FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 31, No 1 (2016), 201–212

LAGRANGE SPACES WITH GENERALIZED (γ, β)-METRIC

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Abstract. The present paper deals with the differential geometry of a Lagrange space endowed with generalized (γ,β)-metric, where γ is an m^{th}-root metric and β is a 1-form. We obtain fundamental tensor, its inverse, Euler-Lagrange equations, semispray coefficients and canonical nonlinear connection for a Lagrange space with generalized (γ,β)-metric. Several other properties of such a space are also discussed.
Keywords: Lagrange space, (γ,β)-metric, fundamental tensor, Euler-Lagrange equation.

1. Introduction

Lagrange spaces with (α,β)-metric were studied by several authors such as Miron [4], Nicolaescu [1, 2], Shukla and Pandey [6]. Recently, Shukla and Pandey [7] discussed Lagrange spaces with (γ,β)-metric and obtained various results. An n-dimensional Lagrange space $L^n = (M, L(x,y))$ is said to be endowed with (γ,β)-metric if Lagrangian $L(x,y)$ is a function of $\gamma(x,y)$ and $\beta(x,y)$, where $\gamma(x,y)$ is a cubic metric and $\beta(x,y)$ is a 1-form, i.e. $\gamma = \sqrt[3]{a_{ijk}(x)y^iy^jy^k}$ and $\beta(x,y) = b_i(x)y^i$. The aim of the present paper is to generalize the notion of (γ,β)-metric by considering $\gamma(x,y)$ as an m^{th}-root metric. We call such metric as generalized (γ,β)-metric.

The paper is organized as follows. Section Two consists of some preliminary results required for the discussion of subsequent sections. It includes the notion of a Lagrange space with generalized (γ,β)-metric. In Section Three, we discuss some properties of a Lagrange space with generalized (γ,β)-metric and obtain the expression for the fundamental metric tensor g_{ij} and its inverse g^{ij}. In Section Four, we consider the variational problem in Lagrange spaces with generalized (γ,β)-metric and obtain various forms of Euler-Lagrange equations. Section Five deals with the semispray of a Lagrange space with generalized (γ,β)-metric. Section Six discusses the nonlinear connection in a Lagrange space with generalized (γ,β)-metric. In Section Seven, we give concluding remarks on the results obtained in the paper and discuss the possibilities of further work on the space under consideration.

Received July 05, 2015; Accepted November 29, 2015
2010 *Mathematics Subject Classification.* Primary 53C60

2. Preliminaries

Let M be an n-dimensional smooth manifold and let TM be its tangent bundle. Let (x^i) and (x^i, y^i) be the local coordinates on M and TM respectively. A Lagrangian is a function $L : TM \to \mathbb{R}$ which is a smooth function on $\widetilde{TM} = TM \setminus \{0\}$ and continuous on the null section. The Lagrangian $L(x, y)$ is said to be regular if $rank\left(g_{ij}(x, y)\right) = n$, where

(2.1) $$g_{ij}(x, y) = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j L$$

is a covariant symmetric tensor called the fundamental tensor of the Lagrangian $L(x, y)$ and $\dot{\partial}_i \equiv \frac{\partial}{\partial y^i}$. A Lagrange space is a pair $L^n = (M, L(x, y))$, $L(x, y)$ being a regular Lagrangian whose fundamental tensor g_{ij} has constant signature on $\widetilde{TM}$.

The integral of action of the Lagrangian $L(x, y)$ along a smooth curve $c : [0, 1] \to M$ leads to the Euler-Lagrange equations:

(2.2) $$E_i(L) \equiv \frac{\partial L}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial y^i}\right) = 0, \quad y^i = \frac{dx^i}{dt}.$$

The coefficients of the semispray S of a Lagrange space $L^n = (M, L(x, y))$ are given by

(2.3) $$G^i(x, y) = \frac{1}{4}g^{ih}\left(y^k\dot{\partial}_h\partial_k L - \partial_h L\right), \quad \partial_k \equiv \frac{\partial}{\partial x^k}.$$

The semispray S is called a canonical semispray as its coefficients depend on $L(x, y)$ only.

