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RIEMANN-STIELTJES INTEGRABILITY OF FINITE DISCONTINUOUS FUNCTIONS ON TIME SCALES

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Abstract. We establish the equivalence of the Riemann-Stieltjes Δ-integral as defined in [13, 14] in terms of the Darboux sum definition and the Riemann sum definition, and provide the definition of the Riemann-Stieltjes ∇-integral in terms of the Riemann sum definition and prove its equivalence with the Riemann-Stieltjes ∇-integral as defined in [13] in terms of the Darboux sum definition. We establish a few results concerning finite discontinuity.
Keywords: Riemann-Stieltjes Delta Integral, Riemann-Stieltjes Nabla Integral, Time Scale.

1. Introduction

The theory of time scale calculus was first introduced in 1988 by the German mathematician Stefan Hilger [9].

As seen in his paper, Hilger's main motivation was the analogy between discrete and continuous analysis and the aim to unify them. The delta derivative was introduced here [9], and a descriptive sense of the integral (named the Cauchy Integral) was given. More than a decade after the so-called delta derivative was formulated, another derivative called the nabla derivative was introduced by Atici and Guseinov [3], which was previously hinted in the works of Calvin and Bohner [2], who introduced a so-called alpha derivative which consisted both the delta and nabla derivative as special cases.

For an excellent introduction to this subject with theoretical developmental summary and rich history, the reader is referred to the following [5, 6, 9, 10, 12].

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Various integration notions, in their constructive sense, is discussed in literature including the Riemann-Stieltjes integral on time scales.

The Riemann integral on time scales was formulated by S. Sailer [5, 7], using the concept of Darboux sum definition of the integral; and by G. Sh. Guseinov and B. Kaymakçalan [7], using the concept of Riemann sum definition of the integral. The latter also proved that the two different approaches of the Riemann integral on time scales are in essence equal [7].

The Riemann-Stieltjes integral on time scales was formulated by S. Sailer [5, 6, 13], using the concept of Darboux sum definition of the integral; and re-investigated by Dorota Mozyrska et al. [13]. The Riemann sum definition of Riemann-Stieltjes Δ-integral is given in [14] by the same authors.
Other studies related to the Riemann-Stieltjes integral on time scales include inequalities and majorization [14], generalization of the integral to deal with discontinuous dynamical equations [4], prove of the Riesz representation theorem on time scales [11].

In this article, we establish the equivalence of the Riemann-Stieltjes integral defined in terms of the Darboux sum definition and the Riemann sum definition, and discuss the Riemann-Stieltjes integrability of finite discontinuous functions, considering three cases. The first case is when the monotone increasing function ψ has finite points of discontinuity while the bounded function f is continuous at those points. The second case is when the bounded function f has finite points of discontinuity while the monotone increasing function ψ is continuous at those points. And finally, the third case is when both bounded function f and monotone increasing function ψ has a common point of discontinuity.

2. Preliminaries

In this section, we recall a few definitions and results on the theory of time scale calculus (one may refer [5, 6, 9] for more insight).

A time scale $\mathbf{T}$ is any non-empty closed subset of $\mathbb{R}$.

Definition 2.1. [9] Forward Jump Operator: The forward jump operator denoted by σ is a mapping, $\sigma : \mathbf{T} \to \mathbf{T}$ defined by $\sigma(t) = \inf\{r \in \mathbf{T} : r > t\}$.

Definition 2.2. [9] Backward Jump Operator: The backward jump operator denoted by ρ is a mapping, $\rho : \mathbf{T} \to \mathbf{T}$ defined by $\rho(t) = \sup\{r \in \mathbf{T} : r < t\}$.

Assuming $p \leq q$, intervals in $\mathbf{T}$ are defined as [5]-
$[p, q] = [p, q]_{\mathbf{T}} = \{t \in \mathbf{T} : p \leq t \leq q\}$; $(p, q) = (p, q)_{\mathbf{T}} = \{t \in \mathbf{T} : p < t < q\}$;
$[p, q) = [p, q)_{\mathbf{T}} = \{t \in \mathbf{T} : p \leq t < q\}$; $(p, q] = (p, q]_{\mathbf{T}} = \{t \in \mathbf{T} : p < t \leq q\}$.

Throughout the article $[p, q]$, $[p, q)$, $(p, q]$ and (p, q) will denote intervals on $\mathbf{T}$.

Let $[p, q]$ be a closed interval on $\mathbf{T}$ such that $p < q$. Let $\mathfrak{P}$ be the collection of all possible partitions of $[p, q]$.

Below we provide the definition of the Riemann Δ-integral and Riemann ∇-integral. Before proceeding we first establish a few preliminary information required for the definition. For the sake of clarity, $\mathcal{V} \in \mathfrak{P}$ will denote the partition for the Δ-integral and $\mathcal{W} \in \mathfrak{P}$ will denote the partition for the ∇-integral.

Let $\mathcal{V} \in \mathfrak{P}$, $\mathcal{V} = \{p = t_0 < t_1 < \ldots < t_n = q\}$, with t_0, t_1, …, t_n being the finite points of division. We consider subintervals of the form $[t_{h-1}, t_h)$, for $1 \leq h \leq n$, and from each subinterval we choose ϑ_h arbitrarily, defined as $\vartheta_h \in [t_{h-1}, t_h)$, and call it the tag point of the respective subinterval.
For $\mathcal{V} \in \mathfrak{P}$, we define a point-interval collection as $\breve{\mathcal{V}} = \{(\vartheta_h, [t_{h-1}, t_h))\}_{h=1}^{n}$, and call it the tagged partition.
We define the mesh of $\mathcal{V}$ as mesh-$(\mathcal{V}) = \max_{1 \leq h \leq n}(t_h - t_{h-1}) > 0$. For some $\delta > 0$, $\mathcal{V}_\delta$ will represent a partition of $[p, q]$ with mesh δ satisfying the property: For each $h = 1, 2, \ldots, n$ we have either- $(t_h - t_{h-1}) \leq \delta$ or $(t_h - t_{h-1}) > \delta \wedge \rho(t_h) = t_{h-1}$. Hence, $\breve{\mathcal{V}}_\delta$ will mean a tagged partition with mesh δ satisfying the above property.

We proceed to give the definition of Riemann Δ-integral on time scales according to G. Sh. Guseinov and B. Kaymakçalan [7, 8].

Definition 2.3. [8] Riemann Δ-integral: A function $f : [p, q]_{\mathbf{T}} \to \mathbb{R}$ is Riemann Δ-integrable if there exists a number $\overline{I} \in \mathbb{R}$ such that, for any $\epsilon > 0$ there exists a $\delta > 0$ such that for any tagged partition $\breve{\mathcal{V}}_\delta$ we have $\left| \sum_{h=1}^{n} f(\vartheta_h)(t_h - t_{h-1}) - \overline{I} \right| < \epsilon$. Here $\overline{I} = \overline{R} \int_p^q f(t)\Delta t$.

