(2n+1)}, 0\right\}$$
$$= \frac{1}{2(2n+1)} \to 0 \text{ as } n \to \infty.$$

This shows that $ft_n \to 0$ as $n \to \infty$.

Also note that

$$S(gt_n, gt_n, 0) = S\left(\frac{1}{4(2n+1)}, \frac{1}{4(2n+1)}, 0\right) = \max\left\{\frac{1}{4(2n+1)}, \frac{1}{4(2n+1)}, 0\right\}$$
$$= \frac{1}{4(2n+1)} \to 0 \text{ as } n \to \infty.$$

This shows that $gt_n \to 0$ as $n \to \infty$. Thus there exists a sequence $\{t_n\}$ in X such that $ft_n \to 0$ and $gt_n \to 0$ as $n \to \infty$. Hence the pair (f, g) satisfies $(E.A)$ property.

Definition 2.10. ([9]) Let (X, S) be an S-metric space and $f, g, R, T: X \to X$ be four self mappings of X. We say that the pairs (f, R) and (g, T) satisfy the common limit range property with respect to R and T if there exist two sequences $\{t_n\}$ and $\{u_n\}$ in X such that

$$\lim_{n\to\infty} Rt_n = \lim_{n\to\infty} ft_n = \lim_{n\to\infty} gu_n = \lim_{n\to\infty} Tu_n = v,$$

for some $v \in R(X) \cap T(X)$ and it is denoted by (CLR_{RT}).

Proposition 2.1. *([1]) Let f and g be weakly compatible self mappings on a set X. If f and g have a unique point of coincidence $w = fx = gx$, then w is the unique common fixed point of f and g.*

In this paper, we prove some common fixed point theorems in the framework of S-metric spaces by using $(E.A)$-property and (CLR_{RT}) property. Also, we give some examples to validate the results. Our results generalize, extend, improve and enrich several existing results in the literature.

3. Common fixed point theorems using $(E.A)$ property

Theorem 3.1. *Let (X, S) be an S-metric space and let $f, g, R, T : X \to X$ be four self-mappings of X satisfying the following conditions:*

(i)

$$S(fx, fy, gz) \leq r \max\Big\{S(Rx, Ry, Tz), S(fx, fx, Rx), S(gz, gz, Tz), S(fy, fy, Tz)\Big\}, \tag{3.1}$$

for all $x, y, z \in X$, where $0 < r < 1$ is a constant;

(ii) *the pairs (f, R) and (g, T) are weakly compatible;*

(iii) *one of the pairs (f, R) and (g, T) satisfies the $(E.A)$ property;*

(iv) *$f(X) \subseteq T(X)$ and $g(X) \subseteq R(X)$.*

If one range of the mappings R and T is a complete subspace of (X, S), then f, g, R and T have a unique common fixed point in X.

Proof. First, we suppose that the pair (f, R) satisfies the $(E.A)$ property. Then by Definition 2.9, there exists a sequence $\{t_n\}$ in X such that $\lim_{n\to\infty} ft_n = \lim_{n\to\infty} Rt_n = t$ for some $t \in X$. Further, since $f(X) \subseteq T(X)$, there exists a sequence $\{w_n\}$ in X such that $\lim_{n\to\infty} ft_n = \lim_{n\to\infty} Tw_n$. Hence $\lim_{n\to\infty} Tw_n = t$. We claim that $\lim_{n\to\infty} gw_n = t$. If not, then putting $x = t_n$ and $y = w_n$ in inequality (3.1) and using Lemma 2.1, we have

$$\begin{aligned} S(ft_n, ft_n, gw_n) &\leq r \max\Big\{S(Rt_n, Rt_n, Tw_n), S(ft_n, ft_n, Rt_n), \\ &\qquad S(gw_n, gw_n, Tw_n), S(ft_n, ft_n, Tw_n)\Big\} \\ &= r \max\Big\{S(Rt_n, Rt_n, ft_n), S(ft_n, ft_n, Rt_n), \\ &\qquad S(gw_n, gw_n, ft_n), S(ft_n, ft_n, Tw_n)\Big\} \\ &= r \max\Big\{S(Rt_n, Rt_n, ft_n), S(ft_n, ft_n, Rt_n), \\ &\qquad S(gw_n, gw_n, ft_n), S(ft_n, ft_n, ft_n)\Big\} \\ &= r \max\Big\{0, 0, S(ft_n, ft_n, gw_n), 0\Big\} \\ &= r\, S(ft_n, ft_n, gw_n), \end{aligned} \tag{3.2}$$

which is a contradiction, since $0 < r < 1$. Hence $S(ft_n, ft_n, gw_n) = 0$, that is, $ft_n = gw_n$. Letting $n \to \infty$, we get $\lim_{n\to\infty} ft_n = \lim_{n\to\infty} gw_n = t$.

Now, first we suppose that $T(X)$ is a complete subspace of (X, S), then $t = Tu$ for some $u \in X$. Consequently, we have

$$\lim_{n\to\infty} gw_n = \lim_{n\to\infty} ft_n = \lim_{n\to\infty} Rt_n = \lim_{n\to\infty} Tw_n = Tu = t.$$

We claim that $gu = Tu$. For this, putting $x = y = t_n$ and $z = u$ in inequality (3.1) and using Lemma 2.1, we have

$$\begin{aligned} S(ft_n, ft_n, gu) \quad \leq \quad & r \max\Big\{S(Rt_n, Rt_n, Tu), S(ft_n, ft_n, Rt_n), \\ & S(gu, gu, Tu), S(ft_n, ft_n, Tu)\Big\}. \end{aligned} \tag{3.3}$$

Letting $n \to \infty$ in (3.3) and using Lemma 2.1, we get

$$\begin{aligned} S(Tu, Tu, gu) &\leq r \max\Big\{S(t,t,t), S(t,t,t), S(gu, gu, Tu), S(Tu, Tu, Tu)\Big\} \\ &= r \max\big\{0, 0, S(gu, gu, Tu), 0\big\} \\ &= r\, S(Tu, Tu, gu), \end{aligned}$$

which is a contradiction, since $0 < r < 1$. Hence $S(Tu, Tu, gu) = 0$, that is, $Tu = gu = t$. Hence u is a coincidence point of the mappings g and T, that is, the pair (g, T). Now, the weak compatibility of the pair (g, T) implies that $gTu = Tgu$ or $gt = Tt$.

