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THE n-DUAL STRUCTURE OF THE SPACE OF p-SUMMABLE SEQUENCE SPACES

Pradeep Kumar Singh and J K Srivastava

Abstract. In this paper, we shall investigate the n-dual structure of the sequence space l^p regarded as normed space and n-normed space, where the given norm is derived by n-norm and they have been studied in [5, 6, 7].
Keywords: Sequence space, normed space, multilinear n-functional, isometric linear bijection.

1. Introduction

Similar to the theory of the space of the bounded linear functionals defined on a normed space, bounded multilinear n-functionals have been defined on an n-normed space and these theories have been studied by White [1], Gunawan [9, 10].

The concept of 2-normed spaces was initially investigated by Gahler [8]. After that, it has been generalized to n-normed spaces and has been studied by many others (see [2, 3, 4, 5, 6, 7]).

Definition 1.1. Let $\mathbf{X}$ be a vector space over $\mathbb{K}(=\mathbb{R}$ or $\mathbb{C})$ of dimension $d \geq n(n \geq 2)$. A non-negative real valued function $\|.,\ldots,.\|$ defined on $\mathbf{X^n}$ satisfying the four conditions:

(N1) $\|x^1, x^2, \cdots, x^n\| = 0$ if and only if $x^1, x^2, \cdots, x^n$ are linearly dependent;

(N2) $\|x^1, x^2, \cdots, x^n\|$ is invariant under any permutation of $x^1, x^2, \cdots, x^n$;

(N3) $\|\alpha \cdot x^1, x^2, \cdots, x^n\| = |\alpha| \cdot \|x^1, x^2, \cdots, x^n\|$;

(N4) $\|x^1 + y, x^2, \cdots, x^n\| \leq \|x^1, x^2, \cdots, x^n\| + \|y, x^2, \cdots, x^n\|$;

for all $x^1, x^2, \cdots, x^n, y \in \mathbf{X}$ and for all $\alpha \in \mathbb{K}$, is called an ***n-norm on X***, and the pair $(\mathbf{X}, \|.,\ldots,.\|)$ is called an ***n-normed space***.

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Definition 1.2. Let $(\mathbf{X}, \|., \ldots, .\|)$ be an *n-normed space* and $\left\{e^1, \ldots, e^n\right\}$ is a linearly independent set of n vectors, let us define:

1. $\|x\|_\infty^d = max\,\{\|x, e^{t_1}, \ldots, e^{t_{n-1}}\| : \{t_1, \ldots, t_{n-1}\} \subset \{1, \ldots, n\}\}$
2. $\|x\|_q^d = \left(\sum_{\{t_1,\ldots,t_{n-1}\}\subset\{1,\ldots,n\}} \|x, e^{t_1}, \ldots, e^{t_{n-1}}\|^q\right)^{1/q}; \quad 1 \le q < \infty.$

In [3, 4], Gunawan proved that these two functions ($\|.\|_\infty^d$ and $\|.\|_q^d$) define *norms* (known as ***derived norms***) on the vector space $\mathbf{X}$ and they are equivalent.

Definition 1.3. Let $(\mathbf{X}, \|.\|)$ is a *normed space* then a linear functional $f : \mathbf{X} \to \mathbb{K}$ is said to be **bounded** if $\exists$ a real number $k > 0$ such that

$$|f(x)| \le k\|x\|, \qquad \text{for all} \quad x \in \mathbf{X}.$$

The linear functional f is said to be **continuous** at a point $x_0 \in \mathbf{X}$ if for every given $\epsilon > 0, \exists\, \delta > 0$ such that

$$x \in \mathbf{X}, \|x - x_0\| < \delta \qquad \Longrightarrow \qquad |f(x) - f(x_0)| < \epsilon.$$

Lemma 1.1. *Let $(\mathbf{X}, \|.\|)$ is a normed space then a linear functional $f : \mathbf{X} \to \mathbb{K}$ is bounded if and only if f is continuous.*

Analogous to the above definitions a bounded multilinear n-functional has been defined on *n-normed space* (for detail see [1, 9, 10]).

Definition 1.4. Let $\mathbf{X}$ be a ***vector space*** then a scalar valued function $f : \mathbf{X}^n \to \mathbb{K}$ is called a ***multilinear n-functional*** if it satisfies:

1. $f(x^1 + y^1, \cdots, x^n + y^n) = \sum_{h^i \in \{x^i, y^i\}, 1 \le i \le n} f(h^1, \cdots, h^n),$
2. $f(\alpha_1 x^1, \cdots, \alpha_n x^n) = \alpha_1 \cdots \alpha_n f(x^1, x^2, \cdots, x^n),$

for every $x^1, x^2, \cdots, x^n \in \mathbf{X}$ and for every $\alpha_i \in \mathbb{K}$.