Now, let $\mathcal{W} \in \mathfrak{P}$, $\mathcal{W} = \{p = t_0 < t_1 < \ldots < t_n = q\}$, with t_0, t_1, …, t_n being the finite points of division. We consider subintervals of the form $(t_{h-1}, t_h]$, for $1 \leq h \leq n$, and from each subinterval we choose ξ_h arbitrarily, defined as $\xi_h \in (t_{h-1}, t_h]$, and call it the tag point of the respective subinterval. For $\mathcal{W} \in \mathfrak{P}$, we define a point-interval collection as $\breve{\mathcal{W}} = \{(\xi_h, (t_{h-1}, t_h])\}_{h=1}^{n}$, and call it the tagged partition.
We define the mesh of $\mathcal{W}$ as mesh-$(\mathcal{W}) = |\mathcal{W}| = \max_{1 \leq h \leq n}(t_h - t_{h-1}) > 0$. For some $\delta > 0$, $\mathcal{W}_\delta$ will represent a partition of $[p, q]$ with mesh δ satisfying the property: For each $h = 1, 2, \ldots, n$ we have either- $(t_h - t_{h-1}) \leq \delta$ or $(t_h - t_{h-1}) > \delta \wedge t_h = \sigma(t_{h-1})$. Hence, $\breve{\mathcal{W}}_\delta$ will mean a tagged partition with mesh δ satisfying the above property.

We proceed to give the definition of Riemann ∇-integral on time scales according to G. Sh. Guseinov and B. Kaymakçalan [7, 8].

Definition 2.4. [8] Riemann ∇-integral: A function $f : [p, q]_{\mathbf{T}} \to \mathbb{R}$ is Riemann ∇-integrable if there exists a number $\underline{I} \in \mathbb{R}$ such that, for any $\epsilon > 0$ there exists a $\delta > 0$ such that for any tagged partition $\breve{\mathcal{W}}_\delta$ we have $\left| \sum_{h=1}^{n} f(\xi_h)(t_h - t_{h-1}) - \underline{I} \right| < \epsilon$. Here $\underline{I} = \underline{R} \int_p^q f(t)\nabla t$.

3. Riemann-Stieltjes integral on time scales

We, again, establish a few preliminary information followed by the definition of the Riemann-Stieltjes Δ-integral and the Riemann-Stieltjes ∇-integral.

Given $\mathcal{V} \in \mathfrak{P}$, $\mathcal{V} = \{p = t_0 < t_1 < \ldots < t_n = q\}$, with t_0, t_1, ..., t_n being the finite points of division. Let ψ be a real-valued monotone increasing function on $[p, q]$. Then for partition $\mathcal{V}$ of $[p, q]$ we define $\psi(\mathcal{V}) = \{\psi(p) = \psi(t_0) < \psi(t_1) < \ldots < \psi(t_n) = \psi(q)\}$ and $\triangle\psi_h = \psi(t_h) - \psi(t_{h-1})$. (Note that unless mentioned otherwise, f will be considered a bounded function and ψ will be considered a monotone increasing function on $[p, q]$). Also for the Δ-integral we consider subintervals of the form $[t_{h-1}, t_h)$.

We now proceed to give the definition of the Riemann-Stieltjes Δ-integral using the Darboux sum definition (which we will call the Darboux-Stieltjes Δ-integral); and using the Riemann sum definition (which we will call the Riemann-Stieltjes Δ-integral) on time scales according to Dorota Mozyrska et al. [13, 14], and we prove the equivalence of these two definitions.

Considering subintervals of the form $[t_{h-1}, t_h)$, we define-
$\overline{D} = \sup\{f(t) : t \in [p, q)\}$; $\overline{d} = \inf\{f(t) : t \in [p, q)\}$; $\overline{D_h} = \sup\{f(t) : t \in [t_{h-1}, t_h)\}$; and $\overline{d_h} = \inf\{f(t) : t \in [t_{h-1}, t_h)\}$.
The upper Darboux-Stieltjes Δ-sum of f with respect to partition $\mathcal{V}$, denoted by $U_\Delta(\mathcal{V}, f, \psi)$ is defined by $U_\Delta(\mathcal{V}, f, \psi) = \sum_{h=1}^{n} \overline{D_h}\ [\psi(t_h) - \psi(t_{h-1})]$.
The lower Darboux-Stieltjes Δ-sum of f with respect to partition $\mathcal{V}$, denoted by $L_\Delta(\mathcal{V}, f, \psi)$ is defined by $L_\Delta(\mathcal{V}, f, \psi) = \sum_{h=1}^{n} \overline{d_h}\ [\psi(t_h) - \psi(t_{h-1})]$.
The upper Darboux-Stieltjes Δ-integral from p to q with respect to ψ is defined as

$$U \int_p^q f(t)\Delta\psi(t) = \inf\{U_\Delta(\mathcal{V}, f, \psi) : \mathcal{V} \in \mathfrak{P}\}.$$

The lower Darboux-Stieltjes Δ-integral from p to q with respect to ψ is defined as

$$L \int_p^q f(t)\Delta\psi(t) = \sup\{L_\Delta(\mathcal{V}, f, \psi) : \mathcal{V} \in \mathfrak{P}\}.$$

Definition 3.1. [13] Darboux-Stieltjes Δ-integral : Let function $f : [p, q]_{\mathbf{T}} \to \mathbb{R}$ be a bounded function and let $\psi : [p, q]_{\mathbf{T}} \to \mathbb{R}$ be a monotone increasing function, then f is said to be Darboux-Stieltjes Δ-integral with respect to ψ on $[p, q]$ provided $U \int_p^q f(t)\Delta\psi(t) = L \int_p^q f(t)\Delta\psi(t)$, and the common value denoted by $\overline{DS} \int_p^q f(t)\Delta\psi(t)$ is called the Darboux-Stieltjes Δ-integral of f with respect to ψ on $[p, q]$.

Now for the Riemann sum definition- for each subinterval of the form $[t_{h-1}, t_h)$, for $1 \leq h \leq n$, we choose ϑ_h arbitrarily, defined as $\vartheta_h \in [t_{h-1}, t_h)$, and call it the tag point of the respective subinterval. As defined above, for $\mathcal{V} \in \mathfrak{P}$, we define a point-interval collection as $\breve{\mathcal{V}} = \{(\vartheta_h, [t_{h-1}, t_h))\}_{h=1}^{n}$, and call it the tagged partition.
We define the mesh of $\mathcal{V}$ as mesh-$(\mathcal{V}) = \max_{1 \leq h \leq n}(t_h - t_{h-1}) > 0$. For some $\delta > 0$, $\mathcal{V}_\delta$ will represent a partition of $[p, q]$ with mesh δ satisfying the property: For each $h = 1, 2, \ldots, n$ we have either- $(t_h - t_{h-1}) \leq \delta$ or $(t_h - t_{h-1}) > \delta \wedge \rho(t_h) = t_{h-1}$. Hence, $\breve{\mathcal{V}}_\delta$ will mean a tagged partition with mesh δ satisfying the above property.

Definition 3.2. [14] Riemann-Stieltjes Δ-integral : Let function $f : [p, q]_{\mathbf{T}} \to \mathbb{R}$ be a bounded function and let $\psi : [p, q]_{\mathbf{T}} \to \mathbb{R}$ be a monotone increasing function.

