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LIFTS OF $F(\alpha, \beta)(3, 2, 1)$-STRUCTURES FROM MANIFOLDS TO TANGENT BUNDLES

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Abstract. The aim of the present paper is to explore the lifts of an $f(\alpha, \beta)(3, 2, 1)$-structure and obtain its partial integrability and integrability conditions on the tangent bundle. Also, the prolongation of an $f(\alpha, \beta)(3, 2, 1)$-structure on the third tangent bundle T_3M is studied.
Keywords: Lifts, Nijenhuis tensor, Partial differential equations, Projection tensors, Integrability.

1. Introduction

The notion of the polynomial structure of degree n

$$Q(F) = F^n + a_n F^{n-1} + ... + a_2 F + a_1 I,$$

where F is the tensor field of type (1,1) and I is the identity tensor field on a differentiable manifold was introduced by Goldenberg et. al. [7, 9]. Recently, Gök et. al. [8] have defined an $f(\alpha, \beta)(3, 2, 1)$-structure on a differentiable manifold, where $\alpha, \beta \in R$ and $\beta \neq 0$ and established its some fundamental properties. They investigated its partial integrability and integrability conditions.

On the other hand, let us consider the tangent bundle TM of a manifold M. Tangent bundle is a primary field of differential geometry used to investigate geometrical structures and their properties such as integrability, curvature, Lie derivative,

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etc. Yano and Ishihara [24] introduced and studied structures like almost complex structures with some basic properties induced in tangent bundles. Das and Khan [3] have researched these lifts of an almost product structure over an almost r-contact structure along with TM. The work of several scholars on various geometric structures and connections have been extremely beneficial, for instance, Dida et. al. [4, 5, 6], Khan and De [11, 19], Khan [14, 15, 16], Omran at el [20], Peyghan et. al. [21], Tekkoyun [22] and Yano and Ishihara [24].

The main purpose of this paper is to study the lifts of an $f(\alpha,\beta)(3,2,1)$-structure on manifolds on the tangent bundle and establish its partial integrability and integrability conditions. Finally, the prolongation of an $f(\alpha,\beta)(3,2,1)$-structure on the third tangent bundle T_3M is studied.

Let M be an n-dimensional differentiable manifold. A non-null tensor field F of type (1, 1) on M is called an $f(\alpha,\beta)(3,2,1)$-structure if it satisfies the equation

$$F^3 = \alpha F^2 + \beta F, \tag{1.1}$$

where $\alpha, \beta \in R$ and $\beta \neq 0$. Hence, the $f(\alpha,\beta)(3,2,1)$-structure F is a polynomial structure of degree 3 with the structure polynomial $x^3 - \alpha x^2 - \beta x = 0$. Also, the pair (M, F) is said to be an $f(\alpha,\beta)(3,2,1)$-manifold. F is of constant rank n everywhere in M

Let l and m be operators defined as

$$\begin{aligned} (a) \quad l &= \frac{F^2 - \alpha F}{\beta}, \\ (b) \quad m &= \frac{-F^2 + \alpha F + \beta I}{\beta}. \end{aligned} \tag{1.2}$$

The operators l and m defined in the equation (1.2) satisfy the following identities:

$$\begin{aligned} l + m &= 0, \\ l^2 &= l, \quad m^2 = m, \quad lm = ml = 0, \\ Fl &= lF = F, \quad Fm = mF = 0. \end{aligned} \tag{1.3}$$

Thus there exist two complementary distributions D_l and D_m corresponding to the projection tensors l and m respectively in M.

2. The complete lift of an $f(\alpha,\beta)(3,2,1)$-structure in the tangent bundle

Let M be an m-dimensional differentiable manifold of class C^∞ and $T_p(M)$ the tangent space at a point p of M then $T(M) = \cup_{p\in M} T_pM$ is a tangent bundle over the manifold M. The tangent bundle TM of M is a differentiable manifold of dimension $2n$. Let $\wp^r_s$ denote the set of tensor field of class C^∞ and type (r,s) in M and $\wp^r_s(T(M))$ denote the corresponding set of tensor fields in $T(M)$ [12, 13, 10].