A *multilinear n-functional* f defined on a *normed space* $(\mathbf{X}, \|.\|)$ is said to be **bounded** (i.e. bounded with respect *norm*)if $\exists\, K > 0$ such that

$$|f(x^1, x^2, \cdots, x^n)| \le K\|x^1\| \cdots \|x^n\|, \qquad \text{for every} \quad x^1, x^2, \cdots, x^n \in \mathbf{X}.$$

Similarly, a *multilinear n-functional* f defined on an *n-normed space* $(\mathbf{X}, \|., \ldots, .\|)$ is said to be **bounded** (i.e. bounded with respect *n-norm*)if $\exists\, K > 0$ such that

$$|f(x^1, x^2, \cdots, x^n)| \le K\|x^1, x^2, \cdots, x^n\|, \qquad \text{for every} \quad x^1, x^2, \cdots, x^n \in \mathbf{X}.$$

As we know that, the set $\mathbf{B}(\mathbf{X}, \mathbb{K})$ of all bounded linear functional f defined on the *normed space* $(\mathbf{X}, \|.\|)$ forms a *normed space* with *norm* defined by

$$\|f\| = sup\,\{|f(x)| : x \in \mathbf{X}, \|x\| = 1\}.$$

The *normed space* $\mathbf{B}(\mathbf{X}, \mathbb{K})$ is called ***dual space*** of the *normed space* $(\mathbf{X}, ||.||)$ and is usually denoted by $\mathbf{X}^*$.

Similarly, the space of *bounded multilinear n-functionals* on $(\mathbf{X}, ||.||)$ [on $(\mathbf{X}, ||., \ldots, .||)$] is called the ***n-dual space*** of the *normed space* $(\mathbf{X}, ||.||)$ [the ***n-dual space*** of the *n-normed space* $(\mathbf{X}, ||., \ldots, .||)$ respectively] see [9, 10], with norms

$$||f||_{n,1} := \sup_{||x^1||\cdots||x^n||\neq 0} \frac{|f(x^1, x^2, \cdots, x^n)|}{||x^1||\cdots||x^n||};$$

$$\left[||f||_{n,n} := \sup_{||x^1, x^2, \cdots, x^n||\neq 0} \frac{|f(x^1, x^2, \cdots, x^n)|}{||x^1, x^2, \cdots, x^n||} \quad \text{respectively}\right].$$

Here, we shall consider the well-known sequence space l^p ,$1 \leq p < \infty$; where

$$l^p = \left\{ x = (x_i)_{i=0}^{\infty} \mid \sum_{i=0}^{\infty} |x_i|^p < \infty \quad \text{and } x_i \in \mathbb{K}; i = 0, 1, 2, \ldots \right\}$$

with norms

$$||x||_p = \left(\sum_{i=0}^{\infty} |x_i|^p \right)^{1/p} \tag{1.1}$$

and

$$||x||_\infty = \sup_{0\leq i<\infty} |x_i| . \tag{1.2}$$

We know that $\left(l^p, ||.||_p\right)$ is a *Banach space* whereas $(l^p, ||.||_\infty)$ is not a *Banach space*.

In [5], for our convenience and need, we have denoted the set of whole numbers as $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ will also be written in the form of a sequence $\mathbb{N} = (0, 1, 2, 3, \ldots)$ as well as in the form of *n-consecutive terms notation* as:

$$\mathbb{N} = (nl, nl + 1, \ldots, nl + (n-1))_{l=0}^{\infty}$$

where "n" is fixed positive integer and refer to the integer "n" of *n-normed space*.

Taking $\overline{\overline{\mathbb{N}}} = \left(\overline{\overline{m}}_{nk}, \overline{\overline{m}}_{nk+1}, \ldots, \overline{\overline{m}}_{nk+(n-1)}\right)_{k=0}^{\infty}$ as a rearrangement of the sequence $\mathbb{N}$. In [5], we have seen that $\left(l^p, \overline{\overline{||., \ldots, .||_p}}\right)$, $1 \leq p < \infty$ is an *n-normed space*, but not complete where

$$\overline{\overline{||x^1, x^2, \ldots, x^n||_p}} = sup\left\{\left|\overline{\overline{x}}^1, \overline{\overline{x}}^2, \ldots, \overline{\overline{x}}^n\right| : \overline{\overline{x}}^1, \overline{\overline{x}}^2, \ldots, \overline{\overline{x}}^n \text{ are parallel rearrangements of } x^1, x^2, \cdots, x^n \quad \text{respectively}\right\}, \tag{1.3}$$