Then function f with respect to ψ on $[p,q]$ is said to be Riemann-Stieltjes Δ-integrable if there exists a number $\overline{I} \in \mathbb{R}$ such that, for any $\epsilon > 0$ there exists a $\delta > 0$, such that for any tagged partition $\breve{\mathcal{V}}_\delta \in \mathfrak{P}$ we have $\left|\overline{RS} - \overline{I}\right| < \epsilon$. Here $\overline{I} = \overline{RS}\int_p^q f(t)\Delta\psi(t)$ and $\overline{RS} = \sum_{h=1}^n f(\vartheta_h)\big[\psi(t_h) - \psi(t_{h-1})\big]$.

The set of all Riemann-Stieltjes Δ-integrable functions on $[p,q]$ will be denoted by $\mathfrak{RS}_\Delta[p,q]$.

Remark 3.1. Taking $\psi(t) = t$ we see that the Riemann-Steiltjes Δ-integral coincides with the Riemann Δ-integral.

Cases when $\mathbf{T} = \mathbb{R}$ and when $\mathbf{T} = \mathbb{Z}$-

1. When $\mathbf{T} = \mathbb{R}$, the Riemann-Stieltjes Δ-integral coincides with the usual Riemann-Stieltjes integral in $\mathbb{R}$.
2. When $\mathbf{T} = \mathbb{Z}$,

$$\begin{aligned}\overline{DS}\int_p^q f(t)\Delta\psi(t) = \overline{RS}\int_p^q f(t)\Delta\psi(t) &= \sum_{h=1}^n f(t_h)\Big(\psi(t_h) - \psi(t_{h-1})\Big) \\ &= \sum_{m=p+1}^q f(m)\Big(\psi(m) - \psi(m-1)\Big).\end{aligned}$$

Theorem 3.1. *If $f \in \mathfrak{RS}_\Delta[p,q]$, then the value of integral, $\overline{I}$, is unique.*

Proof. Let us assume that f with respect to ψ has two integral values, say $\overline{I}'$ and $\overline{I}''$, both satisfy the definition and let $\epsilon > 0$.
Then, there exists $\delta'_{\frac{\epsilon}{2}} > 0$ such that for any tagged partition $\breve{\mathcal{V}}_{\delta'_{\frac{\epsilon}{2}}}$, the respective Riemann-Stieltjes Δ-sum, $\overline{RS}'$ satisfies $\left|\overline{RS}' - \overline{I}'\right| < \frac{\epsilon}{2}$.
Also, there exists $\delta''_{\frac{\epsilon}{2}} > 0$ such that for any tagged partition $\breve{\mathcal{V}}_{\delta''_{\frac{\epsilon}{2}}}$, the respective Riemann-Stieltjes Δ-sum, $\overline{RS}''$ satisfies $\left|\overline{RS}'' - \overline{I}''\right| < \frac{\epsilon}{2}$.
Now, let $\delta_\epsilon = \min\left\{\delta'_{\frac{\epsilon}{2}}, \delta''_{\frac{\epsilon}{2}}\right\} > 0$ and let $\breve{\mathcal{V}}_{\delta_\epsilon}$ be the tagged partition. Since length of the partition of $\breve{\mathcal{V}}_{\delta_\epsilon}$ is lesser or equal to the length of the partitions of $\breve{\mathcal{V}}_{\delta'_{\frac{\epsilon}{2}}}$ and $\breve{\mathcal{V}}_{\delta''_{\frac{\epsilon}{2}}}$, thus taking $\overline{RS}$ to be the respective Riemann-Stieltjes Δ-sum we have $\left|\overline{RS} - \overline{I}'\right| < \frac{\epsilon}{2}$ and $\left|\overline{RS} - \overline{I}''\right| < \frac{\epsilon}{2}$, whence it follows from triangle inequality that,

$$\begin{aligned}\left|\overline{I}' - \overline{I}''\right| &= \left|\overline{I}' - \overline{RS} + \overline{RS} - \overline{I}''\right| \\ &\leq \left|\overline{I}' - \overline{RS}\right| + \left|\overline{RS} - \overline{I}''\right| < \epsilon.\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we conclude that $\overline{I}' = \overline{I}''$. $\square$

Theorem 3.2. *A bounded function f with respect to a monotone increasing function ψ over $[p, q]$ is Riemann-Stieltjes Δ-integrable if and only if it is Darboux-Stieltjes Δ-integrable.*

Proof. First, we suppose that f with respect to ψ is Darboux-Stieltjes Δ-integrable from p to q i.e., $U\int_p^q f(t)\Delta\psi(t) = \overline{DS}\int_p^q f(t)\Delta\psi(t) = L\int_p^q f(t)\Delta\psi(t)$.
Let us choose an arbitrary $\epsilon > 0$ and a respective $\delta > 0$ such that the Cauchy Criterion of integrability holds i.e., $\forall\ \mathcal{V}_\delta \in \mathfrak{P}$ implies,

$$U_\Delta(\mathcal{V}_\delta, f, \psi) - L_\Delta(\mathcal{V}_\delta, f, \psi) < \epsilon. \tag{3.1}$$

We are to show that,

$$\left|\overline{RS} - \overline{DS}\int_p^q f(t)\Delta\psi(t)\right| < \epsilon, \tag{3.2}$$

for every Riemann-Stieltjes Δ-sum ,$\overline{RS}$, with partition $\mathcal{V}_\delta \in \mathfrak{P}$.
From Eq. (3.1) we have,

$$U_\Delta(\mathcal{V}_\delta, f, \psi) \quad < \quad \epsilon + L_\Delta(\mathcal{V}_\delta, f, \psi) \le \epsilon + L\int_p^q f(t)\Delta\psi(t) = \epsilon + \overline{DS}\int_p^q f(t)\Delta\psi(t).$$

Similarly,

$$L_\Delta(\mathcal{V}_\delta, f, \psi) \quad \ge \quad U_\Delta(\mathcal{V}_\delta, f, \psi) - \epsilon \ge U\int_p^q f(t)\Delta\psi(t) - \epsilon = \overline{DS}\int_p^q f(t)\Delta\psi(t) - \epsilon.$$

It is clear that $L_\Delta(\mathcal{V}, f, \psi) \le \overline{RS} \le U_\Delta(\mathcal{V}, f, \psi)$.
Hence taking-

$$\begin{aligned} \overline{RS} &\le U_\Delta(\mathcal{V}_\delta, f, \psi) \\ 0 &\ge \overline{RS} - U_\Delta(\mathcal{V}_\delta, f, \psi) > \overline{RS} - \left(\epsilon + \overline{DS}\int_p^q f(t)\Delta\psi(t)\right) \end{aligned}$$

$$\epsilon \quad \ge \quad \overline{RS} - \overline{DS}\int_p^q f(t)\Delta\psi(t). \tag{3.3}$$

Also taking-

$$\begin{aligned} L_\Delta(\mathcal{V}_\delta, f, \psi) &\le \overline{RS} \\ 0 &\le \overline{RS} - L_\Delta(\mathcal{V}_\delta, f, \psi) \le \overline{RS} - \left(\overline{DS}\int_p^q f(t)\Delta\psi(t) - \epsilon\right) \end{aligned}$$

$$-\epsilon \quad \le \quad \overline{RS} - \overline{DS}\int_p^q f(t)\Delta\psi(t). \tag{3.4}$$

From Eq. (3.3) and Eq. (3.4) we conclude that Eq. (3.2) is true, i.e.,

$$\left|\overline{RS} - \overline{DS}\int_p^q f()\Delta\psi(t)\right| < \epsilon.$$

This shows that $\overline{RS}\int_p^q f(t)\Delta\psi(t) = \overline{I} = \overline{DS}\int_p^q f(t)\Delta\psi(t)$.
Thus, if f with respect to ψ is Darboux-Stieltjes Δ-integrable then it is also Riemann-Stieltjes Δ-integrable.