Let F, G be elements of $\wp^1_1(M)$. Then we have [23]

$$(FG)^C = F^C G^C. \tag{2.1}$$

Putting $F = G$ in the equation (2.3), we obtain

$$(F^2)^C = (F^C)^2. \tag{2.2}$$

Also,

$$(F+G)^C = F^C + G^C. \tag{2.3}$$

Operating the complete lifts of both sides of the equation (1.1), we get

$$\begin{aligned} (F^3)^C &= (\alpha F^2 + \beta F)^C, \\ (F^3)^C &= (\alpha F^2)^C + (\beta F)^C. \end{aligned}$$

In the view of (2.2) and $I^C = I$, we get

$$(F^C)^3 = \alpha(F^C)^2 + \beta F^C. \tag{2.4}$$

In the view of equations (1.1), (2.4) and [23], we can easily say that the rank of F^C is $2n$ if and only if the rank of F is n. Therefore, we have the following theorems:

Theorem 2.1. *Let $F \in \wp_1^1(M)$ be a $f(\alpha,\beta)(3,2,1)$-structure in M, then its complete lift F^C is also an $f(\alpha,\beta)(3,2,1)$-structure in TM.*

Theorem 2.2. *The $f(\alpha,\beta)(3,2,1)$-structure F of rank n in M if and only if its complete lift F^C is of rank $2n$ in TM.*

Let F be a $f(\alpha,\beta)(3,2,1)$-structure of rank n in M. Then the complete lift l^C of l and m^C of m are complementary projection tensors in TM. Thus there exist two complementary distributions D_{l^C} and D_{m^C} determined by l^C and m^C respectively in TM. The distributions D_{l^C} and D_{m^C} are respectively the complete lifts of D_l^C and D_m^C of D_l and D_m [3].

3. Integrability conditions of an $f(\alpha,\beta)(3,2,1)$-structure in the tangent bundle

Let F be the $f(\alpha,\beta)(3,2,1)$-structure that is $F^3 = \alpha F^2 + \beta F$. Then the Nijenhuis tensor N of F is a tensor of type (1,2) given by [17, 18]

$$N(X,Y) = [FX,FY] - F[FX,Y] - F[X,FY] + F^2[X,Y]. \tag{3.1}$$

Let N^C be the Nijenhuis tensor of F^C in TM, then we have

$$\begin{aligned} N^C(X^C,Y^C) &= [F^C X^C, F^C Y^C] - F^C[F^C X^C, Y^C] \\ &- F^C[X^C, F^C Y^C] + (F^2)^C[X^C,Y^C]. \end{aligned} \tag{3.2}$$

Let $X, Y \in \mathrm{Im}_0^1(M)$ and $F \in \wp_1^1(M)$, we have

$$\begin{aligned} [X^C,Y^C] &= [X,Y]^C, \\ (X+Y)^C &= X^C + Y^C, \\ F^C X^C &= (FX)^C. \end{aligned} \tag{3.3}$$

In the view of equations (1.3) and (3.3), we get

$$\begin{aligned} F^C l^C &= (Fl)^C = F^C, \\ F^C m^C &= (Fm)^C = 0. \end{aligned} \tag{3.4}$$

Theorem 3.1. *The following identities hold:*

$$\begin{aligned} N^C(m^C X^C, m^C Y^C) &= \alpha F^C[m^C X^C, m^C Y^C] \\ &+ \beta[m^C X^C, m^C Y^C], \end{aligned} \tag{3.5}$$

$$m^C N^C(X^C, Y^C) = m^C[F^C X^C, F^C Y^C], \tag{3.6}$$

$$m^C(l^C X^C, l^C Y^C) = m^C[F^C X^C, F^C Y^C], \tag{3.7}$$

$$m^C N^C((F^2 - \alpha F)^C X^C, (F^2 - \alpha F)^C Y^C) = \beta^2 m^C N^C(l^C X^C, l^C Y^C). \tag{3.8}$$

Proof: The proof of equations (3.5) to (3.8) is followed by virtue of equations (1.3), (3.4) and (3.1).

Theorem 3.2. *Let $X, Y \in \wp_0^1(M)$, the following conditions are equivalent*

$$\begin{aligned} (a) \quad m^C N^C(X^C, Y^C) &= 0, \\ (b) \quad m^C N^C(l^C X^C, l^C Y^C) &= 0, \\ (c) \quad m^C N^C((F^2 - \alpha F)^C X^C, (F^2 - \alpha F)^C Y^C) &= 0. \end{aligned}$$

Proof: In consequence of the equation (3.8), we have

$$N^C(l^C X^C, l^C Y^C) = 0 \leftrightarrow N^C((F^2 - \alpha F)^C X^C, (F^2 - \alpha F)^C Y^C) = 0.$$

Now the right sides of the equations (3.6), (3.7) are equal which in view of the last equation shows that conditions (a), (b), and (c) are equivalent.