and

(1.4) $$\left|\overline{\overline{x}}^1, \overline{\overline{x}}^2, \ldots, \overline{\overline{x}}^n\right| = \left(\sum_{k=0}^{\infty}\left|det\begin{pmatrix} x^1_{\overline{\overline{m}}_{nk}} & x^1_{\overline{\overline{m}}_{nk+1}} & \cdots x^1_{\overline{\overline{m}}_{nk+(n-1)}} \\ x^2_{\overline{\overline{m}}_{nk}} & x^2_{\overline{\overline{m}}_{nk+1}} & \cdots x^2_{\overline{\overline{m}}_{nk+(n-1)}} \\ \cdots & \cdots & \cdots \\ x^n_{\overline{\overline{m}}_{nk}} & x^n_{\overline{\overline{m}}_{nk+1}} & \cdots x^n_{\overline{\overline{m}}_{nk+(n-1)}} \end{pmatrix}\right|^p\right)^{1/p};$$

$$\overline{\overline{x}}^t = \left(x^t_{\overline{\overline{m}}_{nk}}, x^t_{\overline{\overline{m}}_{nk+1}}, \cdots, x^t_{\overline{\overline{m}}_{nk+(n-1)}}\right)_{k=0}^{\infty};$$

$$x^t = \left(x^t_{nl}, x^t_{nl+1}, \ldots, x^t_{nl+(n-1)}\right)_{l=0}^{\infty}; \quad t = 1, 2, \ldots, n.$$

Besides it, the function

$$\|x^1, x^2, \cdots, x^n\|_\infty := \sup_{i_1,\ldots,i_n}\left|det\begin{pmatrix} x^1_{i_1} & x^1_{i_2} & \ldots & x^1_{i_n} \\ x^2_{i_1} & x^2_{i_2} & \ldots & x^2_{i_n} \\ \ldots & \ldots & \ldots & \ldots \\ x^n_{i_1} & x^n_{i_2} & \ldots & x^1_{i_n} \end{pmatrix}\right|$$

also defines an *n-norm* on l^p , where $i_1, \ldots, i_n \in \mathbb{N}$.

In [5, 7], we have already proved that these two *n-norms* $\overline{\overline{\|., \ldots, .\|_p}}$ and $\|. \ldots, .\|_\infty$ are non-equivalent. Where as their ***derived norms*** with respect to the linearly independent set $\{e^1, \cdots, e^n\}$ are equivalent to $\|.\|_\infty$, where $e^t = (\delta^t_i)_{i=0}^{\infty}$. For details see [5, 6, 7].

Here we shall consider the *normed space* $(l^p, \|.\|_\infty)$ and *n-normed space* $\left(l^p, \overline{\overline{\|., \ldots, .\|_p}}\right)$.

It is well known that, $l^p \subset C_0$ and $\|.\|_\infty \le \|.\|_p$, by applying usual methods for finding *dual spaces* of sequence spaces we have the following Lemma:

Lemma 1.2. *The* **dual space** *of* $(l^p, \|.\|_\infty)$ $1 \le p < \infty$ *is identified by* $\left(l^1, \|.\|_1\right)$. *Moreover, the mapping* $f \to \left(f\left(e^i\right)\right)_{i=0}^{\infty}$ *is a linear isometric bijection.*

Proof. It is easy to check that the sequence $\left(e^i\right)_{i=0}^{\infty}$ constitutes a Schauder basis for the space $(l^p, \|\cdot\|_\infty)$ also where $e^i = \left(\delta^i_j\right)_{j=0}^{\infty}$; $i = 0, 1, 2, \ldots$. Therefore, every $x = (x_i)_{i=0}^{\infty} \in l^p$ can be uniquely expressed as

$$x = \sum_{i=0}^{\infty} x_i e^i .$$

i.e. $\|s_n - x\|_\infty \to 0$ as $n \to \infty$, where $s_n = \sum_{i=0}^{n} x_i e^i$. Let f be bounded linear functional on $(l^p, \|\cdot\|_\infty)$ [It should be noted that f is bounded with respect to $\|\cdot\|_\infty$

]. Since f is continuous and $s_n \to x$ it follows that $f(s_n) \to f(x)$. Hence, $f(x)$ can be uniquely expressed as

$$f(x) = \sum_{i=0}^{\infty} x_i f(e^i) .$$

Now, we shall show that $\left(f\left(e^i\right)\right)_{i=0}^{\infty} \in \left(l^1, \|\cdot\|_1\right)$. Let $n \in \mathbb{N}$ be arbitrary, define $y^n = (y_i)_{i=0}^{\infty} \in l^p$ as follows

$$y_i = \begin{cases} \frac{\overline{f(e^i)}}{|f(e^i)|} & ; \quad f(e^i) \neq 0 \quad \text{and} \quad 0 \leq i \leq n \\ 0 & ; \quad \text{otherwise} . \end{cases}$$

