FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 38, No 1 (2023), 199 - 208
https://doi.org/10.22190/FUMI221215013D
Original Scientific Paper

ON GENERALIZED BERTRAND CURVES IN EUCLIDEAN 3-SPACE

Demet Demir[1] **and Kazım İlarslan**[1]

[1] **Faculty of Sciences and Arts, Department of Mathematics, Kırıkkale University, 71450 Kırıkkale, Türkiye**

Abstract. In this paper, we generalize the notion of Bertrand curve in Euclidean 3-space analogously as in Minkowski 3-space. According to this generalization, the Bertrand curve conditions of a given space curve are obtained and the relations between Frenet vectors and curvature functions are revealed. A curve that meets the given condition is constructed as an example.
Keywords: Bertrand curve, Euclidean 3-space, Frenet vectors, curvature functions.

1. Introduction

In the theory of curves in Euclidean space, one of the important and interesting problems is characterization of a regular curve. In the solution of the problem, the curvature functions k_1 (or $\varkappa$) and k_2 (or τ) of a regular curve have an effective role. For example: if $k_1 = 0 = k_2$, then the curve is a geodesic or if k_1 =constant$\neq 0$ and $k_2 = 0$, then the curve is a circle with radius $1/k_1$, etc. Another way in the solution of the problem is the relationship between the Frenet vectors and Frenet planes of the curves [6, 10]. An interesting example of relations between Frenet vectors belonging to pairs of space curves is Bertrand curves.

A Bertrand curve is a curve in the Euclidean space such that its principal normal is the principal normal of the second curve [2, 12]. The study of this kind of curves has been extended to many other ambient spaces. In [9], Pears studied this problem

Received December 15, 2022. accepted March 27, 2023.
Communicated by Mića Stanković
Corresponding Author: Demet Demir, Faculty of Sciences and Arts, Department of Mathematics, Kırıkkale University, 71450 Kırıkkale, Türkiye | E-mail: demet_ _uzunyol_ _13@hotmail.com
2010 *Mathematics Subject Classification.* Primary 53A04

for curves in the n-dimensional Euclidean space $\mathbb{E}^n$, $n > 3$, and showed that a Bertrand curve in $\mathbb{E}^n$ must belong to a three-dimensional subspace $\mathbb{E}^3 \subset \mathbb{E}^n$. This result is restated by Matsuda and Yorozu [8]. They proved that there was not any special Bertrand curves in $\mathbb{E}^n$ $(n > 3)$ and defined a new kind, which is called $(1, 3)$-type Bertrand curves in 4-dimensional Euclidean space. Bertrand curves and their characterizations were studied by many researchers in Minkowski 3-space and Minkowski space-time [1, 3, 5] as well as in Euclidean space. In addition, $(1, 3)$-type Bertrand curves were studied in semi-Euclidean 4-space with index 2 [11].

A new generalization for Bertrand curves is given by Zhang and Pei in 2020 [13]. In this study, instead of classical condition for Bertrand curves, generalized Bertrand curves are defined such that the principal normal of a given curve belongs to a normal space of another curve.

In this paper, we generalize the notion of Bertrand curve in Euclidean 3-space analogously as in Minkowski 3-space. According to this generalization, the Bertrand curve conditions of a given space curve are obtained and the relations between Frenet vectors and curvature functions are revealed. An example of a curve is constructed that satisfies the given conditions.

2. Preliminaries

In this section, we give some well known results from Euclidean geometry [4, 7]. Let $\mathbb{E}^3$ be the 3-dimensional Euclidean space equipped with the inner product $\langle X, Y\rangle = x_1y_1 + x_2y_2 + x_3y_3$, where $X = (x_1, x_2, x_3)$ and $Y = (y_1, y_2, y_3) \in \mathbb{E}^3$. The norm of X is given by $\|X\| = \sqrt{\langle X, X\rangle}$ and the vector product is given by

$$X \times Y = \begin{pmatrix} e_1 & e_2 & e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix}$$

where $\{e_1, e_2, e_3\}$ is the canonical basis of $\mathbb{E}^3$.