Theorem 3.3. *The complete lift D_m^C in TM of a distribution D_m in M is integral if D_m is integrable in M.*

Proof: The distribution D_m is integral if and only if [23]

$$l[mX, mY] = 0, \tag{3.9}$$

for all $X, Y \in \wp(M)$, where $l = I - m$. Operating complete lift of both sides and using (3.5), we get

$$l^C[m^C X^C, m^C Y^C] = 0, \tag{3.10}$$

for all $X, Y \in \wp(M)$, where $l^C = (I - m)^C = I - m^C$ is the projection tensor complementary to m^C. Thus the condition (3.9) implies (3.10).

Theorem 3.4. *The complete lift D_m^C in TM of a distribution D_m in M is integral if $l^C N^C(m^C X^C, m^C Y^C) = 0$, or equivalently $N^C(m^C X^C, m^C Y^C) = 0$, for all $X, Y \in \wp_0^1(M)$.*

Proof: The distribution D_m is integral in M if and only if [23]

$$N(mX, mY) = 0, \tag{3.11}$$

for all $X, Y \in \wp(M)$. By virtue of condition (3.5), we have

$$N^C(m^C X^C, m^C Y^C) = (F^2)^C(m^C X^C, m^C Y^C)$$

Multiplying throughout by l^C, we get

$$l^C N^C(m^C X^C, m^C Y^C) = (F^2)^C l^C(m^C X^C, m^C Y^C).$$

In view of (3.10), the above relation becomes

$$l^C N^C(m^C X^C, m^C Y^C) = 0. \tag{3.12}$$

Also, we have

$$m^C N^C(m^C X^C, m^C Y^C) = 0. \tag{3.13}$$

Adding equations (3.12) and (3.13), we get

$$(l^C + m^C)N^C(m^C X^C, m^C Y^C) = 0.$$

Since $l^C + m^C = I^C = I$, we have

$$N^C(m^C X^C, m^C Y^C) = 0.$$

Theorem 3.5. *Let the distribution D_l be integrable in M, that is $mN(X,Y) = 0$ for all $X, Y \in \wp_0^1(M)$. Then the distribution D_l^C is integrable in TM if and only if the one of the conditions of Theorem (3.2) is satisfied.*

Proof: The distribution D_l is integral in M if and only if

$$mN(lX, lY) = 0. \tag{3.14}$$

Thus distribution D_l^C is integrable in TM if and only if

$$m^C N^C(l^C X^C, l^C Y^C) = 0.$$

Thus the theorem follows by making use of the equation (3.8).

Theorem 3.6. *Let complete lift F^C of a $f(\alpha,\beta)(3,2,1)$-structure F in M is partially integrable in TM if and only if F is partially integrable in M.*

Proof: The $f(\alpha,\beta)(3,2,1)$-structure F in M is partially integrable if and only if

$$N(lX, lY) = 0, \forall X, Y \in \wp_0^1(M). \tag{3.15}$$

In view of the equations (1.3) and (3.1), we obtain

$$N^C(l^C X^C, l^C Y^C) = (N(lX, lY))^C$$

which implies

$$N^C(l^C X^C, l^C Y^C) = 0 \Leftrightarrow N(lX, lY) = 0.$$

Also from Theorem (3.2), $N^C(l^C X^C, l^C Y^C) = 0$ is equivalent to

$$N^C((F^2 - \alpha F)^C X^C, (F^2 - \alpha F)^C Y^C) = 0.$$

Theorem 3.7. *Let complete lift F^C of a $f(\alpha,\beta)(3,2,1)$-structure F in M is partially integrable in TM if and only if F is partially integrable in M.*

Proof: A necessary and sufficient condition for a $f(\alpha,\beta)(3,2,1)$-structure in M to be integrable is that

$$(N(X,Y)) = 0 \tag{3.16}$$

for all $X, Y \in \wp_0^1(M)$.

In view of the equation (3.1), we get

$$N^C(X^C, Y^C) = (N(X,Y))^C.$$

Therefore, with the help of the equation (3.16) we obtain the result.

4. The horizontal lift of an $f(\alpha,\beta)(3,2,1)$-structure in the tangent bundle

Now, we shall prove some theorems on horizontal lift of the $f(\alpha,\beta)(3,2,1)$-structure. Suppose that there are tensor fields S and $\nabla_\gamma S$ in M and TM respectively with affine connection ∇ in the term of partial differential equations are given by [1, 2, 23]

$$S = S^{i\ldots h}_{k\ldots j}\frac{\partial}{\partial x^i}\otimes\ldots\otimes\frac{\partial}{\partial x^h}\otimes dx^k\otimes\ldots\otimes dx^j, \tag{4.1}$$