Obviously $y^n \in l^p$ and

$$\|y^n\|_\infty = \sup_{0 \leq i < \infty} |y_i| \leq 1$$

(But $\|y^n\|_p \leq (n+1)^{1/p}$.) Now,

$$f(y^n) = \sum_{i=0}^{\infty} y_i f(e^i) = \sum_{i=0}^{n} |f(e^i)| \leq \|f\| \cdot \|y^n\|_\infty \leq \|f\|.$$

Thus, for all $n \in \mathbb{N}$ we have $\sum_{i=0}^{n} |f(e^i)| \leq \|f\|$ therefore

$$\sum_{i=0}^{\infty} |f(e^i)| \leq \|f\|$$

and hence

$$\left(f\left(e^i\right)\right)_{i=0}^{\infty} \in \left(l^1, \|\cdot\|_1\right) .$$

Now, define a mapping $T : (l^p, \|\cdot\|_\infty)^* \to \left(l^1, \|\cdot\|_1\right)$ as follows

$$T(f) = \left(f\left(e^i\right)\right)_{i=0}^{\infty}$$

where $(l^p, \|\cdot\|_\infty)^*$ is dual space of $(l^p, \|\cdot\|_\infty)$. Clearly T is well-defined and linear and from above it follows that $T(f) = 0 \Rightarrow f = 0$ therefore T is one-one.

To prove T is onto, let $\lambda = (\lambda_i)_{i=0}^{\infty} \in \left(l^1, \|\cdot\|_1\right)$ and $x = (x_i)_{i=0}^{\infty} \in l^p$ be arbitrary, clearly x is bounded in $\mathbb{K}$, therefore

$$\sum_{i=0}^{\infty} |x_i \lambda_i| < \infty \quad \Rightarrow \quad \sum_{i=0}^{\infty} x_i \lambda_i < \infty .$$

Define $f : l^p \to \mathbb{K}$ as

$$f(x) = \sum_{i=0}^{\infty} x_i \lambda_i < \infty .$$

Obviously, f is linear and for every $x \in l^p$:

$$|f(x)| \leq \sum_{i=0}^{\infty} |x_i \lambda_i| \leq \left(\sum_{i=0}^{\infty} |\lambda_i| \right) ||x||_\infty < \infty$$

i.e. f is bounded and $T(f) = \left(f(e^i) \right)_{i=0}^{\infty} = (\lambda_i)_{i=0}^{\infty}$ and

$$||f|| \leq \sum_{i=0}^{\infty} |f(e^i)| \, .$$

Moreover, above inequalities give

$$||f|| = \sum_{i=0}^{\infty} |f(e^i)| \, .$$

Thus T is a linear isometric bijection. □

Remark : Moreover, above Lemma1.2 says that if $1 \leq p \leq q < \infty$ then

$$(l^p, ||\cdot||_\infty)^* \quad \cong \quad (l^q, ||\cdot||_\infty)^*$$

and

$$(c_0, ||\cdot||_\infty)^* \quad \cong \quad (c, ||\cdot||_\infty)^* \quad \cong \quad (l^p, ||\cdot||_\infty)^* \quad \cong \quad \left(l^1, ||\cdot||_1 \right) .$$

But it need not be true for $\left(l^p, ||\cdot||_p \right))^*$ **and** $\left(l^q, ||\cdot||_q \right))^*$ **.**

From [9, 10], we have the following results:

Lemma 1.3. *Every bounded multilinear n-functional f defined on the n-normed space $(\mathbf{X}, ||., \ldots, .||)$ satisfies*

1. $f(x^1, x^2, \cdots, x^n) = 0\,;$ *whenever* $x^1, x^2, \cdots, x^n$ *are linearly dependent*
2. $f(x^1, x^2, \cdots, x^n) = sgn(\sigma) f(x^{\sigma(1)}, \ldots, x^{\sigma(n)})$ *for every* $x^1, x^2, \cdots, x^n \in \mathbf{X}$

for every permutation σ of $(1, 2, \ldots, n)$ where $sgn(\sigma) = 1$ if σ is an even permutation and $sgn(\sigma) = -1$ if σ is an odd permutation.

Lemma 1.4. *The norm of every bounded multilinear n-functional f defined on an n-normed space $(\mathbf{X}, ||., \ldots, .||)$ is given by:*

$$||f||_{n,n} := \sup_{||x^1, x^2, \cdots, x^n|| \neq 0} \frac{|f(x^1, x^2, \cdots, x^n)|}{||x^1, x^2, \cdots, x^n||}$$

or equivalently

$$||f||_{n,n} := \sup_{||x^1, x^2, \cdots, x^n|| = 1} |f(x^1, x^2, \cdots, x^n)|$$

or equivalently

$$\|f\|_{n,n} := \sup_{\|x^1,x^2,\cdots,x^n\|\leq 1} |f(x^1,x^2,\cdots,x^n)|$$

or equivalently

$$\|f\|_{n,n} := inf\left\{K : |f(x^1,x^2,\cdots,x^n)| \leq K\|x^1,x^2,\cdots,x^n\|, \quad \text{for every} \quad x^1,x^2,\cdots,x^n \in \mathbf{X}\right\}.$$