The coefficients of canonical nonlinear connection $N\left(N^i_j(x, y)\right)$ of a Lagrange space $L^n = (M, L(x, y))$ are given by

(2.4) $$N^i_j = \dot{\partial}_j G^i.$$

A Lagrangian $L(x, y)$ is said to be a generalized (γ, β)-metric if it is a function of γ and β, i.e.

(2.5) $$L(x, y) = \overline{L}(\gamma, \beta),$$

where

(2.6) $$\gamma^m = a_{i_1 i_2 \ldots i_m}(x) y^{i_1} y^{i_2} \ldots y^{i_m}$$

and

(2.7) $$\beta(x, y) = b_i(x) y^i.$$

We call the space $L^n = (M, L(x, y))$ determined by the Lagrangian (2.5) a *Lagrange space with generalized* (γ, β)*-metric*.

In particular, for $m = 3$, $\gamma(x, y)$ is a cubic metric and the space becomes a *Lagrange space with* (γ, β)*-metric* (*cf.* [7]). For $m = 2$, the space becomes the well known *Lagrange space with* (α, β)*-metric* , where $\alpha(x, y) = \sqrt{a_{ij}(x)y^i y^j}$ and $\beta(x, y) = b_i(x)y^i$ (*cf.* [1, 2]).

For basic notations and terminology related to a Lagrange space, we refer to the books [3] and [5].

3. Fundamental tensor

If we differentiate (2.6) partially with respect to y^j and use the symmetry of $a_{i_1 i_2 \ldots i_m}$ in its indices, we obtain

$$\dot\partial_j \gamma = \gamma^{-(m-1)} a_j(x, y), \tag{3.1}$$

where $a_j(x, y) = a_{j i_2 \ldots i_m}(x) y^{i_2} \ldots y^{i_m}$.

Again differentiating (3.1) partially with respect to y^h, using symmetry of $a_{i_1 i_2 \ldots i_m}(x)$ in its indices and simplifying, we find

$$\dot\partial_j \dot\partial_h \gamma = (m-1)\gamma^{-(m-1)} a_{jh}(x, y) - (m-1)\gamma^{-(2m-1)} a_j a_h, \tag{3.2}$$

where $a_{jh} = a_{j h i_3 \ldots i_m}(x) y^{i_3} \ldots y^{i_m}$.

Further differentiation of (3.2) with respect to y^l yields

$$\begin{aligned} \dot\partial_j \dot\partial_h \dot\partial_l \gamma = {} & (m-1)(2m-1)\gamma^{-(3m-1)} a_j a_h a_l \\ & - (m-1)^2 \gamma^{-(2m-1)} \underset{jhl}{\mathfrak{S}} \{a_j a_{hl}\} + (m-1)(m-2) a_{jhl}\,, \end{aligned} \tag{3.3}$$

where $a_{jhl}(x, y) = a_{j h l i_4 \ldots i_m}(x) y^{i_4} \ldots y^{i_m}$ and $\underset{jhl}{\mathfrak{S}}$ represents the cyclic sum with respect to the indices j, h & l.

Differentiating (2.7) partially with respect to y^j, we have

$$\dot\partial_j \beta = b_j(x). \tag{3.4}$$

Further differentiating (3.4) partially with respect to y^h, we get

$$\dot\partial_j \dot\partial_h \beta = 0. \tag{3.5}$$

Thus, we have

Proposition 3.1. *In a Lagrange space L^n with generalized (γ, β)-metric, the following hold good:*

$$\begin{aligned}
&\dot{\partial}_j\gamma = \gamma^{-(m-1)}a_j(x,y),\\
&\dot{\partial}_j\dot{\partial}_h\gamma = (m-1)\gamma^{-(m-1)}a_{jh}(x,y) - (m-1)\gamma^{-(2m-1)}a_j\, a_h\,,\\
&\dot{\partial}_j\dot{\partial}_h\dot{\partial}_l\gamma = (m-1)\,(2m-1)\gamma^{-(3m-1)}a_j\, a_h\, a_l\\
&\qquad - (m-1)^2\gamma^{-(2m-1)} \underset{j\,h\,l}{\mathfrak{S}}\{a_j\, a_{hl}\} + (m-1)\,(m-2)a_{jhl}\,,\\
&\dot{\partial}_j\beta = b_j(x), \quad \dot{\partial}_j\dot{\partial}_h\beta = 0,
\end{aligned}$$

where

$$\begin{aligned}
a_j(x,y) &= a_{j\,i_2\,\ldots\,i_m}(x)y^{i_2}\ \ldots\ y^{i_m}\,,\\
a_{jh} &= a_{j\,h\,i_3\,\ldots\,i_m}(x)y^{i_3}\ \ldots\ y^{i_m}\,,\\
a_{j\,h\,l}(x,y) &= a_{j\,h\,l\,i_4\,\ldots\,i_m}(x)y^{i_4}\ \ldots\ y^{i_m}.
\end{aligned}$$