Secondly, we suppose that f with respect to ψ is Riemann-Stieltjes Δ-integrable from p to q, and show that it is also Darboux-Stieltjes Δ-integrable for the same. Let $\mathcal{V} = \{p = t_0 < t_1 < \ldots < t_n = q\}$ be a partition of $[p, q]$, then for any $\epsilon > 0$ there exists a $\delta > 0$ such that we consider partitions $\mathcal{V}_\delta \in \mathfrak{P}$. For each $h = \{1, 2, .., n\}$ we choose the tag point $\vartheta_h \in, [t_{h-1}, t_h)$ so that, $f(\vartheta_h) < \overline{d_h} + \epsilon$, where $\overline{d_h} = \inf\{f(t) : t \in [t_{h-1}, t_h)\}$.
The Riemann-Stieltjes Δ-sum, $\overline{RS}$, for these choice of ϑ_h's gives,

$$\begin{aligned}
\overline{RS} &= \sum_{h=1}^{n} f(\vartheta_h)\Big(\psi(t_h) - \psi(t_{h-1})\Big) < \sum_{h=1}^{n}(\overline{d_h} + \epsilon)\Big(\psi(t_h) - \psi(t_{h-1})\Big)\\
L_\Delta(\mathcal{V}_\delta, f, \psi) &> \overline{RS} - \epsilon\big(\psi(q) - \psi(p)\big).
\end{aligned}$$

By definition, we have $\left|\overline{RS} - \overline{I}\right| < \epsilon \;\Rightarrow\; -\epsilon + \overline{I} < \overline{RS} < \overline{I} + \epsilon$.
Hence,

$$\begin{aligned}
L\int_p^q f(t)\Delta\psi(t) &\geq L_\Delta(\mathcal{V}_\delta, f, \psi) > \overline{RS} - \epsilon\big(\psi(q) - \psi(p)\big)\\
&> \big(-\epsilon + \overline{I}\big) - \epsilon\big(\psi(q) - \psi(p)\big) = \overline{I} - \epsilon - \epsilon\big(\psi(q) - \psi(p)\big).
\end{aligned}$$

Since $\epsilon > 0$ is arbitrarily chosen, we conclude that,

$$L\int_p^q f(t)\Delta\psi(t) \geq \overline{I}. \tag{3.5}$$

Similarly, for each $h = \{1, 2, .., n\}$ we choose the tag point $\zeta_h \in [t_{h-1}, t_h)$ so that, $f(\zeta_h) > \overline{D_h} - \epsilon$, where $\overline{D_h} = \sup\{f(t) : t \in [t_{h-1}, t_h)\}$.
The Riemann-Stieltjes Δ-sum, $\overline{RS}$, for these choice of ζ_h's gives,

$$\begin{aligned}
\overline{RS} &= \sum_{h=1}^{n} f(\zeta_h)\Big(\psi(t_h) - \psi(t_{h-1})\Big) > \sum_{h=1}^{n}(\overline{D_h} - \epsilon)\Big(\psi(t_h) - \psi(t_{h-1})\Big)\\
U_\Delta(\mathcal{V}_\delta, f, \psi) &< \overline{RS} + \epsilon\big(\psi(q) - \psi(p)\big).
\end{aligned}$$

By definition, we have $\left|\overline{RS} - \overline{I}\right| < \epsilon \;\Rightarrow -\epsilon + \overline{I} < \overline{RS} < \overline{I} + \epsilon$.
Hence,

$$\begin{aligned}
U\int_p^q f(t)\Delta\psi(t) &\leq U_\Delta(\mathcal{V}_\delta, f, \psi) < \overline{RS} + \epsilon\big(\psi(q) - \psi(p)\big)\\
&< (\overline{I} + \epsilon) - \epsilon\big(\psi(q) - \psi(p)\big) = \overline{I} + \epsilon - \epsilon\big(\psi(q) - \psi(p)\big).
\end{aligned}$$

Since $\epsilon > 0$ is arbitrarily chosen, we conclude that,

$$U\int_p^q f(t)\Delta\psi(t) \leq \overline{I}. \tag{3.6}$$

Thus from Eq. (3.5) and Eq. (3.6) we get $\overline{I} \leq L\int_p^q f(t)\Delta\psi(t) \leq U\int_p^q f(t)\Delta\psi(t) \leq \overline{I}$, which implies $L\int_p^q f(t)\Delta\psi(t) = U\int_p^q f(t)\Delta\psi(t) = \overline{I}$. This proves that $\overline{DS}\int_p^q f(t)\Delta\psi(t) = \overline{I} = \overline{RS}\int_p^q f(t)\Delta\psi(t)$.
Thus if f with respect to ψ is Riemann-Stieltjes Δ-integrable then it is also Darboux-Stieltjes Δ-integrable. □

This concludes that the Riemann-Stieltjes Δ-integral defined in terms of the Riemann sum and the Darboux sum are equivalent, given f is bounded and ψ is monotonically increasing on a closed interval.

Now let $\mathcal{W} \in \mathfrak{P}$, $\mathcal{W} = \{p = t_0 < t_1 < \ldots < t_n = q\}$, with $t_0, t_1, \ldots, t_n$ being the finite points of division. Let ψ be a real-valued monotone increasing function on $[p, q]$. Then for partition $\mathcal{W}$ of $[p, q]$ we define $\psi(\mathcal{W}) = \{\psi(p) = \psi(t_0) < \psi(t_1) < \ldots < \psi(t_n) = \psi(q)\}$ and $\triangle\psi_h = \psi(t_h) - \psi(t_{h-1})$. (Note that unless mentioned otherwise, f will be considered a bounded function and ψ will be considered a monotone increasing function on $[p, q]$). Also for the ∇-integral we consider subintervals of the form $(t_{h-1}, t_h]$.

We now proceed to give the definition of the Riemann-Stieltjes ∇-integral using the Darboux sum definition (which we will call the Darboux-Stieltjes ∇-integral) according to Dorota Mozyrska et al. [13], and define the Riemann-Stieltjes ∇-integral using the Riemann sum definition (which we will call the Riemann-Stieltjes ∇-integral), and we prove the equivalence of these two definitions.