A similar result can be obtained for *bounded multilinear n-functional defined on a normed space.*

2. Results

In [5, 7], we have already investigated the equivalence relations between different *norms and n-norms* defined on l^p as

1. $\overline{\overline{\|x^1,x^2,\ldots,x^n\|_p}} \leq n!\|x^1\|_p\|x^2\|_p\cdots\|x^n\|_p$

2. $\overline{\overline{\|x^1,x^2,\ldots,x^n\|_p}} \leq (n!)^{1/p}\|x^1,x^2,\cdots,x^n\|_p.$

for every $x^1,x^2,\cdots,x^n \in l^p$; where the *n-norm* $\|.,\ldots,.\|_p$ is defined by Gunawan [4] as

$$\|x^1,x^2,\cdots,x^n\|_p = \left[\frac{1}{n!}\sum_{j_1}\cdots\sum_{j_n}|det(x^t_{j_k})|^p\right]^{1/p} \qquad t = 1,2,\ldots,n.$$

The above relations give the following propositions:

Proposition 2.1. *A bounded multilinear n-functional defined on* $\left(l^p, \overline{\overline{\|.,\ldots,.\|_p}}\right)$ *is a bounded multilinear n-functional on* $(l^p, \|.\|_p)$.

Proposition 2.2. *A bounded multilinear n-functional defined on* $\left(l^p, \overline{\overline{\|.,\ldots,.\|_p}}\right)$ *is a bounded multilinear n-functional on* $(l^p, \|.,\ldots,.\|_p)$.

In [10], Gunawan investigated the *n-dual* structure of the *Banach space* $(l^p, \|.\|_p)$ and *n-Banach space* $\left(l^p, \|.,\ldots,.\|_p\right)$, respectively. In [7] we have investigated that the two *n-normed spaces* $\left(l^p, \|.,\ldots,.\|_p\right)$ and $\left(l^p, \overline{\overline{\|.,\ldots,.\|_p}}\right)$ are non-equivalent. Inspired by Gunawan [10] here, we shall investigate the *n-dual spaces* of the *normed space* $(l^p, \|.\|_\infty)$ and *n-normed space* $\left(l^p, \overline{\overline{\|.,\ldots,.\|_p}}\right)$, respectively.

We shall begin our investigations by finding the *2-dual structure* of l^p with respect to *norm* $\|.\|_\infty$ and *2-norm* $\overline{\overline{\|.,.\|}}_p$, respectively.

First of all, let us define the *normed space* $\left(l^1_{\mathbb{N}X\mathbb{N}}, \|.\|^1_2\right)$ of *double indexed sequences* as follows:

$$\Theta := (\theta_{ij})^\infty_{i,j=0} \in l^1_{\mathbb{N}X\mathbb{N}}; \quad \theta_{ij} \in \mathbb{K} \qquad \text{if and only if}$$

$$\|\Theta\|^1_2 = \sup_{\|x\|_\infty=1} \left(\sum_{j=0}^\infty \left| \sum_{i=0}^\infty x_i \theta_{ij} \right| \right) < \infty, \qquad \text{where} \quad x = (x_i)^\infty_{i=0}. \tag{2.1}$$

and the *normed space* $\left(l^A_{\mathbb{N}X\mathbb{N}}, \|.\|^A_2\right)$ as follows:

$$\Theta := (\theta_{ij})^\infty_{i,j=0} \in l^A_{\mathbb{N}X\mathbb{N}}; \quad \theta_{ij} \in \mathbb{K} \qquad \text{if and only if}$$

$$\|\Theta\|^A_2 = \sup_{\overline{\overline{\|x,y\|}}_p \neq 0} \left(\frac{\left|\sum_{j=0}^\infty \sum_{i=0}^\infty x_i y_j \theta_{ij}\right|}{\overline{\overline{\|x, y\|}}_p} \right) < \infty \quad \text{and} \quad \theta_{ij} = -\theta ji \tag{2.2}$$

where $x = (x_i)^\infty_{i=0}$ and $y = (y_j)^\infty_{j=0}$.