On the other hand, since $g(X) \subseteq R(X)$, there exists $v \in X$ such that $gu = Rv$. Thus $Tu = gu = Rv = t$. Let us show that v is a coincidence point of the pair (f, R), that is, $fv = Rv = t$. If not, then putting $x = y = v$ and $z = u$ in inequality (3.1) and using Lemma 2.1, we have

$$\begin{aligned} S(fv, fv, gu) &\leq r \max\Big\{S(Rv, Rv, Tu), S(fv, fv, Rv), \\ &\qquad S(gu, gu, Tu), S(fv, fv, Tu)\Big\} \\ &= r \max\Big\{S(t,t,t), S(fv, fv, gu), \\ &\qquad S(t,t,t), S(t,t,t)\Big\} \\ &= r \max\big\{0, 0, S(fv, fv, gu), 0\big\} \\ &= r\, S(fv, fv, gu), \end{aligned}$$

which is a contradiction, since $0 < r < 1$. Hence $S(fv, fv, gu) = 0$, that is, $S(fv, fv, Rv) = 0$ and hence $fv = Rv = t$. Thus v is a coincidence point of f and R. Further, the weak compatibility of the pair (f, R) implies that $fRv = Rfv$ or $ft = Rt$. Thus t is a common coincidence point of f, g, R and T.

In order to show that t is a common fixed point of f, g, R and T, let us put $x = y = v$ and $z = t$ in (3.1) and using Lemma 2.1, we get

$$\begin{aligned} S(t,t,gt) &= S(fv,fv,gt) \\ &\le r\max\Big\{S(Rv,Rv,Tt), S(fv,fv,Rv), \\ &\qquad S(gt,gt,Tt), S(fv,fv,Tt)\Big\} \\ &= r\max\Big\{S(t,t,gt), S(fv,fv,fv), S(gt,gt,gt), S(t,t,gt)\Big\} \\ &= r\max\big\{S(t,t,gt), 0, 0, S(t,t,gt)\big\} \\ &= r\,S(t,t,gt), \end{aligned}$$

which is a contradiction, since $0 < r < 1$. Hence $S(t,t,gt) = 0$. Thus $gt = t$. Consequently, $ft = gt = Rt = Tt = t$. This shows that t is a common fixed point of the mappings f, g, R and T.

Similar argument arises if we assume that $R(X)$ is a complete subspace of (X,S).

Similarly, the property $(E.A)$ of the pair (g,T) will give the similar result.

Now, we show the uniqueness of the common fixed point. For this, let us assume that q is another common fixed point of f, g, R and T with $q \neq t$. Then using inequality (3.1) and Lemma 2.1 for $x = y = q$ and $z = t$, we have

$$\begin{aligned} S(q,q,t) &= S(fq,fq,gt) \\ &\le r\max\Big\{S(Rq,Rq,Tt), S(fq,fq,Rq), S(gt,gt,Tt), \\ &\qquad S(fq,fq,Tt)\Big\} \\ &= r\max\Big\{S(q,q,t), S(q,q,q), S(t,t,t), S(q,q,t)\Big\} \\ &= r\max\big\{S(q,q,t), 0, 0, S(q,q,t)\big\} \\ &= r\,S(q,q,t), \end{aligned}$$

which is a contradiction, since $0 < r < 1$. Hence $S(q,q,t) = 0$. We conclude that $q = t$. This shows that the common fixed point of f, g, R and T is unique. This completes the proof. □

Corollary 3.1. *Let (X,S) be an S-metric space and let $f, R\colon X \to X$ be two self-mappings of X satisfying the following conditions:*

(*i*)

$$\begin{aligned} S(fx,fy,fz) &\le r\max\Big\{S(Rx,Ry,Rz), S(fx,fx,Rx), S(fz,fz,Rz), \\ &\qquad S(fy,fy,Rz)\Big\}, \end{aligned}$$

for all $x,y,z \in X$, where $0 < r < 1$ is a constant;

(ii) the pair (f, R) is weakly compatible;

(iii) the pair (f, R) satisfies $(E.A)$ property;

(iv) $f(X) \subseteq R(X)$.

If the range of the mapping R is a complete subspace of (X, S), then f and R have a unique common fixed point in X.

Proof. Putting $f = g$ and $R = T$ in inequality (3.1). Then all conditions of Theorem 3.1 are satisfied and hence the result follows. □

Corollary 3.2. *Let (X, S) be an S-metric space and let $f, g: X \to X$ be two self-mappings of X satisfying the following conditions:*

(i)

$$S(fx, fy, gz) \leq r \max\Big\{S(x, y, z), S(fx, fx, x), S(gz, gz, z), S(fy, fy, z)\Big\},$$

for all $x, y, z \in X$ and for some $r \in (0, 1)$;

(ii) one of the pairs (f, I) and (g, I) satisfies the $(E.A)$ property, where I is an identity map on X.

If the one range of the mappings f and g is a complete subspace of (X, S), then f and g have a unique common fixed point in X.

Proof. Follows from Theorem 3.1 by setting $R = T = I$, where I is an identity map on X. □

Now, we get the special cases of Theorem 3.1 as follows.