The moments of Lagrangian $L(x, y)$ are given by

$$p_i := \frac{1}{2}\dot{\partial}_i L. \tag{3.6}$$

In our case, the Lagrangian $L(x, y)$ is a function of γ and β only (*vide* (2.5)). Therefore, we have

$$p_i = \frac{1}{2}(\overline{L}_\gamma\dot{\partial}_i\gamma + \overline{L}_\beta\dot{\partial}_i\beta), \tag{3.7}$$

where $\overline{L}_\gamma = \frac{\partial\overline{L}}{\partial\gamma}$, $\overline{L}_\beta = \frac{\partial\overline{L}}{\partial\beta}$.

Using (3.1) and (3.4) in (3.7), we obtain

$$p_i = \frac{1}{2}(\gamma^{-(m-1)}\overline{L}_\gamma a_i + \overline{L}_\beta b_i). \tag{3.8}$$

Thus, we have

Theorem 3.1. *In a Lagrange space L^n with generalized (γ, β)-metric, the moments of Lagrangian $L(x, y)$ are given by*

$$p_i = \rho a_i + \rho_1 b_i, \tag{3.9}$$

where

$$\rho = \frac{1}{2}\gamma^{-(m-1)}\overline{L}_\gamma \tag{3.10}$$

and

$$\rho_1 = \frac{1}{2}\overline{L}_\beta. \tag{3.11}$$

Remarks 3.2. *The scalars ρ and ρ_1 appearing in Theorem 3.1 are called the* principal invariants *of the space L^n.*

Differentiating (3.10) and (3.11) partially with respect to y^j and simplifying, we respectively have

$$\dot{\partial}_j \rho = \frac{1}{2}\gamma^{-2(m-1)}\left(\overline{L}_{\gamma\gamma} - (m-1)\gamma^{-1}\overline{L}_\gamma\right)a_j + \frac{1}{2}\gamma^{-(m-1)}\overline{L}_{\gamma\beta}b_j \tag{3.12}$$

and

$$\dot{\partial}_j \rho_1 = \frac{1}{2}\gamma^{-(m-1)}\overline{L}_{\beta\gamma}a_j + \frac{1}{2}\overline{L}_{\beta\beta}b_j, \tag{3.13}$$

where

$$\overline{L}_{\gamma\gamma} = \frac{\partial^2\overline{L}}{\partial\gamma^2},\ \ \overline{L}_{\gamma\beta} = \frac{\partial^2\overline{L}}{\partial\gamma\partial\beta} = \frac{\partial^2\overline{L}}{\partial\beta\partial\gamma} = \overline{L}_{\beta\gamma},\ \ \overline{L}_{\beta\beta} = \frac{\partial^2\overline{L}}{\partial\beta^2}.$$

Thus, we have the following:

Proposition 3.2. *The derivatives of the principal invariants of a Lagrange space L^n with generalized (γ, β)-metric are given by*

$$\dot{\partial}_j\rho = \rho_{-2}a_j + \rho_{-1}b_j,\ \ \dot{\partial}_j\rho_1 = \rho_{-1}a_j + \rho_0 b_j, \tag{3.14}$$

with

$$\rho_{-2} = \frac{1}{2}\gamma^{-2(m-1)}\left(\overline{L}_{\gamma\gamma} - (m-1)\gamma^{-1}\overline{L}_\gamma\right),\ \ \rho_{-1} = \frac{1}{2}\gamma^{-(m-1)}\overline{L}_{\gamma\beta} \tag{3.15}$$

and

$$\rho_0 = \frac{1}{2}\overline{L}_{\beta\beta}. \tag{3.16}$$

The energy of Lagrangian $L(x, y)$ is defined as

$$E_L := y^i\dot{\partial}_i L - L. \tag{3.17}$$

Using (2.5) in (3.17), we have

$$E_{\overline{L}} = y^i(\overline{L}_\gamma\,\dot{\partial}_i\gamma + \overline{L}_\beta\,\dot{\partial}_i\beta) - \overline{L}. \tag{3.18}$$