Theorem 2.1. *The 2-dual space of* $(l^p, \|.\|_\infty); 1 \leq p < \infty$ *is identified by* $\left(l^1_{\mathbb{N}X\mathbb{N}}, \|.\|^1_2\right)$. *Moreover, the mapping* $f \to \Theta := (f(e^i, e^j))^\infty_{i,j=0}$ *is an isometric linear bijection.*

Proof. Let f is a *bounded bilinear 2-functional* on $(l^p, \|.\|_\infty)$, then for every $x = (x_i)^\infty_{i=0}$ and $y = (y_j)^\infty_{j=0}$ $f(x, y)$ can be expressed as

$$f(x, y) = \sum_{j=0}^\infty y_j \sum_{i=0}^\infty x_i f(e^i, e^j) \tag{2.3}$$

We shall first show that $(f(e^i, e^j))^\infty_{i,j=0} \in l^1_{\mathbb{N}X\mathbb{N}}$. Since for any arbitrary $x \in l^p$ with $\|x\|_\infty = 1$ the function f_x defined on l^p as $f_x(y) = f(x, y)$ is bounded linear functional on $(l^p, \|.\|_\infty)$,that is

$$|f_x(y)| = |f(x, y)| \leq \|f\|_{2,1}\|x\|_\infty\|y\|_\infty = \|f\|_{2,1}\|y\|_\infty \tag{2.4}$$

Therefore by lemma 1.2, f_x can be identified as $f_x \equiv (f_x(e^j))^\infty_{j=0}$ with *norm* $\|f_x\| = \sum^\infty_{j=0} |f_x(e^j)| = \sum^\infty_{j=0} |f(x, e^j)| = \sum^\infty_{j=0} |\sum^\infty_{i=0} x_i f(e^i, e^j)|$, as well as $\|f_x\| = sup\{|f_x(y)| : \|y\|_\infty = 1\}$, therefore from (2.4), we have

$$\sum_{j=0}^\infty \left|\sum_{i=0}^\infty x_i f(e^i, e^j)\right| \leq \|f\|_{2,1}; \quad \text{for every arbitrary} \quad \|x\|_\infty = 1.$$

Which shows that, $\Theta := (f(e^i, e^j))^\infty_{i,j=0} \in l^1_{\mathbb{N}X\mathbb{N}}$ with

$$\|\Theta\|^1_2 = \|f(e^i, e^j)\|^1_2 \leq \|f\|_{2,1}. \tag{2.5}$$

Next, let us define a function T on the *2-dual space* of $(l^p, ||.||_\infty)$ to $l^1_{\mathbb{N}X\mathbb{N}}$ such that $T(f) = (f(e^i, e^j))_{i,j=0}^{\infty}$ then obviously, T is well defined and linear. From (2.3), it is clear that f is zero function, whenever $T(f) = O$, thus T is one-one.

Next, let $\Theta := (\theta_{ij})_{i,j=0}^{\infty} \in l^1_{\mathbb{N}X\mathbb{N}}$ is arbitrary, for $x = (x_i)_{i=0}^{\infty}$ and $y = (y_j)_{j=0}^{\infty}$ define $f : l^p X l^p \to \mathbb{K}$ as follows:

$$f(x, y) = \sum_{j=0}^{\infty} y_j \sum_{i=0}^{\infty} x_i \theta_{ij};$$

obviously $f(e^i, e^j) = \theta_{ij}$. For $x, y \in l^p$ with $||x||_\infty = 1$ and $||y||_\infty = 1$, we have

$$|f(x, y)| \leq \sum_{j=0}^{\infty} |y_j \sum_{i=0}^{\infty} x_i \theta_{ij}| \leq ||y||_\infty \sum_{j=0}^{\infty} |\sum_{i=0}^{\infty} x_i \theta_{ij}| \leq ||\Theta||_2^1.$$

Therefore for every $||x||_\infty \neq 0$ and $||y||_\infty \neq 0$

$$\frac{|f(x, y)|}{||x||_\infty ||y||_\infty} \leq ||\Theta||_2^1$$

or equivalently,

$$|f(x, y)| \leq ||\Theta||_2^1 ||x||_\infty ||y||_\infty, \tag{2.6}$$

which exhibits that f is *bounded bilinear 2-functional* on $(l^p, ||.||_\infty)$ and $T(f) = \Theta$ with

$$||f||_{2,1} \leq ||\Theta||_2^1 = ||f(e^i, e^j)||_2^1. \tag{2.7}$$

From (2.5) and (2.7) it is clear that, T is isometric linear bijection. □

Theorem 2.2. *The 2-dual space of the 2-normed space* $\left(l^p, \overline{\overline{||., .||_p}}\right)$ *is identified as* $\left(l^A_{\mathbb{N}X\mathbb{N}}, ||.||_2^A\right)$. *Moreover, the mapping* $f \to \Theta := (f(e^i, e^j))_{i,j=0}^{\infty}$ *is an isometric linear bijection.*

Proof. Since $\overline{\overline{||x, y||_p}} \leq 2||x||_p||||y||_p$ see[5], therefore $f(x, y)$ can be expressed as

$$f(x, y) = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} y_j x_i f(e^i, e^j). \tag{2.8}$$

Since f is bounded therefore

$$|f(x, y)| = |\sum_{j=0}^{\infty} \sum_{i=0}^{\infty} y_j x_i f(e^i, e^j)| \leq ||f||_{2,2} \overline{\overline{||x, y||_p}}.$$