Let I be an interval of $\mathbb{R}$ and let $\gamma : I \to \mathbb{E}^3$ be a regular space curve, that is, $\gamma^{'}(t) \neq 0$ for all $t \in I$, where $\gamma^{'}(t) = \frac{d\gamma}{dt}(t)$. We say that γ is nondegenerate condition if $\gamma^{'}(t) \times \gamma^{''}(t) \neq 0$ for all $t \in I$. If we take the arc-length parameter s, that is, $\left\|\gamma^{'}(s)\right\| = 1$ for all s, then the tangent vector ,the principal normal vector ,and the binormal vector are given by

$$\begin{aligned} T(s) &= \gamma^{'}(s), \\ N(s) &= \frac{\gamma^{''}(s)}{\left\|\gamma^{''}(s)\right\|}, \\ B(s) &= T(s) \times N(s) \end{aligned}$$

where $\gamma^{'}(s) = \frac{d\gamma}{ds}(s)$. Then $\{T(s), N(s), B(s)\}$ is a moving frame of $\gamma(s)$ and we have the Frenet-Serret formula:

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{2.1}$$

where

$$\begin{aligned} \kappa(s) &= \left\| \gamma''(s) \right\|, \\ \tau(s) &= \frac{\det\left(\gamma'(s), \gamma''(s), \gamma'''(s)\right)}{\kappa^2(s)}. \end{aligned}$$

If we take general parameter t ,then the tangent vector ,the principal normal vector and the binormal vector are given by

$$\begin{aligned} T(t) &= \tfrac{\dot{\gamma}(t)}{\|\dot{\gamma}(t)\|}, \\ N(t) &= B(t) \times T(t), \\ B(t) &= \tfrac{\dot{\gamma}(t)\times\ddot{\gamma}(t)}{\|\dot{\gamma}(t)\times\ddot{\gamma}(t)\|} \end{aligned}$$

where $\dot{\gamma}(t) = \frac{d\gamma}{dt}(t)$. Then $\{T(t), N(t), B(t)\}$ is a moving frame of $\gamma(t)$ and we have the Frenet-Serret formula:

$$\begin{bmatrix} \dot{T}(t) \\ \dot{N}(t) \\ \dot{B}(t) \end{bmatrix} = \begin{bmatrix} 0 & \|\dot{\gamma}(t)\| \kappa(t) & 0 \\ -\|\dot{\gamma}(t)\| \kappa(t) & 0 & \|\dot{\gamma}(t)\| \tau(t) \\ 0 & -\|\dot{\gamma}(t)\| \tau(t) & 0 \end{bmatrix} \begin{bmatrix} T(t) \\ N(t) \\ B(t) \end{bmatrix} \tag{2.2}$$

where

$$\begin{aligned} \kappa(t) &= \frac{|\ \dot{\gamma}(t) \times \ddot{\gamma}(t)\ |}{|\ \dot{\gamma}(t)\ |^3}, \\ \tau(t) &= \frac{\det\left(\dot{\gamma}(t), \ddot{\gamma}(t),\ \dddot{\gamma}(t)\right)}{|\ \dot{\gamma}(t) \times \ddot{\gamma}(t)\ |^2}. \end{aligned}$$

3. Generalized Bertrand curves in Euclidean 3-space

In this section, generalized Bertrand curve concept given by Zhang and Pein (see [13]) Minkowski 3-space will be defined in 3-dimensional Euclidean space $\mathbb{E}^3$and generalized Bertrand curve conditions will be obtained for a curve in this space.

Definition 3.1. Let $\alpha(s)$ and $\alpha^*(s^*)$ be two curves in 3-dimensional Euclidean space. If the principal normal $N(s)$ of $\alpha(s)$ lies in the normal plane of $\alpha^*(s^*)$ and the angle between $N(s)$ and $N^*(s^*)$ is θ at the corresponding points, then we call $\alpha(s)$ a generalized Bertrand curve, $\alpha^*(s^*)$ is a generalized Bertrand mate of $\alpha(s)$. Also $(\alpha(s), \alpha^*(s^*))$ is called a pair of generalized Bertrand curves.

According to the above definition if $\alpha(s)$ is a generalized Bertrand Curve in $\mathbb{E}^3$ then the following holds :

$$\begin{array}{ll} (i) & N = \cos\theta N^*(s^*) + \sin\theta B^*(s^*), \\ (ii) & \langle N(s), N^*(s^*)\rangle = \cos\theta = \text{constant} \end{array}$$

where N^* and B^* are principal normal and binormal vectors of α^*.