Since γ and β are positively homogeneous of degree one in y^i, by virtue of Euler's theorem on homogeneous functions, we have

$$y^i\dot{\partial}_i\gamma = \gamma \quad \text{and} \quad y^i\dot{\partial}_i\beta = \beta. \tag{3.19}$$

In view of (3.19), (3.18) takes the form

$$E_{\overline{L}} = \gamma\overline{L}_\gamma + \beta\overline{L}_\beta - \overline{L}. \tag{3.20}$$

Thus, we have

Theorem 3.3. *In a Lagrange space with generalized (γ, β)-metric, the energy of the Lagrangian $L(x, y)$ is given by (3.20).*

Now, we find expression for the fundamental tensor $g_{ij}(x, y)$ of a Lagrange space with generalized (γ, β)-metric. Using (2.5) in (2.1), we have

$$g_{ij} = \frac{1}{2}\Big[\left(\overline{L}_{\gamma\gamma}\,\dot\partial_i\gamma + \overline{L}_{\gamma\beta}\,\dot\partial_i\beta\right)\dot\partial_j\gamma + \overline{L}_\gamma\,\dot\partial_i\,\dot\partial_j\,\gamma + \left(\overline{L}_{\beta\gamma}\,\dot\partial_i\gamma + \overline{L}_{\beta\beta}\,\dot\partial_i\beta\right)\dot\partial_j\beta + \overline{L}_\beta\,\dot\partial_i\,\dot\partial_j\beta\Big]. \tag{3.21}$$

In view of Proposition 3.1, (3.21) takes the form

$$g_{ij}(x, y) = (m-1)\rho a_{ij} + \rho_{-2}a_ia_j + \rho_{-1}(a_ib_j + a_jb_i) + \rho_0 b_ib_j. \tag{3.22}$$

Equation (3.22) can be written as

$$g_{ij}(x, y) = (m-1)\rho a_{ij} + c_ic_j, \tag{3.23}$$

where

$$c_i = q_{-1}a_i + q_0b_i \tag{3.24}$$

and q_{-1}, q_0 satisfy

$$(a)\ \ q_0q_{-1} = \rho_{-1}, \quad (b)\ \ (q_{-1})^2 = \rho_{-2}, \quad (c)\ \ q_0^2 = \rho_0. \tag{3.25}$$

Thus, we have

Theorem 3.4. *The fundamental tensor of a Lagrange space with generalized (γ, β)-metric is given by (3.23).*

The following result gives the expression for the inverse of g_{ij}.

Theorem 3.5. *The inverse g^{ij} of the fundamental tensor g_{ij} of a Lagrange space with generalized (γ, β)-metric is given by*

$$g^{ij} = \frac{1}{(m-1)\,\rho}\left(a^{ij} - \frac{1}{(m-1)\,\rho + c^2}c^ic^j\right), \tag{3.26}$$

where

$$(a)\ \ c^i = a^{ir}c_r, \quad (b)\ \ c^2 = a^{ij}\,c_ic_j. \tag{3.27}$$

Proof. Let (a^{ij}) be the inverse of the nonsingular matrix (a_{ij}). Consider the matrix (g^{ij}) given by (3.26). Now

$$\begin{aligned}
g_{ij}\,g^{jk} &= \left[(m-1)\,\rho\,a_{ij} + c_i\,c_j\right]\frac{1}{(m-1)\,\rho}\left(a^{jk} - \frac{c^j\,c^k}{(m-1)\,\rho + c^2}\right)\\
&= \delta^j_k - \frac{a_{ij}\,c^j\,c^k}{(m-1)\rho + c^2} + \frac{a^{jk}\,c_i\,c_j}{(m-1)\rho} - \frac{c_i\,c_j\,c^j\,c^k}{(m-1)\rho\{(m-1)\,\rho + c^2\}}\\
&= \delta^j_k - \frac{c_i\,c^k}{(m-1)\,\rho + c^2} + \frac{c_i\,c^k}{(m-1)\rho} - \frac{c^2\,c_i\,c^k}{(m-1)\,\rho\{(m-1)\,\rho + c^2\}}\\
&= \delta^j_k\,.
\end{aligned}$$