Defining $\Theta := (f(e^i, e^j))_{i,j=0}^{\infty}$ above equation exhibits that $\Theta := (f(e^i, e^j))_{i,j=0}^{\infty} \in l^A_{\mathbb{N}X\mathbb{N}}$ and

$$||\Theta||_2^A = ||f(e^i, e^j)||_2^A \leq ||f||_{2,2} \tag{2.9}$$

Now for any arbitrary $\Theta := (\theta_{ij})_{i,j=0}^{\infty} \in l^{A}_{\mathbb{N}X\mathbb{N}}$ define *bilinear functional*

$$f(x,y) = \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} y_j x_i \theta_{ij}\,;$$

it is easy to show that f is *bounded bilinear 2-functional* on $\left(l^p, \overline{\overline{\|.,.\|_p}}\right)$ with $f(e^i, e^j) = \theta_{ij}$ and $\|f\|_{2,2} \le \|\Theta\|_2^A = \|f(e^i,e^j)\|_2^A$

Now proceeding as in *theorem*2.1, we have the result. □

To achieve the *n-dual spaces* of $(l^p, \|.\|_\infty)\,; 1 \le p < \infty$ and $\left(l^p, \overline{\overline{\|.,\ldots,.\|_p}}\right)$. Let us generalizes the definitions of $\left(l^1_{\mathbb{N}X\mathbb{N}}, \|.\|_2^1\right)$ and $\left(l^A_{\mathbb{N}X\mathbb{N}}, \|.\|_2^A\right)$ to following *normed space of n-indexed sequence spaces* as follows:

The *normed space* $\left(l^1_{\mathbb{N}^n}, \|.\|_n^1\right)$ of *n-indexed sequences* $\Theta := (\theta_{i_1,\ldots,i_n})_{i_1,\ldots,i_n=0}^{\infty}$ with $\theta_{i_1,\ldots,i_n} \in \mathbb{K}$ as follows:

$$\Theta := (\theta_{i_1,\ldots,i_n}) \in l^1_{\mathbb{N}^n}\,; \qquad \text{if and only if}$$

$$\|\Theta\|_n^1 = \sup_{\|x^1\|_\infty,\ldots,\|x^{n-1}\|_\infty=1} \left(\sum_{i_n=0}^{\infty}\left|\sum_{i_{n-1}=0}^{\infty},\cdots,\sum_{i_1=0}^{\infty} x^1_{i_1},\ldots,x^{n-1}_{i_{n-1}}\theta_{i_1,\ldots,i_n}\right|\right) < \infty,$$

$$\text{where} \quad x^t = (x_i^t)_{i=0}^{\infty};\, t = 1,\ldots,n-1. \tag{2.10}$$

and the *normed space* $\left(l^A_{\mathbb{N}^n}, \|.\|_n^A\right)$ of *n-indexed sequences* $\Theta := (\theta_{i_1,\ldots,i_n})_{i_1,\ldots,i_n=0}^{\infty}$ with $\theta_{i_1,\ldots,i_n} \in \mathbb{K}$ as follows:

$$\Theta := (\theta_{i_1,\ldots,i_n}) \in l^A_{\mathbb{N}^n}\,; \qquad \text{if and only if} \quad \theta_{i_1,\ldots,i_n} = sgn(\sigma)\theta_{\sigma(i_1),\ldots,\sigma(i_n)} \quad \text{and}$$

$$\|\Theta\|_n^A = \sup_{\overline{\overline{\|x^1,x^2,\ldots,x^n\|_p}}\neq 0} \frac{\left(\left|\sum_{i_n=0}^{\infty}\sum_{i_{n-1}=0}^{\infty},\cdots,\sum_{i_1=0}^{\infty} x^1_{i_1},\ldots,x^n_{i_n}\theta_{i_1,\ldots,i_n}\right|\right)}{\overline{\overline{\|x^1,x^2,\ldots,x^n\|_p}}} < \infty,$$

$$\text{where} \quad x^t = (x_i^t)_{i=0}^{\infty};\, t = 1,\ldots,n. \tag{2.11}$$

and for every permutation σ of $(i_1, i_2, \ldots, i_n)$ where $sgn(\sigma) = 1$ if σ is an even permutation and $sgn(\sigma) = -1$ if σ is an odd permutation.