Considering subintervals of the form $(t_{h-1}, t_h]$, we define-
$\underline{D} = \sup\{f(t) : t \in (p, q]\}$; $\underline{d} = \inf\{f(t) : t \in (p, q]\}$; $\underline{D_h} = \sup\{f(t) : t \in (t_{h-1}, t_h]\}$; and $\underline{d_h} = \inf\{f(t) : t \in (t_{h-1}, t_h]\}$.
The upper Darboux-Stieltjes ∇-sum of f with respect to partition $\mathcal{W}$, denoted by $U_\nabla(\mathcal{W}, f, \psi)$ is defined by $U_\nabla(\mathcal{W}, f, \psi) = \sum_{h=1}^{n} \underline{D_h}\ \left[\psi(t_h) - \psi(t_{h-1})\right]$.
The lower Darboux-Stieltjes ∇-sum of f with respect to partition $\mathcal{W}$, denoted by $L_\nabla(\mathcal{W}, f, \psi)$ is defined by $L_\nabla(\mathcal{W}, f, \psi) = \sum_{h=1}^{n} \underline{d_h}\ \left[\psi(t_h) - \psi(t_{h-1})\right]$.
The upper Darboux-Stieltjes ∇-integral from p to q with respect to ψ is defined as

$$U\int_p^q f(t)\nabla\psi(t) = \inf\{U_\nabla(\mathcal{W}, f, \psi) : \mathcal{W} \in \mathfrak{P}\}.$$

The lower Darboux-Stieltjes ∇-integral from p to q with respect to ψ is defined as

$$L\int_p^q f(t)\nabla\psi(t) = \sup\{L_\nabla(\mathcal{W}, f, \psi) : \mathcal{W} \in \mathfrak{P}\}.$$

Definition 3.3. [13] Darboux-Stieltjes Δ-integral : Let function $f : [p, q]_{\mathbf{T}} \to \mathbb{R}$ be a bounded function and let $\psi : [p, q]_{\mathbf{T}} \to \mathbb{R}$ be a monotone increasing function, then f is said to be Darboux-Stieltjes ∇-integral with respect to ψ on $[p, q]$

provided $U\int_p^q f(t)\nabla\psi(t) = L\int_p^q f(t)\nabla\psi(t)$, and the common value denoted by $\underline{DS}\int_p^q f(t)\nabla\psi(t)$ is called the Darboux-Stieltjes ∇-integral of f with respect to ψ on $[p, q]$.

Now for the Riemann sum definition- for each subinterval of the form $(t_{h-1}, t_h]$, for $1 \leq h \leq n$, we choose ξ_h arbitrarily, defined as $\xi_h \in (t_{h-1}, t_h]$, and call it the tag point of the respective subinterval. As defined above, for $\mathcal{W} \in \mathfrak{P}$, we define a point-interval collection as $\breve{\mathcal{W}} = \big\{\big(\xi_h, (t_{h-1}, t_h]\big)\big\}_{h=1}^n$, and call it the tagged partition. We define the mesh of $\mathcal{W}$ as mesh-$(\mathcal{W}) = \max_{1\leq h\leq n}(t_h - t_{h-1}) > 0$. For some $\delta > 0$, $\mathcal{W}_\delta$ will represent a partition of $[p, q]$ with mesh δ satisfying the property: For each $h = 1, 2, \ldots, n$ we have either- $(t_h - t_{h-1}) \leq \delta$ or $(t_h - t_{h-1}) > \delta \wedge \rho(t_h) = t_{h-1}$. Hence, $\breve{\mathcal{W}}_\delta$ will mean a tagged partition with mesh δ satisfying the above property.

Definition 3.4. [14] Riemann-Stieltjes ∇-integral: Let function $f : [p, q]_{\mathbf{T}} \to \mathbb{R}$ be a bounded function and let $\psi : [p, q]_{\mathbf{T}} \to \mathbb{R}$ be a monotone increasing function. Then function f with respect to ψ on $[p, q]$ is said to be Riemann-Stieltjes ∇-integrable if there exists a number $\underline{I} \in \mathbb{R}$ such that, for any $\epsilon > 0$ there exists a $\delta > 0$, such that for any tagged partition $\breve{\mathcal{W}}_\delta \in \mathfrak{P}$ we have $|\underline{RS} - \underline{I}| < \epsilon$. Here $\underline{I} = \underline{RS}\int_p^q f(t)\nabla\psi(t)$ and $\underline{RS} = \sum_{h=1}^n f(\xi_h)\big[\psi(t_h) - \psi(t_{h-1})\big]$.

The set of all Riemann-Stieltjes ∇-integrable functions on $[p, q]$ will be denoted by $\mathfrak{RS}_\nabla[p, q]$.

Remark 3.2. Taking $\psi(t) = t$ we see that the Riemann-Steiltjes ∇-integral coincides with the Riemann ∇-integral.

Cases when $\mathbf{T} = \mathbb{R}$ and when $\mathbf{T} = \mathbb{Z}$-

1. When $\mathbf{T} = \mathbb{R}$, the Riemann-Stieltjes ∇-integral coincides with the usual Riemann-Stieltjes integral in $\mathbb{R}$.

2. When $\mathbf{T} = \mathbb{Z}$,

$$\begin{aligned}\underline{DS}\int_p^q f(t)\nabla\psi(t) = \underline{RS}\int_p^q f(t)\nabla\psi(t) &= \sum_{h=1}^n f(t_h)\Big(\psi(t_h) - \psi(t_{h-1})\Big)\\ &= \sum_{m=p+1}^q f(m)\Big(\psi(m) - \psi(m-1)\Big).\end{aligned}$$

Theorem 3.3. *If $f \in \mathfrak{RS}_\nabla[p, q]$, then the value of integral, $\underline{I}$, is unique.*

Theorem 3.4. *A bounded function f with respect to a monotone increasing function ψ over $[p, q]$ is Riemann-Stieltjes ∇-integrable if and only if it is Darboux-Stieltjes ∇-integrable.*

This concludes that the Riemann-Stieltjes ∇-integral defined in terms of the Riemann sum and the Darboux sum are equivalent, given f is bounded and ψ is monotonically increasing on a closed interval.

Henceforth, all definitions and results will be in terms of the Δ-integral. The case of the ∇-integral can be obtained in a similar manner using the above ∇-integral definition.

Below we discuss the Riemann-Stieltjes integrability of finite discontinuous functions, considering three cases. Case when the monotone increasing function ψ has finite points of discontinuity while the bounded function f is continuous at those points. Case when the bounded function f has finite points of discontinuity while the monotone increasing function ψ is continuous at those points. And finally the case when both bounded function f and monotone increasing function ψ has a common point of discontinuity.

4. Results

Theorem 4.1. *Let $\psi : [p, q] \to \mathbb{R}$ be a step function with discontinuities at $r_1 < \ldots < r_s$, in $[p, q]$. Let $f : [p, q] \to \mathbb{R}$ be continuous at each r_j, $1 \leq j \leq s$. Then $f \in \mathfrak{R}\mathfrak{S}_\Delta[p, q]$ and,*

$$\overline{RS} \int_p^q f(t)\Delta\psi(t) = \sum_{j=1}^{s} f(r_j)\Big[\psi(r_j^+) - \psi(r_j^-)\Big],$$

where $\psi(r_j^+) = \lim_{r_j \to t^+} \psi(t)$ and $\psi(r_j^-) = \lim_{r_j \to t^-} \psi(t)$. Also $\psi(p^-) = \psi(p)$ and $\psi(q^+) = \psi(q)$.