This shows that the matrix (g_{ij}) given by (3.23) is nondegenerate and its inverse (g^{ij}) is given by (3.26). □

Remarks 3.6. *Substituting $m = 3$ and $m = 2$ in the expressions obtained in Proposition 3.1, Theorem 3.1, Proposition 3.2, Theorem 3.4 and Theorem 3.5, we obtain the corresponding results for a Lagrange space with (γ, β)- and (α, β)-metrics, respectively (cf.* [1, 2, 7]*)*

4. Euler-Lagrange equations

Using (2.5) in (2.2), we obtain

$$E_i(\overline{L}) \equiv \frac{\partial \overline{L}}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial \overline{L}}{\partial y^i}\right) = 0, \quad y^i = \frac{dx^i}{dt}. \tag{4.1}$$

For the Lagrangian $\overline{L}$ given by (2.5), we have

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial \overline{L}}{\partial y^i}\right) =& \left(\overline{L}_{\gamma\gamma}\frac{d\gamma}{dt} + \overline{L}_{\gamma\beta}\frac{d\beta}{dt}\right)\frac{\partial \gamma}{\partial y^i} + \left(\overline{L}_{\beta\gamma}\frac{d\gamma}{dt} + \overline{L}_{\beta\beta}\frac{d\beta}{dt}\right)\frac{\partial \beta}{\partial y^i}\\ &+ \overline{L}_{\gamma}\frac{d}{dt}\left(\frac{\partial \gamma}{\partial y^i}\right) + \overline{L}_{\beta}\frac{d}{dt}\left(\frac{\partial \beta}{\partial y^i}\right).\end{aligned} \tag{4.2}$$

In view of $\partial_i\overline{L} = \overline{L}_\gamma\partial_i\gamma + \overline{L}_\beta\partial_i\beta$ and (4.2), (4.1) takes the form

$$\begin{aligned}E_i(\overline{L}) =& \overline{L}_\gamma E_i(\gamma) + \overline{L}_\beta E_i(\beta) - \left(\overline{L}_{\gamma\gamma}\frac{d\gamma}{dt} + \overline{L}_{\gamma\beta}\frac{d\beta}{dt}\right)\frac{\partial \gamma}{\partial y^i}\\ &- \left(\overline{L}_{\beta\gamma}\frac{d\gamma}{dt} + \overline{L}_{\beta\beta}\frac{d\beta}{dt}\right)\frac{\partial \beta}{\partial y^i}.\end{aligned} \tag{4.3}$$

Since

$$E_i(\gamma^m) = m\gamma^{m-1}E_i(\gamma) - m\,\frac{\partial \gamma}{\partial y^i}\frac{d\gamma^{m-1}}{dt},$$

we get

$$E_i(\gamma) = \frac{1}{m}\gamma^{-(m-1)}E_i(\gamma^m) + \gamma^{-(m-1)}\frac{\partial \gamma}{\partial y^i}\frac{d\gamma^{m-1}}{dt}. \tag{4.4}$$

From $E_i(\beta) = \frac{\partial \beta}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial \beta}{\partial y^i}\right)$, we have

$$E_i(\beta) = 2F_{ir}y^r, \quad y^r = \frac{dx^r}{dt}, \tag{4.5}$$

where

$$F_{ir} = \frac{1}{2}\left(\frac{\partial b_r}{\partial x^i} - \frac{\partial b_i}{\partial x^r}\right) \tag{4.6}$$

is the electromagnetic tensor field of the potentials b_i.