Theorem 2.3. *The n-dual space of* $(l^p, \|.\|_\infty)$ *is identified by* $\left(l^1_{\mathbb{N}^n}, \|.\|_n^1\right)$. *Moreover, the mapping* $f \to \Theta := (f(e^{i_1},\ldots,e^{i_n}))_{i_1,\ldots,i_n=0}^{\infty}$ *is an isometric linear bijection.*

Proof. The proof is similar to case *n=2*. For any $x^1, x^2, \cdots, x^n \in l^p\,; x^t = (x_i^t)_{i=0}^{\infty}\;\; ; 1 \le t \le n$ the *bounded multilinear n-functional* $f(x^1, x^2, \cdots, x^n)$ can be expressed as

$$f(x^1,x^2,\cdots,x^n) = \sum_{i_n=0}^{\infty}\sum_{i_{n-1}=0}^{\infty},\ldots,\sum_{i_1=0}^{\infty} x^1_{i_1},\ldots,x^n_{i_n} f(e^{i_1},\ldots,e^{i_n}).$$

First of all we shall show that $(f(e^{i_1}, e^{i_2}, \ldots, e^{i_n}))_{i_1,\ldots,i_n=0}^{\infty} \in l^1_{\mathbb{N}^n}$. To do this we shall use mathematical induction on n. For *n=2*, we have already showed it. Let us assume that it is true for *n-1*, we have to prove it for n. Let f is *bounded multilinear n-functional* and $x^1 \in l^p$ with $\|x^1\|_\infty = 1$, if we define $f_{x^1} : l^p \mathrm{x} l^p \cdots \mathrm{x} l^p \quad (n-1 \; times) \to \mathbb{K}$ as

$$f_{x^1}(x^2, \ldots, x^n) = f(x^1, x^2, \cdots, x^n),$$

then f_{x^1} is *bounded multilinear (n-1)-functional* on l^p and

$$|f_{x^1}(x^2, \ldots, x^n)| = |f(x^1, x^2, \cdots, x^n)| \leq \|f\|_{n,1}\|x^2\|_\infty \cdots \|x^n\|_\infty;$$

which implies that

$$\|f_{x^1}\|_{n-1,1} \leq \|f\|_{n,1}$$

therefore it can be identified by $(f_{x^1}(e^{i_2}, \ldots, e^{i_n})) \in l^1_{\mathbb{N}^{n-1}}$ and

$$\begin{aligned}\|f_{x^1}\|_{n-1,1} &= \sup_{\|x^2\|_\infty,\ldots,\|x^{n-1}\|_\infty=1}\left(\sum_{i_n=0}^{\infty}\left|\sum_{i_{n-1}=0}^{\infty},\cdots,\sum_{i_2=0}^{\infty} x^2_{i_2},\ldots,x^{n-1}_{i_{n-1}} f_{x^1}(e^{i_2},\ldots,e^{i_n})\right|\right)\\ &= \sup_{\|x^2\|_\infty,\ldots,\|x^{n-1}\|_\infty=1}\left(\sum_{i_n=0}^{\infty}\left|\sum_{i_{n-1}=0}^{\infty},\cdots,\sum_{i_1=0}^{\infty} x^1_{i_1}x^2_{i_2},\ldots,x^{n-1}_{i_{n-1}} f(e^{i_1},e^{i_2},\ldots,e^{i_n})\right|\right).\end{aligned}$$

That is,

$$\sup_{\|x^2\|_\infty,\ldots,\|x^{n-1}\|_\infty=1}\left(\sum_{i_n=0}^{\infty}\left|\sum_{i_{n-1}=0}^{\infty},\cdots,\sum_{i_1=0}^{\infty} x^1_{i_1}x^2_{i_2},\ldots,x^{n-1}_{i_{n-1}} f(e^{i_1},\ldots,e^{i_n})\right|\right) \leq \|f\|_{n,1}$$

for every arbitrary $\|x^1\|_\infty = 1$ therefore

$$\|(f(e^{i_1}, e^{i_2}, \ldots, e^{i_n}))\|^1_n \leq \|f\|_{n,1}. \tag{2.12}$$

Thus $(f(e^{i_1}, e^{i_2}, \ldots, e^{i_n}))_{i_1,\ldots,i_n=0}^{\infty} \in l^1_{\mathbb{N}^n}$. Rest part is similar to the case *n=2*. □

Theorem 2.4. *The n-dual space of the n-normed space* $\left(l^p, \overline{\overline{\|.,\ldots,.\|_p}}\right)$ *is identified by* $\left(l^A_{\mathbb{N}^n}, \|.\|^A_n\right)$. *Moreover, the mapping* $f \to \Theta := (f(e^{i_1}, \ldots, e^{i_n}))_{i_1,\ldots,i_n=0}^{\infty}$ *is an isometric linear bijection.*

Proof. The proof is similar to the proof of *theorem* 2.2. □

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Pradeep Kumar Singh
Faculty of Science
Department of Mathematics and Statistics
D D U Gorakhpur University, Gorakhpur
UP India 273009
pradeep3789@gmail.com

J K Srivastava
Faculty of Science
Department of Mathematics and Statistics
D D U Gorakhpur University, Gorakhpur
UP India 273009
jks_ddugu@yahoo.com