Proof. Let $\epsilon > 0$. Choose partition $\mathcal{V}_\delta = \big\{p = \tilde{t}_0 < \tilde{t}_1 < \ldots < \tilde{t}_m = q\big\}$ such that $\big\{r_1, r_2, \ldots, r_s\big\} \subset \mathcal{V}_\delta$ and $\delta = \min\big\{|r_2 - r_1|, \ldots, |r_s - r_{s-1}|, \delta_0\big\}$, δ_0 given by $(\star)$. (The explanation of $(\star)$ is mentioned below).
Now let $\mathcal{V} = \big\{p = t_0 < t_1 < \ldots < t_n = q\big\} \supset \mathcal{V}_\delta$ and from each subinterval we choose $\vartheta_h \in [t_{h-1}, t_h)$ arbitrarily, then $\overline{RS}$ is given by,

$$\overline{RS} = \sum_{h=1}^{n} f(\vartheta_h)\Big[\psi(t_h) - \psi(t_{h-1})\Big].$$

Clearly no r_j can be strictly between t_{h-1} and t_h. Furthermore, since $\mathcal{V} \supset \mathcal{V}_\delta$ where $\delta = \min\big\{|r_2 - r_1|, ..., |r_s - r_{s-1}|, \delta_0\big\}$ we cannot have both t_{h-1} and t_h in $\big\{r_1, r_2, \ldots, r_s\big\}$. So either,

1. neither t_h nor t_{h-1} is in $\{r_1, r_2, \ldots, r_s\}$, in which case, as no point of discontinuity r_j can lie between t_{h-1} and t_h we have $\psi(t_h) - \psi(t_{h-1}) = 0$,

Or

2. $t_h = r_j$ for some $1 \leq j \leq s$. In this case t_{h-1} cannot be r'_j and no r'_j can be between t_{h-1} and r_j so that,

$$\psi(t_h) - \psi(t_{h-1}) = \psi(r_j) - \psi(r_j^-),$$

Or

3. $t_{h-1} = r_j$ for some $1 \leq j \leq s$. In this case t_h cannot be r'_j and no r'_j can be between r_j and t_h so that,

$$\psi(t_h) - \psi(t_{h-1}) = \psi(r_j^+) - \psi(r_j).$$

So,

$$\begin{aligned}\overline{RS} &= \sum_{h=1}^{n} f(\vartheta_h)\Big[\psi(t_h) - \psi(t_{h-1})\Big] \\ &= \sum_{j=1}^{s} \Big\{ f(\vartheta_{h_j})\big[\psi(r_j) - \psi(r_j^-)\big] + f(\vartheta_{h_{j'}})\big[\psi(r_j^+) - \psi(r_j)\big]\Big\}.\end{aligned}$$

Here ϑ_{h_j} lies in the subinterval of $\mathcal{V}$ whose right hand end point is r_j and so is within a distance less than δ or r_j.
Similarly, $\vartheta_{h_{j'}}$ lies in the subinterval of $\mathcal{V}$ whose left hand end point is r_j and so is within a distance less than δ or r_j.
Now,

$$\sum_{j=1}^{s} f(r_j)\Big[\psi(r_j^+) - \psi(r_j^-)\Big] = \sum_{j=1}^{s} \Big\{ f(r_j)\big[\psi(r_j) - \psi(r_j^-)\big] + f(r_j)\big[\psi(r_j^+) - \psi(r_j)\big]\Big\}.$$

Hence,

$$\begin{aligned}\left|\overline{RS} - \sum_{j=1}^{s} f(r_j)\Big[\psi(r_j^+) - \psi(r_j^-)\Big]\right| \leq \sum_{j=1}^{s} \Big\{ &\big|f(\vartheta_{h_j}) - f(r_j)\big|\big|\psi(r_j) - \psi(r_j^-)\big| \\ &+ \big|f(\vartheta_{h_{j'}}) - f(r_j)\big|\big|\psi(r_j^+) - \psi(r_j)\big|\Big\}.\end{aligned}$$

Now $(\star)$ for each $1 \leq j \leq s$, f is continuous at r_j so that there is a $\delta_j > 0$ such that,

$$\Big|f(\vartheta) - f(r_j)\Big| < \frac{\epsilon}{\sum_{k=1}^{s} \big\{\big|\psi(r_k) - \psi(r_k^-)\big| + \big|\psi(r_k^+) - \psi(r_k)\big|\big\}},$$

for all ϑ with $|\vartheta - r_j| < \delta_j$.
Choose $\delta_0 = \min\big\{\delta_1, \delta_2, ..., \delta_s\big\}$. Then,

$$\left|\overline{RS} - \sum_{j=1}^{s} f(r_j)\Big[\psi(r_j^+) - \psi(r_j^-)\Big]\right| < \epsilon,$$

as desired. □

Theorem 4.2. *Suppose that f is bounded on $[p,q]$, and has only finitely many points of discontinuity in $[p,q]$, and that the monotonically increasing function ψ is continuous at each point of discontinuity of f. Then $f \in \mathfrak{RS}_{\triangle}[p,q]$.*

Proof. Let $\epsilon > 0$. Suppose that f is bounded on $[p,q]$ and continuous on $[p,q] - \mathcal{A}$ where $\mathcal{A} = \{r_1, r_2, \ldots, r_s\}$ is the non-empty finite set of points of discontinuity of f in $[p,q]$, and suppose ψ is a monotonically increasing function on $[p,q]$ that is continuous at each element of $\mathcal{A}$.
Given $\mathcal{A}$ is finite and ψ is continuous at each $r_j \in \mathcal{A}$, we find s pairwise disjoint intervals $[x_j, y_j]$, $j = 1, 2, \ldots, s$ such that,

$$\mathcal{A} \subset \bigcup_{j=1}^{s} [x_j, y_j] \subsetneq [p,q] \ \ \textit{and} \ \ \sum_{j=1}^{s} \Big[\psi(y_j) - \psi(x_j)\Big] < \epsilon^*,$$

for any $\epsilon^* > 0$; furthermore, the intervals can be chosen in such a way that $r_m \in \mathcal{A} \cap (p,q)$ is an element of the interior of the corresponding interval, $[x_m, y_m]$. Let

$$\mathcal{K} = [p,q] - \bigcup_{j=1}^{s} [x_j, y_j].$$

Then $\mathcal{K}$ is compact and f is continuous on $\mathcal{K}$ implies that f is uniformly continuous there. Thus, corresponding to $\epsilon^* > 0$, $\exists\ \delta > 0$ such that,

$$\forall\ \tilde{t}_1, \tilde{t}_2 \in \mathcal{K}, \qquad \big|\tilde{t}_1 - \tilde{t}_2\big| < \delta \Rightarrow \big|f(\tilde{t}_1) - f(\tilde{t}_2)\big| < \epsilon^*.$$

Now let $\mathcal{V} = \{p = t_0, t_1, \ldots, t_n = q\}$ be a partition of $[p,q]$ satisfying the following conditions:

- $x_j, y_j \in \mathcal{V}\ \forall\ j \in \{1, 2, \ldots, s\}$.
- $(x_j, y_j) \cap \mathcal{V} = \phi$.
- $t_{h-1} \neq x_j$ where $h \in \{1, 2, \ldots, n\}$ and $j \in \{1, 2, \ldots, s\} \Rightarrow \triangle t_h < \delta$.