Corollary 3.3. *Let (X, S) be a complete S-metric space and let $f, g: X \to X$ be two self-mappings of X satisfying the following condition:*

$$S(fx, fy, gz) \leq r \max\Big\{S(x, y, z), S(fx, fx, x), S(gz, gz, z), S(fy, fy, z)\Big\},$$

for all $x, y, z \in X$ with $r \in (0, 1)$. Then there exists a unique point $\mu \in X$ such that $f\mu = g\mu = \mu$.

Proof. If we take R and T as an identity map on X, then Theorem 3.1 follows that f and g have a unique common fixed point. □

Corollary 3.4. *Let (X, S) be a complete S-metric space and let $f: X \to X$ be a self-mapping of X satisfying the following condition:*

$$S(fx, fy, fz) \leq r \max\Big\{S(x, y, z), S(fx, fx, x), S(fz, fz, z), S(fy, fy, z)\Big\},$$

for all $x, y, z \in X$ with $r \in (0, 1)$. Then f has a unique fixed point in X.

Proof. If we take R and T as an identity map on X and $f = g$, then Theorem 3.1 follows that f has a unique fixed point. □

Theorem 3.2. *Let (X, S) be an S-metric space and let $f, g, R, T \colon X \to X$ be four mappings satisfying the following conditions:*

(i)

$$S(fx, fy, gz) \leq h_1 S(Rx, Ry, Tz) + h_2 S(fx, fx, Rx) + h_3 S(gz, gz, Tz) + h_4 S(fy, fy, Tz), \tag{3.4}$$

for all $x, y, z \in X$, where h_1, h_2, h_3, h_4 are nonnegative constants with $h_1 + h_2 + h_3 + h_4 < 1$;

(ii) the pairs (f, R) and (g, T) are weakly compatible;

(iii) one of the pairs (f, R) and (g, T) satisfies the $(E.A)$ property;

(iv) $f(X) \subseteq T(X)$ and $g(X) \subseteq R(X)$.

If the one range of the mappings R and T is a complete subspace of (X, S), then f, g, R and T have a unique common fixed point in X.

Proof. First, we suppose that the pair (f, R) satisfies the $(E.A)$ property. Then by Definition 2.9, there exists a sequence $\{t_n\}$ in X such that $\lim_{n\to\infty} ft_n = \lim_{n\to\infty} Rt_n = t$ for some $t \in X$. Further, since $f(X) \subseteq T(X)$, there exists a sequence $\{w_n\}$ in X such that $\lim_{n\to\infty} ft_n = \lim_{n\to\infty} Tw_n$. Hence $\lim_{n\to\infty} Tw_n = t$. We claim that $\lim_{n\to\infty} gw_n = t$. If not, then putting $x = y = t_n$ and $z = w_n$ in inequality (3.4) and using Lemma 2.1, we have

$$\begin{aligned} S(ft_n, ft_n, gw_n) &\leq h_1 S(Rt_n, Rt_n, Tw_n) + h_2 S(ft_n, ft_n, Rt_n), \\ &\quad + h_3 S(gw_n, gw_n, Tw_n) + h_4 S(ft_n, ft_n, Tw_n) \\ &= h_1 S(Rt_n, Rt_n, ft_n) + h_2 S(ft_n, ft_n, Rt_n), \\ &\quad + h_3 S(gw_n, gw_n, ft_n) + h_4 S(ft_n, ft_n, Tw_n) \\ &= h_1 S(Rt_n, Rt_n, ft_n) + h_2 S(ft_n, ft_n, Rt_n), \\ &\quad + h_3 S(gw_n, gw_n, ft_n) + h_4 S(ft_n, ft_n, ft_n) \\ &= (h_1 + h_2) S(Rt_n, Rt_n, ft_n) + h_3 S(gw_n, gw_n, ft_n). \end{aligned} \tag{3.5}$$

This implies

$$\begin{aligned} S(ft_n, ft_n, gw_n) &\leq \Big(\frac{h_1 + h_2}{1 - h_3}\Big) S(Rt_n, Rt_n, ft_n) \\ &= q\, S(Rt_n, Rt_n, ft_n), \end{aligned} \tag{3.6}$$

where $q = \Big(\frac{h_1+h_2}{1-h_3}\Big) < 1$, since $h_1 + h_2 + h_3 + h_4 < 1$. Now, letting $n \to \infty$ in (3.6), we get $\lim_{n\to\infty} S(ft_n, ft_n, gw_n) = 0$, that is, $\lim_{n\to\infty} ft_n = \lim_{n\to\infty} gw_n = t$. The rest of the proof is similar to that of Theorem 3.1, so we omit it. □

Remark 3.1. Completeness of the space X is relaxed in Theorem 3.1 and Theorem 3.2.

4. Common fixed point theorems using (CLR_{RT}) property

Theorem 4.1. *Let (X, S) be an S-metric space and let $f, g, R, T\colon X \to X$ be four self-mappings of X satisfying the following conditions:*

(i)

$$S(fx, fy, gz) \leq r \max\Big\{S(Rx, Ry, Tz), S(fx, fx, Rx), S(gz, gz, Tz), S(fy, fy, Tz)\Big\}, \tag{4.1}$$

for all $x, y, z \in X$, where $0 < r < 1$ is a constant;

(ii) the pairs (f, R) and (g, T) are weakly compatible.

If the pairs (f, R) and (g, T) satisfy (CLR_{RT}) property, then the mappings f, g, R and T have a unique common fixed point in X.