Theorem 3.1. *Let $\alpha(s)$ be a generalized Bertrand curve in Euclidean 3-space, parametrized by its arc-length s and $\alpha^*(s^*)$ be the generalized Bertrand mate curve of $\alpha(s)$ in $\mathbb{E}^3$ such that the principal normal $N(s)$ of $\alpha(s)$ lies in the normal plane spanned by $\{N^*, B^*\}$ and the angle between N and N^* is θ at the corresponding points. The curvatures and Frenet vector of α and α^* are related as follows:*

$$\begin{aligned} T^* &= \left(\frac{1-\lambda\kappa}{f'}\right)T + \frac{\lambda\tau}{f'}B, \\ N^* &= eT + \cos\theta N + hB, \\ B^* &= \frac{D_1}{\sqrt{D_1^2+D_2^2+D_3^2}}T + \frac{D_2}{\sqrt{D_1^2+D_2^2+D_3^2}}N + \frac{D_3}{\sqrt{D_1^2+D_2^2+D_3^2}}B \end{aligned}$$

and

$$\kappa^*(s^*) = \frac{\kappa - \lambda\left(\kappa^2+\tau^2\right)}{\left[(1-\lambda\kappa)^2 + \lambda^2\tau^2\right]\cos\theta}, \qquad \tau^*(s^*) = \frac{\sqrt{D_1^2+D_2^2+D_3^2}}{f'}$$

where $\cos\theta \neq 0$ *and* $\lambda \in \mathbb{R}_0$, $\left(f'\right)^2 = (1-\lambda\kappa)^2 + \lambda^2\tau^2$,

$$e = \frac{-\cos\theta\left(\lambda\kappa' + \frac{f''}{f'}(1-\lambda\kappa)\right)}{\kappa - \lambda\left(\kappa^2+\tau^2\right)} \quad , \quad h = \frac{\cos\theta\left(\lambda\tau' + \frac{f''}{f'}\lambda\tau\right)}{\kappa - \lambda\left(\kappa^2+\tau^2\right)}$$

and

$$D_1 = e' - \kappa\cos\theta + \kappa^*(1-\lambda\kappa), \quad D_2 = e\kappa - h\tau, \quad D_3 = \tau\cos\theta + h' + \kappa^*\lambda\tau.$$

Proof. Assume that there exists the generalized Bertrand curve α in $\mathbb{E}^3$ and its α^* generalized Bertrand mate α^* in $\mathbb{E}^3$.Then α^*can be parametrized by

$$\alpha^*(s^*) = \alpha(s) + \lambda(s)N(s) \tag{3.1}$$

where $s^* = s^*(s)$. Differentiating equation (3.1) with respect to s and using Frenet frame (2.1), we get

$$T^*f' = (1-\lambda\kappa)T + \lambda' N + \lambda\tau B.$$

By taking the inner product of the last relation by $N = \cos\theta N^* + \sin\theta B^*$, we have $\lambda' = 0$.Substituting this in the last relation, we find

$$T^{*}f' = (1-\lambda\kappa)T - \lambda\tau B. \tag{3.2}$$

From equation (3.2), we obtain

$$\langle T^{*}f^{'}, T^{*}f^{'}\rangle = (f')^{2} = (1+\lambda\kappa)^{2} + (\lambda\tau)^{2}. \tag{3.3}$$

Differentianting equation (3.2) with respect to s and using Frenet frame (3.2), we obtain

$$\kappa^{*}N^{*}(f^{'})^{2} + f^{''}T^{*} = (-\lambda\kappa^{'})T + (\kappa - \lambda\kappa^{2} - \lambda\tau^{2})N + (\lambda\tau^{'})B. \tag{3.4}$$

By taking the inner product of the last relation with $N = \cos\theta N^{*} + \sin\theta B^{*}$, we get

$$\kappa^{*}(f^{'})^{2}\cos\theta = \kappa - \lambda(\kappa^{2}+\tau^{2}). \tag{3.5}$$

Then, by using equation (3.3), we find

$$\kappa^{*} = \frac{\kappa - \lambda(\kappa^{2}+\tau^{2})}{\left[(1-\lambda\kappa)^{2} + \lambda^{2}\tau^{2}\right]\cos\theta}. \tag{3.6}$$

Putting the equations (3.2) and (3.6) in (3.4), we get

$$N^{*} = eT + \cos\theta N + hB \tag{3.7}$$

where, $e = \dfrac{-\cos\theta\left(\lambda\kappa^{'} + \frac{f^{''}}{f'}(1-\lambda\kappa)\right)}{\kappa - \lambda(\kappa^{2}+\tau^{2})}$ and $h = \dfrac{\cos\theta\left(\lambda\tau^{'} + \frac{f^{''}}{f'}\lambda\tau\right)}{\kappa - \lambda(\kappa^{2}+\tau^{2})}$.