Note that under the conditions established, $t_{m-1} = x_j \Rightarrow t_m = y_j$.
If,

$$\overline{D} = \sup_{t \in [p,q]} \big|f(t)\big|,$$

$$\overline{D_h} = \sup_{t_{h-1} \leq t \leq t_h} f(t) \ \ \textit{and} \ \ \overline{d_h} = \sup_{t_{h-1} \leq t \leq t_h} f(t),$$

then for each h, $\overline{D_h} - \overline{d_h} \leq 2\overline{D}$.
Furthermore, $\overline{D_h} - \overline{d_h} < \epsilon^*$ as long as $t_{h-1} \neq x_j$. Hence,

$$\begin{aligned} U_{\triangle}(\mathcal{V}, f, \psi) - L_{\triangle}(\mathcal{V}, f, \psi) &= \sum_{j=1}^{s} \big(\overline{D_j} - \overline{d_j}\big)\Big(\psi(t_j) - \psi(t_{j-1})\Big) \\ &\leq \big[\psi(q) - \psi(p)\big]\epsilon^* + 2\overline{D}\epsilon^* < \epsilon, \end{aligned}$$

where,

$$\epsilon^* < \frac{\epsilon}{2\overline{D} + \big[\psi(q) - \psi(p)\big]}.$$

Since $\epsilon > 0$ is arbitrary, we conclude $f \in \mathfrak{RS}_\Delta[p, q]$. □

Theorem 4.3. *If f and ψ have a common discontinuity, say $r \in [p, q]$ then f is not Riemann-Stieltjes Δ-integrable with respect to ψ on $[p, q]$.*

Proof. By contradiction. Suppose $f \in \mathfrak{RS}_\Delta[p, q]$, then given $\epsilon > 0 \ \exists \ \delta > 0$ for partition $\mathcal{V}_\delta \in \mathfrak{P}$ of $[p, q]$ such that,

$$\left|\overline{RS}_{\mathcal{V}_\delta} - \overline{RS}\int_p^q f(t)\Delta\psi(t)\right| < \epsilon, \tag{4.1}$$

where, $\overline{RS}_{\mathcal{V}_\delta} = \sum_{h=1}^{n} f(\vartheta_h)\big(\psi(t_h) - \psi(t_{h-1})\big)$ and $\vartheta_h \in [t_{h-1}, t_h)$. Let,

$$\mathcal{V} = \mathcal{V}_\delta \cup r \quad \Rightarrow \mathcal{V}_\delta \subset \mathcal{V}. \tag{4.2}$$

Choose, $\epsilon_f > 0$ such that for all $\delta_f > 0 \ \exists \ \vartheta_f$ such that,

$$\big|\vartheta_f - r\big| < \delta_f \ \ and \ \ \big|f(\vartheta_f) - f(r)\big| \geq \sqrt{\epsilon_f}.$$

Choose $\epsilon_\psi > 0$ such that for all $\delta_\psi > 0 \ \exists \ \vartheta_\psi$ such that,

$$\big|\vartheta_\psi - r\big| < \delta_\psi \ \ and \ \ \big|\psi(\vartheta_\psi) - \psi(r)\big| \geq \sqrt{\epsilon_\psi}.$$

Since $p < r < q$, there exists k, $1 \leq k \leq n$, such that $t_{k-1} < r < t_k$.
Let $\epsilon^* = \inf\big\{\epsilon_f, \epsilon_\psi\big\}$. Choose $\epsilon = \epsilon^*$ for $\delta^* = \min\big(t_k - r, r - t_{k-1}\big)$ so there exists ϑ^* such that,

$$\big|\vartheta_f^* - r\big| < \delta^* \ \ and \ \ \big|f(\vartheta_f^*) - f(r)\big| \geq \sqrt{\epsilon^*} \ \ and,$$

$$\big|\vartheta_\psi^* - r\big| < \delta^* \ \ and \ \ \big|\psi(\vartheta_\psi^*) - f(r)\big| \geq \sqrt{\epsilon^*}.$$

Therefore,

$$\left|\overline{RS}_{\mathcal{V}} - \overline{RS}\int_p^q f(t)\Delta\psi(t)\right| < \epsilon. \tag{4.3}$$

From Eq. 4.1, 4.3, and Eq. 4.2 and Theorem 3.1 we get,

$$\left|\overline{RS}_{\mathcal{V}_\delta} - \overline{RS}_{\mathcal{V}}\right| \geq \sqrt{\epsilon^*}\sqrt{\epsilon^*} \geq \epsilon,$$

which is a contradiction, thus f is not integrable with respect to ψ in case of common discontinuity. □

5. Example

Example 5.1. Let $\mathbb{T} = \overline{l^{\mathbb{Z}}}$, $l > 1$, $f(t) = t$, $\psi(t) = t^2$, $[p, q] = [0, 1]_{\mathbb{T}}$ and $\mathfrak{P}$ be the collection of all possible partitions of $[p, q]$. Let $\mathcal{V} \in \mathfrak{P}$, considering the partition $\mathcal{V} = \{0, l^{-n+1}, \cdots, l^{-1}, 1 : t_0 = 0 < l^{-n+1} < \cdots < l^{-n+h} < \cdots < l^{-1} < 1 = t_n,\ where\, t_h = l^{-n+h}\ for\, h = 1, \cdots, n\}$, we have $\triangle\psi_h = \psi(t_h) - \psi(t_{h-1}) = t_h^2 - t_{h-1}^2 = l^{2(-n+h-1)}(l^2 - 1)$ for $h = 2, \cdots, n$, and $\triangle\psi_1 = t_1^2 - 0 = l^{2(-n+1)}$. According to [13], we have

$$\overline{D_h} = \sup\Big\{f(t) : t \in [t_{h-1}, t_h)\Big\} = \rho(t_h);$$

$$\overline{d_h} = \inf\Big\{f(t) : t \in [t_{h-1}, t_h)\Big\} = t_{h-1}.$$

For our partition, we have $\overline{D_h} = t_{h-1} = \overline{d_h},\ for\, h = 2, \cdots, n$ and $\overline{D_1} = \rho(t_1) = l^{-n}$, $\overline{d_1} = 0$. Then,

$$L_\Delta(\mathcal{V}, f, \psi) = \sum_{h=1}^{n} \overline{d_h}\Big(\psi(t_h) - \psi(t_{h-1})\Big) = \sum_{h=2}^{n} t_{h-1}\triangle\psi_h = \frac{l+1}{l^2+l+1}(1 - l^{3(-n+1)}),$$

and

$$U_\Delta(\mathcal{V}, f, \psi) = \sum_{h=1}^{n} \overline{D_h}\Big(\psi(t_h) - \psi(t_{h-1})\Big) = l^{2-3n} + \sum_{h=2}^{n} t_{h-1}\triangle\psi_h = \frac{l+1+l^{2-3n}}{l^2+l+1}.$$