Proof. As (f, R) and (g, T) satisfy (CLR_{RT}) property, we can find two sequences $\{t_n\}$ and $\{w_n\}$ in X such that

$$\lim_{n\to\infty} R(t_n) = \lim_{n\to\infty} f(t_n) = \lim_{n\to\infty} g(w_n) = \lim_{n\to\infty} T(w_n) = \lambda$$

for some $\lambda \in R(X) \cap T(X)$. Then $\lambda = T\alpha_1 = R\alpha_2$ for some $\alpha_1, \alpha_2 \in X$. Now, we show that $g\alpha_1 = T\alpha_1$. For each $n \in \mathbb{N}$, from equation (4.1) and using Lemma 2.1, we have

$$\begin{aligned} S(ft_n, ft_n, g\alpha_1) &\leq r \max\Big\{S(Rt_n, Rt_n, T\alpha_1), S(ft_n, ft_n, Rt_n), \\ &\qquad S(g\alpha_1, g\alpha_1, T\alpha_1), S(ft_n, ft_n, T\alpha_1)\Big\}. \end{aligned}$$

Now, letting $n \to \infty$ in the above inequality, we get

$$\begin{aligned} S(T\alpha_1, T\alpha_1, g\alpha_1) &\leq r \max\Big\{S(T\alpha_1, T\alpha_1, T\alpha_1), S(T\alpha_1, T\alpha_1, T\alpha_1), \\ &\qquad S(g\alpha_1, g\alpha_1, T\alpha_1), S(T\alpha_1, T\alpha_1, T\alpha_1)\Big\} \\ &= r \max\Big\{0, 0, S(T\alpha_1, T\alpha_1, g\alpha_1), 0\Big\} \\ &= r\, S(T\alpha_1, T\alpha_1, g\alpha_1). \end{aligned}$$

That is,

$$S(T\alpha_1, T\alpha_1, g\alpha_1) \leq r\, S(T\alpha_1, T\alpha_1, g\alpha_1),$$

which is a contradiction, since $0 < r < 1$. Hence we conclude that $S(T\alpha_1, T\alpha_1, g\alpha_1) = 0$. It follows that $g\alpha_1 = T\alpha_1$. Therefore, we can prove the result as in Theorem 3.1 and hence we omit the rest of the proof. □

Theorem 4.2. *Let (X,S) be an S-metric space and let $f,g,R,T\colon X\to X$ be four mappings satisfying the following conditions:*

(i)

$$S(fx,fy,gz) \le h_1\,S(Rx,Ry,Tz)+h_2\,S(fx,fx,Rx) + h_3\,S(gz,gz,Tz)+h_4\,S(fy,fy,Tz), \tag{4.2}$$

for all $x,y,z\in X$, where h_1,h_2,h_3,h_4 are nonnegative constants with $h_1+h_2+h_3+h_4<1$;

(ii) *the pairs (f,R) and (g,T) are weakly compatible.*

If the pairs (f,R) and (g,T) satisfy (CLR_{RT}) property, then the mappings f, g, R and T have a unique common fixed point in X.

Proof. Since (f,R) and (g,T) satisfy (CLR_{RT}) property, so we can find two sequences $\{t_n\}$ and $\{w_n\}$ in X such that

$$\lim_{n\to\infty}R(t_n)=\lim_{n\to\infty}f(t_n)=\lim_{n\to\infty}g(w_n)=\lim_{n\to\infty}T(w_n)=\lambda$$

for some $\lambda\in R(X)\cap T(X)$. Then $\lambda=T\alpha_1=R\alpha_2$ for some $\alpha_1,\alpha_2\in X$. Now, we show that $g\alpha_1=T\alpha_1$. For each $n\in\mathbb{N}$, from equation (4.2) and using Lemma 2.1, we have

$$\left.\begin{aligned} S(ft_n,ft_n,g\alpha_1) \le\ & h_1\,S(Rt_n,Rt_n,T\alpha_1)+h_2\,S(ft_n,ft_n,Rt_n)\\ & +h_3\,S(g\alpha_1,g\alpha_1,T\alpha_1)+h_4\,S(ft_n,ft_n,T\alpha_1)\end{aligned}\right\}.$$

Now, letting $n\to\infty$ in the above inequality, we get

$$\begin{aligned} S(T\alpha_1,T\alpha_1,g\alpha_1) &\le h_1\,S(T\alpha_1,T\alpha_1,T\alpha_1)+h_2\,S(T\alpha_1,T\alpha_1,T\alpha_1)\\ &\quad \left. +h_3\,S(g\alpha_1,g\alpha_1,T\alpha_1)+h_4\,S(T\alpha_1,T\alpha_1,T\alpha_1)\right\}\\ &= h_3\,S(T\alpha_1,T\alpha_1,g\alpha_1)\\ &< (h_1+h_2+h_3+h_4)\,S(T\alpha_1,T\alpha_1,g\alpha_1),\end{aligned}$$

which is a contradiction, since by hypothesis $h_1+h_2+h_3+h_4<1$. Therefore, we conclude that $S(T\alpha_1,T\alpha_1,g\alpha_1)=0$ and hence it follows that $g\alpha_1=T\alpha_1$. Now, we show that $f\alpha_2=R\alpha_2$. For each $n\in\mathbb{N}$, from equation (4.2), we have

$$\begin{aligned} S(f\alpha_2,f\alpha_2,gw_n) \le\ & h_1\,S(R\alpha_2,R\alpha_2,Tw_n)+h_2\,S(f\alpha_2,f\alpha_2,R\alpha_2)\\ & +h_3\,S(gw_n,gw_n,Tw_n)+h_4\,S(f\alpha_2,f\alpha_2,Tw_n).\end{aligned}$$