Differentianting (3.7) with respect to s and using Frenet frame (2.1), we obtain

$$(-\kappa^{*}T^{*} + \tau^{*}B^{*})f^{'} = (e^{'} - \kappa\cos\theta)T + (e\kappa - h\tau)N + (\tau\cos\theta + h^{'})B. \tag{3.8}$$

By using (3.2) and (3.8), we get

$$\tau^{*}B^{*} = \frac{D_1}{f'}T + \frac{D_2}{f'}N + \frac{D_3}{f'}B \tag{3.9}$$

where

$$\begin{aligned} D_1 &= e^{'} - \kappa\cos\theta + \kappa^{*}(1-\lambda\kappa), \\ D_2 &= e\kappa - h\tau, \\ D_3 &= \tau\cos\theta + h^{'} + \kappa^{*}\lambda\tau. \end{aligned}$$

By taking the inner product of equation (3.9) with itself, we get

$$T^* = \frac{\sqrt{D_1^2 + D_2^2 + D_3^2}}{f'}. \tag{3.10}$$

By using equation (3.10) in equation (3.9), we obtain

$$B^* = \frac{D_1}{\sqrt{D_1^2 + D_2^2 + D_3^2}} T + \frac{D_2}{\sqrt{D_1^2 + D_2^2 + D_3^2}} N + \frac{D_3}{\sqrt{D_1^2 + D_2^2 + D_3^2}} B. \tag{3.11}$$

This completes the proof. □

Theorem 3.2. *Let α be a unit speed curve with N principal normal vector in $\mathbb{E}^3$. α^* be a regular curve with N^* principal normal vector, then (α, α^*) is a pair of generalized Bertrand curve if and only if the curvature $\kappa(s)$ and torsion $\tau(s)$ of $\alpha(s)$ satisfy;*

(3.12)

$$\kappa - \lambda(\kappa^2 + \tau^2) = \cos\theta \frac{\left\{ \left[\kappa - \lambda(\kappa^2 + \tau^2)\right]^2 \left[\lambda^2\tau^2 + (1 - \lambda\kappa)^2\right] + \left[\lambda\tau' - \lambda^2(\kappa\tau' - \kappa'\tau)\right]^2 \right\}^{\frac{1}{2}}}{\left[(1 - \lambda\kappa)^2 + \lambda^2\tau^2\right]^{\frac{1}{2}}},$$

where θ is the angle between the vectors $N(s)$, $N^*(s^*)$ and λ is a non-zero constant , $\kappa' = \dfrac{d\kappa}{ds}$ and $\tau' = \dfrac{d\tau}{ds}$.

Proof. We assume that (α, α^*) is a pair of generalized Bertrand curve in $\mathbb{E}^3$, then we have

$$\alpha^*(s^*) = \alpha(s) + \lambda(s)N(s). \tag{3.13}$$

From Theorem 1, we get

$$T^* f' = (1 - \lambda\kappa)T + \lambda\tau B \tag{3.14}$$

and we have known,

$$N = \cos\theta N^* + \sin\theta B^*. \tag{3.15}$$

Differentiating (3.14) with respect to s and using Frenet frame (2.1), we find

$$-\kappa T + \tau B = -\kappa^* \cos\theta T^* f' - \tau^* \sin\theta N^* f' + \tau^* \cos\theta B^* f'. \tag{3.16}$$

By inner product (3.16) with (3.14), we reached

$$\kappa - \lambda(\kappa^2 + \tau^2) = \kappa^* \cos\theta (f^{'})^2. \tag{3.17}$$

The curvature $\kappa^*(s^*)$ of the curve $\alpha^*(s^*)$ is

$$\kappa^*(s^*) = \frac{\left\| \alpha^{*'} \times \alpha^{*''} \right\|}{\left\| \alpha^{*'} \right\|^3}$$

where, $\alpha^{*'} = \dfrac{d\alpha^*}{ds^*}\dfrac{ds^*}{ds} = \dfrac{d\alpha^*}{ds}$, $\left\| \alpha^{*'} \right\| = \left\| \dfrac{d\alpha^*}{ds} \right\| = \left\| f^{'} \right\|$.