Thus, the lower Darboux-Stieltjes Δ-integral from p to q with respect to ψ is

$$L\int_p^q f(t)\Delta\psi(t) = \sup\Big\{L_\Delta(\mathcal{V}, f, \psi) : \mathcal{V} \in \mathfrak{P}\Big\} = \lim_{n\to\infty} \frac{l+1}{l^2+l+1}(1 - l^{3(-n+1)}) = \frac{l+1}{l^2+l+1},$$

and the upper Darboux-Stieltjes Δ-integral from p to q with respect to ψ is

$$U\int_p^q f(t)\Delta\psi(t) = \inf\Big\{U_\Delta(\mathcal{V}, f, \psi) : \mathcal{V} \in \mathfrak{P}\Big\} = \lim_{n\to\infty} \frac{l+1+l^{2-3n}}{l^2+l+1} = L\int_p^q f(t)\Delta\psi(t).$$

Consequently, $\overline{DS}\int_p^q f(t)\Delta\psi(t) = \frac{l+1}{l^2+l+1}$. $\forall\vartheta_h \in [t_{h-1}, t_h)$, we have Riemann-Stieltjes Δ-sum, $\overline{RS}$, of the function f with respect to ψ as

$$\overline{RS} = \sum_{h=1}^{n} f(\vartheta_h)\Bigg(\psi(t_h) - \psi(t_{h-1})\Bigg) = \sum_{h=1}^{n} f(\vartheta_h)\Delta\psi_h.$$

As the length of the subinterval $[t_{h-1}, t_h)$ tends to 0, we have $\vartheta_h \to t_{h-1}$ and

$$\sum_{h=1}^{n} f(\vartheta_h)\Bigg(\psi(t_h) - \psi(t_{h-1})\Bigg) \to \frac{l+1}{l^2+l+1},$$

i.e., the Riemann-Stieltjes Δ-integral $\overline{I} = \overline{RS}\int_p^q f(t)\Delta\psi(t) = \frac{l+1}{l^2+l+1} = \overline{DS}\int_p^q f(t)\Delta\psi(t)$.

Example 5.2. Let $\mathbb{T} = \overline{l^{\mathbb{Z}}}$, $l > 1$, $f(t) = t$, $\psi(t) = t^2$, $[p, q] = [0, 1]_{\mathbb{T}}$ and $\mathfrak{P}$ be the collection of all possible partitions of $[p, q]$. Let $\mathcal{W} \in \mathfrak{P}$, considering the partition $\mathcal{W} = \{0, l^{-n+1}, \cdots, l^{-1}, 1 : t_0 = 0 < l^{-n+1} < \cdots < l^{-n+h} < \cdots < l^{-1} < 1 = t_n,\ where\, t_h =$

$l^{-n+h}\ for\ h=1,\cdots,n\}$, we have $\triangle\psi_h=\psi(t_h)-\psi(t_{h-1})=t_h^2-t_{h-1}^2=l^{2(-n+h-1)}(l^2-1)$ for $h=2,\cdots,n$, and $\triangle\psi_1=t_1^2-0=l^{2(-n+1)}$. According to [13], we have

$$\underline{D_h}=\sup\Big\{f(\mathrm{t}):\mathrm{t}\in(\mathrm{t}_{h-1},\mathrm{t}_h]\Big\}=t_h;$$

$$\underline{d_h}=\inf\Big\{f(\mathrm{t}):\mathrm{t}\in(\mathrm{t}_{h-1},\mathrm{t}_h]\Big\}=\sigma(t_{h-1}).$$

For our partition, we have $\underline{D_h}=t_h=\underline{d_h},\ for\ h=2,\cdots,n$ and $\underline{D_1}=t_1=l^{-n+1}$, $\underline{d_1}=\sigma(0)=0$. Then,

$$L_\nabla(\mathcal{W},f,\psi)=\sum_{h=1}^{n}\underline{d_h}\Big(\psi(\mathrm{t}_h)-\psi(\mathrm{t}_{h-1})\Big)=\sum_{h=2}^{n}t_h\triangle\psi_h=l\frac{l+1}{l^2+l+1}(1-l^{3(-n+1)}),$$

and

$$U_\nabla(\mathcal{W},f,\psi)=\sum_{h=1}^{n}\underline{D_h}\Big(\psi(\mathrm{t}_h)-\psi(\mathrm{t}_{h-1})\Big)=l^{3-3n}+\sum_{h=2}^{n}t_h\triangle\psi_h=l\frac{l+1+l^{2-3n}}{l^2+l+1}.$$

Thus, the lower Darboux-Stieltjes ∇-integral from p to q with respect to ψ is

$$L\int_p^q f(\mathrm{t})\nabla\psi(\mathrm{t})=\sup\Big\{L_\nabla(\mathcal{W},f,\psi):\mathcal{W}\in\mathfrak{P}\Big\}=\lim_{n\to\infty}l\frac{l+1}{l^2+l+1}(1-l^{3(-n+1)})=\frac{l^2+l}{l^2+l+1},$$

and the upper Darboux-Stieltjes ∇-integral from p to q with respect to ψ is

$$U\int_p^q f(\mathrm{t})\nabla\psi(\mathrm{t})=\inf\Big\{U_\nabla(\mathcal{W},f,\psi):\mathcal{W}\in\mathfrak{P}\Big\}=\lim_{n\to\infty}l\frac{l+1+l^{2-3n}}{l^2+l+1}=L\int_p^q f(\mathrm{t})\nabla\psi(\mathrm{t}).$$

Consequently, $\underline{DS}\int_p^q f(\mathrm{t})\nabla\psi(\mathrm{t})=\frac{l^2+l}{l^2+l+1}$. $\forall\xi_h\in(\mathrm{t}_{h-1},\mathrm{t}_h]$, we have Riemann-Stieltjes ∇-sum, $\underline{RS}$, of the function f with respect to ψ as

$$\underline{RS}=\sum_{h=1}^{n}f(\xi_h)\Big(\psi(\mathrm{t}_h)-\psi(\mathrm{t}_{h-1})\Big).$$

As length of the subinterval $(\mathrm{t}_{h-1},\mathrm{t}_h]$ tends to 0, then we have $\xi_h\to t_h$ and

$$\sum_{h=1}^{n}f(\xi_h)\Big(\psi(\mathrm{t}_h)-\psi(\mathrm{t}_{h-1})\Big)\to\frac{l^2+l}{l^2+l+1},$$

i.e., the Riemann-Stieltjes ∇-Integral $\underline{I}=\underline{RS}\int_p^q f(\mathrm{t})\nabla\psi(\mathrm{t})=\frac{l^2+l}{l^2+l+1}=\underline{DS}\int_p^q f(\mathrm{t})\nabla\psi(\mathrm{t})$.

6. Conclusion

We establish the equivalence of the Riemann-Stieltjes Δ-(resp. ∇-) integral on time scales as defined in [13, 14] in terms of the Darboux sum definition and the Riemann sum definition, and discuss the Riemann-Stieltjes integrability of finite discontinuous functions, considering three cases. Case one is when the monotone increasing function ψ has finite points of discontinuity while the bounded function f is continuous at those points. Case two is when the bounded function f has finite points of discontinuity while the monotone increasing function ψ is continuous at those points. And finally the case three is when both bounded function f and monotone increasing function ψ has a common point of discontinuity. We establish that the integrability holds for two of the above three cases, failing when both bounded function f and monotone increasing function ψ has a common point of discontinuity.

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