Now, letting $n\to\infty$ in the above inequality and using Lemma 2.1 and $(S1)$, we get

$$\begin{aligned} S(f\alpha_2,f\alpha_2,\lambda) \le\ & h_1\,S(R\alpha_2,R\alpha_2,\lambda)+h_2\,S(f\alpha_2,f\alpha_2,R\alpha_2)\\ & +h_3\,S(\lambda,\lambda,\lambda)+h_4\,S(f\alpha_2,f\alpha_2,Tw_n)\end{aligned}$$

$$
\begin{aligned}
&= h_1\, S(\lambda,\lambda,\lambda) + h_2\, S(f\alpha_2, f\alpha_2, \lambda) \\
&\quad + h_3\, S(\lambda,\lambda,\lambda) + h_4\, S(f\alpha_2, f\alpha_2, \lambda) \\
&= h_1(0) + h_3(0) + (h_2 + h_4)\, S(f\alpha_2, f\alpha_2, \lambda) \\
&= (h_2 + h_4)\, S(f\alpha_2, f\alpha_2, \lambda) \\
&< (h_1 + h_2 + h_3 + h_4)\, S(f\alpha_2, f\alpha_2, \lambda),
\end{aligned}
$$

which is a contradiction, since by hypothesis $h_1 + h_2 + h_3 + h_4 < 1$. Hence, we conclude that $S(f\alpha_2, f\alpha_2, \lambda) = 0$ and thus it follows that $f\alpha_2 = \lambda$, so $f\alpha_2 = R\alpha_2 = g\alpha_1 = T\alpha_1 = \lambda$. Since the pair (f, R) is weakly compatible and $f\alpha_2 = R\alpha_2$ implies that $fR\alpha_2 = Rf\alpha_2$ and hence $f\lambda = R\lambda$. Now since the pair (g, T) is weakly compatible and $g\alpha_1 = T\alpha_1$ implies that $Tg\alpha_1 = gT\alpha_1$ and hence $g\lambda = T\lambda$.

Now, we show that λ is a common fixed point of f and R. For this, we consider

$$
\begin{aligned}
S(f\lambda, f\lambda, g\alpha_1) &\le h_1\, S(R\lambda, R\lambda, T\alpha_1) + h_2\, S(f\lambda, f\lambda, R\lambda) \\
&\quad + h_3\, S(g\alpha_1, g\alpha_1, T\alpha_1) + h_4\, S(f\lambda, f\lambda, T\alpha_1) \\
&= h_1\, S(f\lambda, f\lambda, \lambda) + h_2\, S(f\lambda, f\lambda, f\lambda) \\
&\quad + h_3\, S(g\alpha_1, g\alpha_1, g\alpha_1) + h_4\, S(f\lambda, f\lambda, \lambda).
\end{aligned}
$$

Using the condition $(S1)$ and $\lambda = g\alpha_1$ in the above inequality, we obtain

$$
\begin{aligned}
S(f\lambda, f\lambda, g\alpha_1) &\le (h_1 + h_4)\, S(f\lambda, f\lambda, g\alpha_1) \\
&< (h_1 + h_2 + h_3 + h_4)\, S(f\lambda, f\lambda, g\alpha_1),
\end{aligned}
$$

which is a contradiction, since by hypothesis $h_1 + h_2 + h_3 + h_4 < 1$. Hence, we conclude that $S(f\lambda, f\lambda, g\alpha_1) = 0$. This will imply that $g\alpha_1 = f\lambda$ and hence $f\lambda = R\lambda = \lambda$. This shows that λ is a common fixed point of f and R.

Now, we show that λ is a common fixed point of g and T. For this, we consider the inequality (4.2) and using Lemma 2.1, we have

$$
\begin{aligned}
S(f\alpha_2, f\alpha_2, g\lambda) &\le h_1\, S(R\alpha_2, R\alpha_2, T\lambda) + h_2\, S(f\alpha_2, f\alpha_2, R\alpha_2) \\
&\quad + h_3\, S(g\lambda, g\lambda, T\lambda) + h_4\, S(f\alpha_2, f\alpha_2, T\lambda) \\
&= h_1\, S(\lambda, \lambda, g\lambda) + h_2\, S(f\alpha_2, f\alpha_2, f\alpha_2) \\
&\quad + h_3\, S(g\lambda, g\lambda, g\lambda) + h_4\, S(\lambda, \lambda, g\lambda).
\end{aligned}
$$

Using the condition $(S1)$ and $\lambda = f\alpha_2$ in the above inequality, we obtain

$$
\begin{aligned}
S(\lambda, \lambda, g\lambda) &\le (h_1 + h_4)\, S(\lambda, \lambda, g\lambda) \\
&< (h_1 + h_2 + h_3 + h_4)\, S(\lambda, \lambda, g\lambda),
\end{aligned}
$$

which is a contradiction, since by hypothesis $h_1 + h_2 + h_3 + h_4 < 1$. Hence, we conclude that $S(\lambda, \lambda, g\lambda) = 0$. This will imply that $g\lambda = \lambda$ and hence $g\lambda = T\lambda = \lambda$. This shows that λ is a common fixed point of g and T. Hence λ is a common fixed point of f, g, R and T.

Now, we show the uniqueness of the common fixed point. Let us assume that μ is another common fixed point of f, g, R and T such that $f\mu = g\mu = R\mu = T\mu = \mu$ with $\mu \neq \lambda$. Again from the given inequality (4.2), we have

$$\begin{aligned} S(\lambda,\lambda,\mu) &= S(f\lambda, f\lambda, g\mu) \\ &\leq h_1\, S(R\lambda, R\lambda, T\mu) + h_2\, S(f\lambda, f\lambda, R\lambda) \\ &\quad + h_3\, S(g\mu, g\mu, T\mu) + h_4\, S(f\lambda, f\lambda, T\mu) \\ &= h_1\, S(\lambda,\lambda,\mu) + h_2\, S(\lambda,\lambda,\lambda) \\ &\quad + h_3\, S(\mu,\mu,\mu) + h_4\, S(\lambda,\lambda,\mu). \end{aligned}$$

Using the condition $(S1)$ in the above inequality, we get

$$\begin{aligned} S(\lambda,\lambda,\mu) &\leq (h_1 + h_4)\, S(\lambda,\lambda,\mu) \\ &< (h_1 + h_2 + h_3 + h_4)\, S(\lambda,\lambda,\mu), \end{aligned}$$

which is a contradiction, since by hypothesis $h_1 + h_2 + h_3 + h_4 < 1$. Hence, we conclude that $S(\lambda,\lambda,\mu) = 0$, that is, $\lambda = \mu$. Thus, the common fixed point of f, g, R and T is unique. This completes the proof. □

Remark 4.1. Completeness of the space X is relaxed in Theorem 4.1 and Theorem 4.2.