Using (4.4) and (4.5) in (4.3), we obtain

$$
\begin{aligned}
E_i(\overline{L}) =& \frac{2}{m}\left(\frac{1}{2}\gamma^{-(m-1)}\overline{L}_\gamma\right)E_i(\gamma^m) + 2\left(\frac{1}{2}\gamma^{-(m-1)}\overline{L}_\gamma\right)\frac{\partial\gamma}{\partial y^i}\frac{d\gamma^{m-1}}{dt} \\
&+ 4\left(\frac{1}{2}\overline{L}_\beta\right)F_{ir}y^r - \frac{\partial\gamma}{\partial y^i}\left(\overline{L}_{\gamma\gamma}\frac{d\gamma}{dt} + \overline{L}_{\gamma\beta}\frac{d\beta}{dt}\right) \\
&- \frac{\partial\beta}{\partial y^i}\left(\overline{L}_{\beta\gamma}\frac{d\gamma}{dt} + \overline{L}_{\beta\beta}\frac{d\beta}{dt}\right).
\end{aligned} \tag{4.7}
$$

Thus, we have

Theorem 4.1. *The Euler-Lagrange equations of a Lagrange space with generalized (γ, β)-metric are of the following form:*

$$
\begin{aligned}
E_i(\overline{L}) \equiv& \frac{2}{m}\rho E_i(\gamma^m) + 2\rho\frac{\partial\gamma}{\partial y^i}\frac{d\gamma^{m-1}}{dt} + 4\rho_1 F_{ir}y^r - \frac{\partial\gamma}{\partial y^i}\left(\overline{L}_{\gamma\gamma}\frac{d\gamma}{dt} + \overline{L}_{\gamma\beta}\frac{d\beta}{dt}\right) \\
&- \frac{\partial\beta}{\partial y^i}\left(\overline{L}_{\beta\gamma}\frac{d\gamma}{dt} + \overline{L}_{\beta\beta}\frac{d\beta}{dt}\right) = 0, \quad y^i = \frac{dx^i}{dt}.
\end{aligned} \tag{4.8}
$$

For the natural parametrization of the curve $c : t \in [0,1] \mapsto x^i(t) \in M$ with respect to the m^{th}-root metric $a_{i_1 \dots i_m}(x)$, $\gamma\left(x, \frac{dx}{dt}\right) = 1$.

Thus, we have the following:

Theorem 4.2. *In the natural parametrization, the Euler-Lagrange equations of a Lagrange space with generalized (γ, β)-metric are*

$$
E_i(\overline{L}) \equiv \frac{2}{m}\rho E_i(\gamma^m) + 4\rho_1 F_{ir}y^r - \frac{\partial\gamma}{\partial y^i}\overline{L}_{\gamma\beta}\frac{d\beta}{ds} - \frac{\partial\beta}{\partial y^i}\overline{L}_{\beta\beta}\frac{d\beta}{ds} = 0. \tag{4.9}
$$

If β is constant on the integral curve c of the Euler-Lagrange equations with natural parametrization, then (4.9) takes the form

$$
E_i(\overline{L}) \equiv \frac{2}{m}\rho E_i(\gamma^m) + 4\rho_1 F_{ir}y^r = 0. \tag{4.10}
$$

Thus, we have

Theorem 4.3. *If β is constant along the integral curve of the Euler-Lagrange equations with natural parametrization, then the Euler-Lagrange equations of the Lagrange space with generalized (γ, β)-metric are given by (4.10).*

Remarks 4.4. *Substituting $m = 3$ and $m = 2$ in the Euler-Lagrange equations in Theorem 4.1, Theorem 4.2 and Theorem 4.3, we obtain corresponding forms of Euler-Lagrange equations in Lagrange space with (γ, β)- and (α, β)-metrics, respectively (cf. [1, 2, 7]).*

5. Canonical semispray

In this section, we obtain the coefficients of the canonical semispray of a Lagrange space with generalized (γ, β)-metric.

Using (2.5) in (2.3), we obtain

$$G^i(x,y) = \frac{1}{4} g^{ih} \left(y^k \dot{\partial}_h \partial_k \overline{L} - \partial_h \overline{L} \right). \tag{5.1}$$

Since $\gamma^m = a_{i_1 i_2 \ldots i_m}(x) y^{i_1} y^{i_2} \ldots y^{i_m}$ and $\beta = b_i(x)\, y^i$, we have

$$\partial_h \gamma = A_h \gamma^{-(m-1)}, \quad \partial_h \beta = B_h, \tag{5.2}$$

where

$$A_h = \frac{1}{m} (\partial_h a_{i_1 i_2 \ldots i_m}) y^{i_1} y^{i_2} \ldots y^{i_m}, \quad B_h = (\partial_h b_i) y^i. \tag{5.3}$$