By using the last relations in (3.17), we easily get

$$\kappa - \lambda(\kappa^2 + \tau^2) = \frac{\left\| \alpha^{*'} \times \alpha^{*''} \right\|}{\left\| \alpha^{*'} \right\|} \cos\theta. \tag{3.18}$$

Also we have

$$\alpha^{*'} = (1 - \lambda\kappa)T + \lambda\tau B$$

and differentiating the last relation with respect to s, we get

$$\alpha^{*''} = \lambda\kappa^{'} T + \left[\kappa - (\kappa^2 + \tau^2)\right] N + \lambda\tau^{'} B.$$

Then if we calculate $\left\| \alpha^{*'} \right\|$ and $\left\| \alpha^{*'} \times \alpha^{*''} \right\|$, we get

$$\begin{aligned} \left\| \alpha^{*'} \right\| &= \left[(1-\lambda\kappa)^2 + \lambda^2\tau^2\right]^{\frac{1}{2}}, \\ \left\| \alpha^{*'} \times \alpha^{*''} \right\| &= \Big\{ \lambda^2\tau^2 \left[\kappa - \lambda(\kappa^2+\tau^2)\right]^2 + \left[\lambda\tau^{'} - \lambda^2(\kappa\tau^{'} - \kappa^{'}\tau)\right]^2 + \\ &\quad \left[(1-\lambda\kappa)^2 \left[\kappa - \lambda(\kappa^2+\tau^2)\right]^2\right] \Big\}^{\frac{1}{2}}. \end{aligned} \tag{3.19}$$

We put equations (3.19) in (3.18), we get equation (3.12) .

Conversely,we will prove that if $\kappa(s)$ and $\tau(s)$ satisfy equation (3.12), the principal normal and binormal of α^* generated by the equation

$$\alpha^*(s^*) = \alpha(s) + \lambda(s)N(s) \tag{3.20}$$

are coplanar with the principal normal of $\alpha(s)$, where $s^* = s^*(s)$. The angle between N and N^* is θ in equation (3.12), we have known that λ is a non-zero constant. Then from (3.20) differentiating with respect to s we easily get

$$T^* f' = (1 - \lambda\kappa)T + \lambda\tau B. \tag{3.21}$$

By taking the inner product of equation (3.21) with N, we get

$$\langle T^*, N\rangle = 0$$

which means that; N is coplanar with N^* and B^*.
Then we prove that

$$\langle N, N^*\rangle = \cos\theta.$$

We assume that

$$N = aN^* + bB^* \qquad a, b \in \mathbb{R}. \tag{3.22}$$

Differentiating equation (3.22) with respect to s, we find

$$-\kappa T + \tau B = -\kappa^* a T^* f^{'} - \tau^* b N^* f^{'} + \tau^* a B^* f^{'}. \tag{3.23}$$

By taking product equation (3.23) with equation (3.21), we get

$$\kappa - \lambda(\kappa^2 + \tau^2) = \kappa^* a (f^{'})^2. \tag{3.24}$$

From equation (3.24) and (3.17) we easily obtain $\cos\theta = a$.
From equation (3.22) we get

$$\langle N, N^*\rangle = a = \cos\theta.$$

This completes the proof. □

Remark 3.1. In Theorem 2, when $\theta = 0$,we have $\cos\theta = 1$. Then we have

$$\kappa-\lambda(\kappa^2+\tau^2) = \frac{\left\{\left[\kappa - \lambda(\kappa^2 + \tau^2)\right]^2 \left[\lambda^2\tau^2 + (1-\lambda\kappa)^2\right] + \left[\lambda\tau^{'} - \lambda^2(\kappa\tau^{'} - \kappa^{'}\tau)\right]^2\right\}^{\frac{1}{2}}}{\left[(1-\lambda\kappa)^2 + \lambda^2\tau^2\right]^{\frac{1}{2}}}.$$

Squaring both sides of this equation, we find

$$\left[\lambda\tau^{'} - \lambda^2(\kappa\tau^{'} - \kappa^{'}\tau)\right] = 0.$$

Therefore ,

$$\begin{aligned} \tau^{'}(\lambda\kappa - 1) &= \lambda\kappa^{'}\tau \\ \frac{d\tau}{\tau} &= \frac{d(\lambda\kappa - 1)}{(\lambda\kappa - 1)} \\ \lambda_1\tau + \lambda_2\kappa &= 1 \end{aligned}$$

where λ is a constant. The equation $\lambda_1\tau + \lambda_2\kappa = 1$ is the necessary and sufficient condition for a curve to be a Bertrand curve in Euclidean 3-space.