The following examples illustrate Theorem 3.1, Theorem 3.2, Theorem 4.1 and Theorem 4.2 respectively.

Example 4.1. Let $X = [0,1]$. We define the function $S\colon X^3 \to [0,\infty)$ by

$$S(x,y,z) = \begin{cases} 0, & \text{if } x{=}y{=}z, \\ \max\{x,y,z\}, & \text{if otherwise}, \end{cases}$$

for all $x, y, z \in X$, then S is an S-metric on X. Define four self-maps $f, g, R.T\colon X \to X$ on X by $f(x) = \frac{x}{4}$, $g(x) = \frac{x}{4}$, $T(x) = x$ and $R(x) = \frac{x}{2}$ for all $x \in X$. Let $x, y \in X$. Now consider the following cases:

Case I. Let $x < y < z$. Then we have

$$S(fx, fy, gz) = S\Big(\frac{x}{4}, \frac{y}{4}, \frac{z}{4}\Big) = \max\Big\{\frac{x}{4}, \frac{y}{4}, \frac{z}{4}\Big\} = \frac{z}{4},$$

$$S(Rx, Ry, Tz) = S\Big(\frac{x}{2}, \frac{y}{2}, z\Big) = \max\Big\{\frac{x}{2}, \frac{y}{2}, z\Big\} = z,$$

$$S(fx, fx, Rx) = S\Big(\frac{x}{4}, \frac{x}{4}, \frac{x}{2}\Big) = \max\Big\{\frac{x}{4}, \frac{x}{4}, \frac{x}{2}\Big\} = \frac{x}{2},$$

$$S(gz, gz, Tz) = S\Big(\frac{z}{4}, \frac{z}{4}, z\Big) = \max\Big\{\frac{z}{4}, \frac{z}{4}, z\Big\} = z,$$

$$S(fy, fy, Tz) = S\Big(\frac{y}{4}, \frac{y}{4}, z\Big) = \max\Big\{\frac{y}{4}, \frac{y}{4}, z\Big\} = z.$$

Now using inequality (3.1), we have

$$\begin{aligned} S(fx, fy, gz) &= \frac{z}{4} \\ &\leq r \max\Big\{S(Rx, Ry, Tz), S(fx, fx, Rx), \\ &\qquad S(gz, gz, Tz), S(fy, fy, Tz)\Big\} \\ &= r \max\{z, \frac{x}{2}, z, z\} = r\, z, \end{aligned}$$

that is,

$$\frac{1}{4} \leq r.$$

If we take $\frac{1}{4} \leq r < 1$, then we have $0 < r < 1$.

Now using inequality (3.4) of Theorem 3.2, we have

$$\begin{aligned} S(fx, fy, gz) &= \frac{z}{4} \\ &\leq h_1\, S(Rx, Ry, Tz) + h_2\, S(fx, fx, Rx) \\ &\quad + h_3\, S(gz, gz, Tz) + h_4\, S(fy, fy, Tz) \\ &= h_1\, z + h_2\, \frac{x}{2} + h_3\, z + h_4\, z. \end{aligned}$$

Putting $x = 0$ and $z = 1$ in the above inequality, we obtain

$$\frac{1}{4} \leq h_1 + h_3 + h_4.$$

The above inequality is satisfied for (i) $h_1 = \frac{1}{4}$ and $h_2 = h_3 = h_4 = 0$, (ii) $h_1 = \frac{1}{5}$, $h_3 = \frac{1}{5}$ and $h_2 = h_4 = 0$, (iii) $h_1 = \frac{1}{8}$, $h_3 = \frac{1}{8}$, $h_4 = \frac{1}{4}$ and $h_2 = 0$ with $h_1 + h_2 + h_3 + h_4 < 1$ etc., that is, it satisfies for $h_i \in [0, 1)$ for $i = 1, 2, 3, 4$ with $h_1 + h_2 + h_3 + h_4 < 1$.

Case II. Note that $f(X) = [0, \frac{1}{4}]$, $g(X) = [0, \frac{1}{4}]$, $T(X) = [0, 1] = X$ and $R(X) = [0, \frac{1}{2}]$. This will imply that $f(X) \subset R(X)$ and $g(X) \subset T(X)$.

Case III. Now we show that the pairs (f, R) and (g, T) are weakly compatible. For this, suppose that $Tx = gx$ for $x \in X$. Then $x = \frac{x}{4}$. It follows that $x = 0$. Now, we consider $Tg(x) = T(gx) = T(0) = 0$ and $gT(x) = g(Tx) = g(0) = 0$. Thus, the pair (g, T) is weakly compatible. Now, let $fx = Rx$ for $x \in X$. This implies that $\frac{x}{4} = \frac{x}{2}$ and hence $x = 0$. Now, we consider $fR(x) = f(Rx) = f(0) = 0$ and $Rf(x) = R(fx) = R(0) = 0$. It follows that the pair (f, R) is also weakly compatible.