Using (3.10), (3.11) and (5.2) in $\partial_k \overline{L} = \overline{L}_\gamma \partial_k \gamma + \overline{L}_\beta \partial_k \beta$, we get

$$\partial_k \overline{L} = 2\rho A_k + 2\rho_1 B_k. \tag{5.4}$$

Differentiating (5.4) partially with respect to y^h and simplifying, we have

$$\dot{\partial}_h \partial_k \overline{L} = 2(\rho_{-2} a_h + \rho_{-1} b_h) A_k + 2\rho A_{kh} + 2(\rho_{-1} a_h + \rho_0 b_h) B_k + 2\rho_1 b_{kh}, \tag{5.5}$$

where

$$(a) \;\; A_{kh} = \dot{\partial}_h A_k, \quad (b) \;\; b_{kh} = \dot{\partial}_h B_k. \tag{5.6}$$

Using (5.4) and (5.5) in (5.1), we obtain

$$\begin{aligned} G^i =& \frac{1}{2} g^{ih} \Big[(\rho_{-2} A_0 + \rho_{-1} B_0) a_h + (\rho_{-1} A_0 + \rho_0 B_0) b_h + \rho A_{0h} \\ & + \rho_1 b_{0h} - (\rho A_h + \rho_1 B_h) \Big], \end{aligned} \tag{5.7}$$

where

$$\begin{aligned} &(i) \quad A_0 = A_k(x,y) y^k, \quad (ii) \quad B_0 = B_k(x,y) y^k, \\ &(iii) \;\; A_{0h} = A_{kh}(x,y) y^k, \quad (iv) \;\; b_{0h} = b_{kh}(x,y) y^k. \end{aligned} \tag{5.8}$$

Thus, we have

Theorem 5.1. *The local coefficients of canonical semispray of a Lagrange space with generalized (γ, β)-metric are given by (5.7).*

6. Canonical nonlinear connection

In this section, we obtain the local coefficients of the canonical nonlinear connection of a Lagrange space with generalized (γ, β)-metric.

Partial differentiation of $g^{ih}g_{is} = \delta^h_s$, with respect to y^j, yields

$$\dot{\partial}_j g^{ih} = -2g^{rh}C^i_{rj}. \tag{6.1}$$

If we partially differentiate the quantities appearing in (3.15) and (5.8) with respect to y^j, we find the following quantities:

$$\begin{cases} \dot{\partial}_j \rho_{-2} = \mu_{-3}a_j + \mu_{-2}b_j, \quad \dot{\partial}_j \rho_{-1} = \mu_{-2}a_j + \mu_{-1}b_j, \\ \dot{\partial}_j \rho_0 = \mu_{-1}a_j + \mu_0 b_j, \\ \dot{\partial}_j A_0 = A_j + A_{0j}, \quad \dot{\partial}_j B_0 = \underset{s\,j}{\mathfrak{S}}\{\partial_s b_j\} y^s, \\ \dot{\partial}_j A_{0h} = 2A_{0hj} + A_{jh}, \quad \dot{\partial}_j b_{0h} = b_{jh}, \end{cases} \tag{6.2}$$

where

$$\begin{cases} \mu_{-3} = \frac{1}{2}\gamma^{-3(m-1)}\left[\overline{L}_{\gamma\gamma\gamma} - 3(m-1)\gamma^{-1}\overline{L}_{\gamma\gamma} + (2m-1)(m-1)\gamma^{-2}\overline{L}_{\gamma}\right], \\ \mu_{-2} = \frac{1}{2}\gamma^{-2(m-1)}\left[\overline{L}_{\gamma\gamma\beta} - (m-1)\gamma^{-1}\overline{L}_{\gamma\beta}\right], \\ \mu_{-1} = \frac{1}{2}\gamma^{-(m-1)}\overline{L}_{\gamma\gamma\beta}, \quad \mu_0 = \frac{1}{2}\overline{L}_{\beta\beta\beta}, \\ A_{0hj} = A_{rhj}y^r, \quad A_{rhj} = \partial_r a_{hj}. \end{cases} \tag{6.3}$$