Example 3.1. Let us take $\theta = \frac{\pi}{4}, \lambda = 2$ and $\kappa = 1$ in Theorem 2, then we obtain $\tau = \frac{3\tanh\left(\sqrt{\frac{5}{2}}s\right)}{\sqrt{10 - 4\tanh\left(\sqrt{\frac{5}{2}}s\right)^2}}$. Thus we have generalized Bertrand curve α with curvatures $\kappa = 1$ and $\tau = \frac{3\tanh\left(\sqrt{\frac{5}{2}}s\right)}{\sqrt{10 - 4\tanh\left(\sqrt{\frac{5}{2}}s\right)^2}}$ in Euclidean 3-space. It is easily check that α is a Salkowski curve.

Remark 3.2. If we take $\theta = \frac{\pi}{2}$ in equation 3.15, we get $N = B^*$ which means that α is a Mannheim curve and (α, α^*) is a pair Mannheim curve. Also from Theorem 2, we get $\kappa = \lambda\left(\kappa^2 + \tau^2\right)$. This equation also shows that α is a Mannheim curve .

Remark 3.3. We know that curves with constant curvatures (circular helix) are Bertrand curves. The equation (3.12) satisfies the circular helix if and only if $\theta = 0$ or $\theta = \pi$ .In this case, circular helices are not generalized by Bertrand curves but only by classical Bertrand curves.

Acknowledgements

As authors, we would like to express our sincere thanks to the referees and the Editor who contributed with their comments and suggestions for improving the paper.

REFERENCES

1. H. Balgetir, M. Bektaş and M. Ergüt: *Bertrand curves for non-null curves in 3-dimensional Lorentzian space.* Hadronic J. **27** (2)(2004), 229-236.
2. J. M. Bertrand: *Mémoire sur la théorie des courbes á double courbure.* Comptes Rendus. **36** (1850).
3. N. Ekmekci and K. İlarslan: *On Bertrand curves and their characterization.* Differ. Geom. Dyn. Syst. **3**(2) (2001), 17-24.
4. S. Honda and M. Takahashi: *Bertrand and Mannheim curves of framed curves in the 3-dimensional Euclidean space.* Turk. J. Math. **44** (2020), 883-899.
5. D. H. Jin: *Null Bertrand curves in a Lorentz manifold.* J. Korea Soc. Math. Educ. Ser. B Pure Appl. Math. **44** (3) (2008), 209-215.
6. S. Özkaldi Karakuş, K. İlarslan and Y. Yayli: *A new approach for characterization of curve couples in Euclidean 3-space.* Honam Mathematical J. **36**(1) (2014), 113-129.
7. W. Kuhnel: *Differential geometry: curves-surfaces-manifolds.* Braunschweig, Wiesbaden, 1999.
8. H. Matsuda and S. Yorozu: *Notes on Bertrand curves.* Yokohama Math. J. **50**(1-2) (2003), 41-58.

9. L. R. Pears: *Bertrand curves in Riemannian space*. J. London Math. Soc. **1-10** (2) (1935), 180-183.
10. A. Uçum, K. İlarslan and S. Özkaldi Karakuş: *Curve couples and spacelike Frenet planes in Minkowski* 3*-space*. Honam Mathematical J. **36** (3) (2014), 475-492.
11. A. Uçum, O. Keçilioğlu and K. İlarslan: *Generalized Bertrand curves with spacelike* $(1, 3)$*-normal plane in Minkowski space-time*. Turk. J. Math. **40** (2016), 487-505.
12. B. Saint Venant: *Mémoire sur les lignes courbes non planes*. Journal de l'Ecole Polytechnique. **18** (1845), 1-76.
13. C. Zhang and D. Pei: *Generalized Bertrand curves in Minkowski 3-space*. Mathematics. **8** (2020), 2199.