Case IV. Now we show that the pairs (g, T) satisfies $(E.A)$ property. For this, consider the sequence $\{t_n\} = \{\frac{1}{2n+1}\}_{n\geq 1}$. Clearly the sequence $\{t_n\}$ is in X and note that $Tt_n = t_n = \frac{1}{2n+1}$ and $gt_n = \frac{t_n}{4} = \frac{1}{4(2n+1)}$ for all $n \in \mathbb{N}$. This will imply that

$$S(Tt_n, Tt_n, 0) = S\left(\frac{1}{2n+1}, \frac{1}{2n+1}, 0\right) = \max\left\{\frac{1}{2n+1}, \frac{1}{2n+1}, 0\right\}$$

$$= \frac{1}{2n+1} \to 0 \text{ as } n \to \infty.$$

This shows that $Tt_n \to 0$ as $n \to \infty$.

Also note that

$$S(gt_n, gt_n, 0) = S\bigg(\frac{1}{4(2n+1)}, \frac{1}{4(2n+1)}, 0\bigg) = \max\bigg\{\frac{1}{4(2n+1)}, \frac{1}{4(2n+1)}, 0\bigg\}$$

$$= \frac{1}{4(2n+1)} \to 0 \text{ as } n \to \infty.$$

This shows that $gt_n \to 0$ as $n \to \infty$.

Thus there exists a sequence $\{t_n\}$ in X such that $gt_n \to 0$ and $Tt_n \to 0$ as $n \to \infty$. Hence the pair (g, T) satisfies $(E.A)$ property.

Similarly, we can show that the pair (f, R) also satisfies $(E.A)$ property.

Case V. As $f(X) = [0, \frac{1}{4}]$, then $f(X)$ is a complete subspace of X.

Thus all the conditions of Theorem 3.1 and Theorem 3.2 are satisfied and hence the mappings f, g, R and T have a unique common fixed point, namely $x = 0 \in X$.

Example 4.2. Let $X = [0, 4]$. We define the function $S\colon X^3 \to [0, \infty)$ by

$$S(x, y, z) = \begin{cases} 0, & \text{if } x{=}y{=}z, \\ \max\{x, y, z\}, & \text{if otherwise,} \end{cases}$$

for all $x, y, z \in X$, then S is an S-metric on X. Define four self-maps $f, g, R.T\colon X \to X$ on X by $f(x) = \frac{x}{2}$, $g(x) = \frac{x}{2}$, $T(x) = x$ and $R(x) = x$ for all $x \in X$. Let $x, y, z \in X$. Now consider the following cases:

Case I. Let $x < y < z$. Then we have

$$S(fx, fy, gz) = S\Big(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\Big) = \max\Big\{\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\Big\} = \frac{z}{2},$$

$$S(Rx, Ry, Tz) = S\Big(x, y, z\Big) = \max\Big\{x, y, z\Big\} = z,$$

$$S(fx, fx, Rx) = S\Big(\frac{x}{2}, \frac{x}{2}, x\Big) = \max\Big\{\frac{x}{2}, \frac{x}{2}, x\Big\} = x,$$

$$S(gz, gz, Tz) = S\Big(\frac{z}{2}, \frac{z}{2}, z\Big) = \max\Big\{\frac{z}{2}, \frac{z}{2}, z\Big\} = z,$$

$$S(fy, fy, Tz) = S\Big(\frac{y}{2}, \frac{y}{2}, z\Big) = \max\Big\{\frac{y}{2}, \frac{y}{2}, z\Big\} = z.$$

Now using inequality (4.1), we have

$$\begin{aligned} S(fx, fy, gz) &= \frac{z}{2} \\ &\le r \max\Big\{S(Rx, Ry, Tz), S(fx, fx, Rx), \\ &\qquad S(gz, gz, Tz), S(fy, fy, Tz)\Big\} \\ &= r \max\{z, x, z, z\} = r\, z, \end{aligned}$$

that is,

$$\frac{1}{2} \leq r.$$

If we take $\frac{1}{2} \leq r < 1$, then we have $0 < r < 1$.

Now using inequality (4.2) of Theorem 4.2, we have

$$\begin{aligned} S(fx, fy, gz) &= \frac{z}{2} \\ &\leq h_1\, S(Rx, Ry, Tz) + h_2\, S(fx, fx, Rx) \\ &\quad + h_3\, S(gz, gz, Tz) + h_4\, S(fy, fy, Tz) \\ &= h_1\, z + h_2\, x + h_3\, z + h_4\, z. \end{aligned}$$

Putting $x = 0$ and $z = 1$ in the above inequality, we obtain

$$\frac{1}{2} \leq h_1 + h_3 + h_4.$$

The above inequality is satisfied for (i) $h_1 = \frac{1}{2}$ and $h_2 = h_3 = h_4 = 0$, (ii) $h_1 = \frac{1}{4}$, $h_3 = \frac{1}{4}$ and $h_2 = h_4 = 0$, (iii) $h_1 = \frac{1}{5}$, $h_3 = \frac{1}{5}$, $h_4 = \frac{1}{4}$ and $h_2 = 0$ with $h_1 + h_2 + h_3 + h_4 < 1$ etc., that is, it satisfies for $h_i \in [0, 1)$ for $i = 1, 2, 3, 4$ with $h_1 + h_2 + h_3 + h_4 < 1$.

Case II. Now we show that the pairs (f, R) and (g, T) satisfy (CLR_{RT}) property. For this, we choose the sequences $\{t_n\} = \{\frac{1}{n}\}_{n \geq 1}$ and $\{w_n\} = \{\frac{1}{2n+3}\}_{n \geq 1}$. Clearly the sequences $\{t_n\}$ and $\{w_n\}$ are in X. Then we have

$$S(Rt_n, Rt_n, 0) = S\left(\frac{1}{n}, \frac{1}{n}, 0\right) = \max\left\{\frac{1}{n}, \frac{1}{n}, 0\right\}$$

$$= \frac{1}{n} \to 0 \text{ as } n \to \infty.$$

This shows that $Rt_n \to 0$ as $n \to \infty$.