$$\nabla_\gamma S = y^l\nabla_\gamma S^{i\ldots h}_{k\ldots j}\frac{\partial}{\partial x^i}\otimes\ldots\otimes\frac{\partial}{\partial y^h}\otimes dx^k\otimes\ldots\otimes dx^j \tag{4.2}$$

corresponding to the induced coordinates (x^h, y^h) in $\pi^{-1}(U)$[23].
Now, we define the horizontal lift S^H of a tensor field S in M to TM by

$$S^H = S^C - \nabla_\gamma S. \tag{4.3}$$

Theorem 4.1. *Let $F \in \wp_1^1$ be an $f(\alpha,\beta)(3,2,1)$-structure in M, then its horizontal lift F^H is also $f(\alpha,\beta)(3,2,1)$-structure in TM.*

Proof: If $P(t)$ is a polynomial in one variable t, then we have [23]

$$(P(F))^H = P(F^H), \tag{4.4}$$

for all $F \in \wp_1^1(M)$.

Operating the horizontal lifts of both sides of the equation (1.1), we get

$$\begin{aligned}(F^3)^H &= (\alpha F^2 + \beta F)^H,\\ (F^3)^H &= (\alpha F^2)^H + (\beta F)^H.\end{aligned}$$

In the view of (4.4) and $I^H = I$, we get

$$(F^H)^3 = (\alpha F^H)^2 + \beta F^H \tag{4.5}$$

which shows that F^H is an $f(\alpha,\beta)(3,2,1)$-structure in TM [13]. In the view of equations (1.1) and (4.5), we can easily say that the rank of F^H is $2n$ if and only if the rank of F is n. Therefore , we have the following theorem:

Theorem 4.2. *The $f(\alpha,\beta)(3,2,1)$-structure F of rank n in M if and only if its complete lift F^H is of rank $2n$ in TM.*

Let m be a projection tensor field of type (1,1) in Mdefined by (1.3), in M there exists a distribution D determined by m. Also

$$m^2 = m.$$

In view of (4.4), we get

$$(m^H)^2 = m^H.$$

Thus, m^H is also a projection in TM. Hence there exists in TM a distribution D^H corresponding to m^H, which is called the horizontal lift of the distribution D.

5. Prolongation of an $f(\alpha,\beta)(3,2,1)$-structure on third tangent bundle T_3M

Let T_3M be the third order tangent bundle over M and let F^{III} be the third lift on F in T_3M. Then for any $F, G \in \wp_1^1(M)$, we have

$$\begin{aligned}
(G^{III}F^{III})X^{III} &= G^{III}(F^{III}X^{III}) \\
&= (G^{III}(FX)^{III}) \\
&= (G(FX))^{III} \\
&= (GF)^{III}X^{III},
\end{aligned} \tag{5.1}$$

for all $X \in \wp_0^1(M)$. Thus we have

$$G^{III}F^{III} = (GF)^{III}$$

If $P(t)$ is a polynomial in one variable t, then we have [23]

$$(P(F))^{III} = P(F^{III}). \tag{5.2}$$

for all $F \in \wp_1^1(M)$.

Theorem 5.1. *Let $F \in \wp_1^1(M)$ be a $f(\alpha,\beta)(3,2,1)$-structure in M, then the third lift F^{III} is also $f(\alpha,\beta)(3,2,1)$-structure in T_3M.*

Proof: If $P(t)$ is a polynomial in one variable t, then we have [23]

$$(P(F))^{III} = P(F^{III}), \tag{5.3}$$

for all $F \in \wp_1^1(M)$. Operating the third lifts of both sides of the equation (1.1), we get

$$\begin{aligned}(F^3 &= \alpha F^2 + \beta F)^{III}, \\ (F^3)^{III} &= (\alpha F^2)^{III} + (\beta F)^{III}.\end{aligned}$$

In the view of (5.3) and $I^{III} = I$, we get

$$(F^{III})^3 = \alpha(F^{III})^2 - \beta F^{III} \tag{5.4}$$

which shows that F^{III} is a $f(\alpha, \beta)(3, 2, 1)$-structure in T_3M.

Theorem 5.2. *The third lift F^{III} is integrable in T_3M if and only if F is integrable in M.*

Proof: Let N^{III} and N be Nijenhuis tensors of F^{III} and F respectively. Then we have

$$N^{III}(X, Y) = (N(X, Y))^{III}. \tag{5.5}$$

since $f(\alpha, \beta)(3, 2, 1)$-structure is integrable in M if and only if $N(X, Y) = 0$. then from (5.5), we get

$$N^{III}(X, Y) = 0. \tag{5.6}$$

Thus F^{III} is integrable if and only if F is integrable in M.

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