Also, we have

$$\dot{\partial}_j a_h = (m-1)\, a_{jh}. \tag{6.4}$$

Now, applying (5.7) in (2.4), we get

$$\begin{aligned} N^i_j =& \frac{1}{2}\left(\dot{\partial}_j g^{ih}\right)\left((\rho_{-2}A_0 + \rho_{-1}B_0)\, a_h + (\rho_{-1}A_0 + \rho_0 B_0)\, b_h + \rho_{-2}(\dot{\partial}_h A_0)\right. \\ & \left. + \rho A_{0h} + \rho_1 b_{0h} - (\rho A_h + \rho_1 B_h)\right) + \frac{1}{2}g^{ih}\Big[((\dot{\partial}_j\rho_{-2})A_0 + (\dot{\partial}_j\rho_{-1})B_0 \\ & + \rho_{-1}(\dot{\partial}_j B_0))a_h + (\rho_{-2}A_0 + \rho_{-1}B_0)\dot{\partial}_j a_h + ((\dot{\partial}_j\rho_{-1})A_0 + \rho_{-1}(\dot{\partial}_j A_0) \\ & + (\dot{\partial}_j\rho_0)B_0 + \rho_0(\dot{\partial}_j B_0))b_h + (\dot{\partial}_j\rho)A_{0h} + (\dot{\partial}_j\rho_1)b_{0h} + \rho\dot{\partial}_j A_{0h} \\ & + \rho_1\dot{\partial}_j b_{0h} - ((\dot{\partial}_j\rho)A_h + \rho(\dot{\partial}_j A_h) + (\dot{\partial}_j\rho_1)B_h + \rho_1(\dot{\partial}_j B_h))\Big]. \end{aligned} \tag{6.5}$$

Using (3.14), (5.6), (5.8), (6.1), (6.2) and (6.4) in (6.5) and simplifying, we obtain

$$
\begin{aligned}
N^i_j = & -2C^i_{rj}G^r + \frac{1}{2}g^{ih}\Big[\rho_{-2}((A_j + A_{0j})a_h + (A_{0h} - A_h)a_j + (m-1)\,A_0 a_{jh}) \\
& + \rho_{-1}((A_j + A_{0j})b_h + (A_{0h} - A_h)b_j + a_j b_{0h} + \underset{sj}{\mathfrak{S}}\{\partial_s b_j\}y^s a_h \\
& + (m-1)\,B_0 a_{jh} - a_j B_h) + \rho_0\left(\underset{sj}{\mathfrak{S}}\{\partial_s b_j\}y^s b_h - b_j B_h + b_j b_{0h}\right) \\
& + \rho_1 \underset{jh}{\mathfrak{M}}\{b_{jh}\} + \rho\left(2A_{0hj} + A_{jh} - A_{hj}\right) + \mu_{-3}A_0 a_j a_h + \mu_{-2}(A_0 b_j a_h \\
& + a_j(B_0 a_h + A_0 b_h)) + \mu_{-1}\left(b_j\,(B_0 a_h + A_0 b_h) + a_j B_0 b_h\right) + \mu_0 B_0 b_j b_h\Big],
\end{aligned} \tag{6.6}
$$

where $\underset{jh}{\mathfrak{M}}$ stands for interchange of indices j & h and difference.

Thus, we have

Theorem 6.1. *The local coefficients of the canonical nonlinear connection of a Lagrange space with generalized* (γ, β)*-metric are given by* (6.6).

Remarks 6.2. *Substituting* $m = 3$ *and* $m = 2$ *in* (6.6)*, we obtain the local coefficients of the nonlinear connection in Lagrange space with* (γ, β)*- and* (α, β)*-metrics, respectively (cf.* [1, 2, 7]*).*

7. Conclusions

In the paper, we have developed the theory of Lagrange spaces with generalized (γ, β)-metric. It presents a significant generalization of the earlier works of Nicolaescu [1, 2], and Shukla and Pandey [7]. The expressions for the geometric objects obtained in the paper may be useful in further work on the spaces under consideration. The importance of the results lies in the study of canonical metrical d-connection, curvatures and torsions in such spaces. The expressions for canonical semispray and nonlinear connection, obtained respectively in Section 5 and Section 6 may be applicable in geodesic correspondences between two Lagrange spaces with different generalized (γ, β)-metrics on the same underlying manifold. It is a matter of later investigations to look into the aforesaid applications of the results obtained in the paper.

Acknowledgments

S. K. Shukla gratefully acknowledges the financial support provided by the Council of Scientific and Industrial Research (CSIR), India.

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