Also we observe that

$$S(ft_n, ft_n, 0) = S\left(\frac{1}{2n}, \frac{1}{2n}, 0\right) = \max\left\{\frac{1}{2n}, \frac{1}{2n}, 0\right\}$$

$$= \frac{1}{2n} \to 0 \text{ as } n \to \infty.$$

This shows that $ft_n \to 0$ as $n \to \infty$.

Similarly, we obtain that

$$S(gw_n, gw_n, 0) = S\left(\frac{1}{2(2n+3)}, \frac{1}{2(2n+3)}, 0\right) = \max\left\{\frac{1}{2(2n+3)}, \frac{1}{2(2n+3)}, 0\right\}$$

$$= \frac{1}{2(2n+3)} \to 0 \text{ as } n \to \infty.$$

This shows that $gw_n \to 0$ as $n \to \infty$.

Also we observe that

$$S(Tw_n, Tw_n, 0) = S\left(\frac{1}{2n+3}, \frac{1}{2n+3}, 0\right) = \max\left\{\frac{1}{2n+3}, \frac{1}{2n+3}, 0\right\}$$

$$= \frac{1}{2n+3} \to 0 \text{ as } n \to \infty.$$

This shows that $Tw_n \to 0$ as $n \to \infty$.

Since $R(0) = 0 = T(0)$, we have $0 \in R(X) \cap T(X)$. Therefore there exist sequences $\{t_n\}$ and $\{w_n\}$ in X such that

$$\lim_{n\to\infty} R(t_n) = \lim_{n\to\infty} f(t_n) = \lim_{n\to\infty} T(w_n) = \lim_{n\to\infty} g(w_n).$$

Therefore the pairs (f, R) and (g, T) satisfy (CLR_{RT}) property.

Case III. Now we show that the pairs (f, R) and (g, T) are weakly compatible. For this, suppose that $Tx = gx$ for $x \in X$. Then $x = \frac{x}{2}$. It follows that $x = 0$. Now, we consider $Tg(x) = T(gx) = T(0) = 0$ and $gT(x) = g(Tx) = g(0) = 0$. Thus, the pair (g, T) is weakly compatible. Now, let $fx = Rx$ for $x \in X$. This implies that $\frac{x}{2} = x$ and hence $x = 0$. Now, we consider $fR(x) = f(Rx) = f(0) = 0$ and $Rf(x) = R(fx) = R(0) = 0$. It follows that the pair (f, R) is also weakly compatible.

Thus all the hypothesis of Theorem 4.1 and Theorem 4.2 are satisfied and hence the mappings f, g, R and T have a unique common fixed point, namely $x = 0 \in X$.

5. Well-Posedness Theorem

In this section, we prove well-posedness of fixed point problem of mapping in Corollary 3.4.

Definition 5.1. ([15]) Let (X, d) be a metric space and let $T\colon X \to X$ be a mapping. The fixed point problem of T is said to be well posed if:

(1) T has a unique fixed point x_0,

(2) for any sequence $\{x_n\} \in X$ with $\lim_{n\to\infty} d(Tx_n, x_n) = 0$, we have $\lim_{n\to\infty} d(x_n, x_0) = 0$.

Now, we define well-posedness of fixed point in S-metric spaces.

Let $FP(f, X)$ denote a fixed point problem of mapping f and let $F(f)$ denote the set of all fixed points of f.

Definition 5.2. Let (X, S) be an S-metric space and let $f\colon X \to X$ be a mapping. $FP(f, X)$ is called well posed if:

(1) f has a unique fixed point x_0,

(2) for any sequence $\{x_n\}$ in X with

$$\lim_{n\to\infty} S(fx_n, fx_n, x_n) = 0 = \lim_{n\to\infty} S(x_n, x_n, fx_n),$$

implies

$$\lim_{n\to\infty} S(x_0, x_0, x_n) = 0 = \lim_{n\to\infty} S(x_n, x_n, x_0).$$

Theorem 5.1. *Let $f: X \to X$ be a self mapping as in Corollary 3.4. Then the fixed point problem for f is well posed.*

Proof. From Corollary 3.4, we know that f has a unique fixed point $v = fv \in X$. Let $\{x_n\} \subset X$ be such that $\lim_{n\to\infty} S(fx_n, fx_n, x_n) = 0 = \lim_{n\to\infty} S(x_n, x_n, fx_n)$. Then, we have

$$\begin{aligned} S(x_n, x_n, v) &\leq 2S(x_n, x_n, fx_n) + S(v, v, fx_n) \\ &= 2S(x_n, x_n, fx_n) + S(fx_n, fx_n, fv) \\ &\leq 2S(x_n, x_n, fx_n) + r \max\Big\{S(x_n, x_n, v), \\ &\quad S(fx_n, fx_n, x_n), S(fv, fv, v), S(fx_n, fx_n, v)\Big\}. \end{aligned}$$

Taking the limit as $n \to \infty$ in the above inequality and using $(S1)$ and Lemma 2.1, we obtain

$$\begin{aligned} S(x_n, x_n, v) &\leq r \max\Big\{S(x_n, x_n, v), 0, 0, 0\Big\} \\ &= r\, S(x_n, x_n, v) \\ &< S(x_n, x_n, v), \end{aligned}$$

which is a contraction, since $0 < r < 1$. Hence $S(x_n, x_n, v) \to 0$ as $n \to \infty$ which is equivalent to saying that $x_n \to v$ as $n \to \infty$. Thus, the fixed point problem of f is well-posed. This completes the proof. □

6. Conclusion

In this paper, we prove some unique common fixed point theorems in the setting of S-metric spaces with the help of weakly compatible condition, $(E.A)$ property and (CLR_{RT}) property of the pair of mappings and give some corollaries of the main results. We validate our results by illustrative examples. We have also proved well-posedness of a fixed point problem. Our results extend, generalize and improve several results from the existing literature (see, for example, [12], [16], [17], [18] and many others).

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