FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 38, No 3 (2023), 577 - 597
https://doi.org/10.22190/FUMI230204037S
Original Scientific Paper

COUPLED FIXED POINT THEOREMS FOR CONTRACTIVE TYPE CONDITION IN PARTIAL METRIC SPACES WITH APPLICATIONS

Gurucharan Singh Saluja

H.N. 3/1005, Geeta Nagar, Raipur,
Raipur-492001 (Chhattisgarh), India

Abstract. This paper deals with a coupled fixed point theorem for a mapping satisfying contractive type condition in the setting of partial metric spaces. Furthermore, we give some consequences of the established result. Also, we give an example to validate the result and state some applications to the main result of a self mapping which is involved in an integral type contraction. Our results extend and generalize several previously published results from the existing literature. Specially, our results generalize the results of *Aydi* [5].
Keywords: Coupled fixed point, contractive type condition, partial metric space.

1. Introduction

In 1922, *Banach* has proved a fixed point theorem for a contraction mapping in a complete metric space. It plays an important role in analysis to find a unique solution of many mathematical problems. It is very popular tool in many branches of mathematics for solving existing problems. Since then there are numerous generalizations [18, 20, 23, 49, 59, 60, 63] of this result by weakening its hypothesis while retaining the convergence property of successive iterates for a unique fixed point of mappings. *Wolk* [62] and *Monjardet* [37] investigated the extension of the Banach contraction principle to partially ordered sets (poset) in order to obtain fixed points under certain conditions. In 2004, *Ran and Reurings* [47] established

Received February 04, 2023. accepted October 21, 2023.
Communicated by Dijana Mosić
Corresponding Author: Gurucharan Singh Saluja, Govt. N.P.G. College of Science, H.N. 3/1005, Geeta Nagar, Raipur, Raipur-492001 (Chhattisgarh), India | E-mail: saluja1963@gmail.com
2010 *Mathematics Subject Classification.* Primary 47H10; Secondary 54H25

fixed points in partially ordered metric spaces (POMS) with some applications to matrix equations. Later on, many researchers [2, 3, 22, 37, 39, 43, 47, 58] settled fixed point results in POMS (see, also [15], [56]).

On the other hand, the concept of coupled fixed points in ordered spaces was introduced by *Bhashkar and Lakshmikantham* [9] and applied their results to boundary value problems for the unique solution. Also, *Ciric and Lakshmikantham* [10] introduced the concept of coupled coincidence, common fixed points to nonlinear contractions in ordered metric spaces. More results on coupled fixed points, coupled coincidence points and common coupled fixed points in various spaces, one can see [5, 7, 11, 12, 14, 16, 17, 25, 27, 28, 29, 40, 42, 48, 51, 54, 55, 57] and many others.

Matthews [35, 36] introduced the concept of partial metric space (PMS) as a part of the study of denotational semantics of data flow networks. It is widely recognized that partial metric spaces play an important role in constructing models in the theory of computation (see, e.g., [21], [44] and some others). Introducing partial metric space, *Matthews* proved the partial metric version of Banach fixed point theorem ([8]). The PMS is a generalization of the usual metric spaces in which the distance of a point in the self may not be zero, that is, $d(x,x)$ may not be zero (for more details, see [4], [34], [45]).

Motivated and inspired by the works of *Aydi* [5] and many others, the aim of this paper is to establish a coupled fixed point result for a mapping satisfying generalized contractive condition in the setting of partial metric spaces and also prove well-posedness of coupled fixed point problem. Furthermore, we give some applications of the established result.

2. Preliminaries

In the sequel, we need the following definitions, lemmas and auxiliary results to prove our main result.

Definition 2.1. ([36]) Let $\mathcal{Y}$ be a nonempty set and $p\colon \mathcal{Y}\times\mathcal{Y}\to\mathbb{R}^+$ be a self mapping of $\mathcal{Y}$ such that for all $\rho,\sigma,\tau\in\mathcal{Y}$ the followings are satisfied:

($\mathcal{P}1$) $\rho=\sigma \Leftrightarrow p(\rho,\rho)=p(\rho,\sigma)=p(\sigma,\sigma)$,

($\mathcal{P}2$) $p(\rho,\rho)\le p(\rho,\sigma)$,

($\mathcal{P}3$) $p(\rho,\sigma)=p(\sigma,\rho)$,

($\mathcal{P}4$) $p(\rho,\sigma)\le p(\rho,\tau)+p(\tau,\sigma)-p(\tau,\tau)$.

Then p is called partial metric on $\mathcal{Y}$ and the pair $(\mathcal{Y},p)$ is called partial metric space (in short *PMS*).

Remark 2.1. It is clear that if $p(\rho,\sigma)=0$, then from ($\mathcal{P}1$), ($\mathcal{P}2$), and ($\mathcal{P}3$), $\rho=\sigma$. But if $\rho=\sigma$, $p(\rho,\sigma)$ may not be 0.

If p is a partial metric on X, then the function $p^s\colon\mathcal{Y}\times\mathcal{Y}\to\mathbb{R}^+$ given by

$$p^s(\rho,\sigma)=2p(\rho,\sigma)-p(\rho,\rho)-p(\sigma,\sigma), \tag{2.1}$$

is a metric on $\mathcal{Y}$.

Example 2.1. ([6]) Let $\mathcal{Y} = \mathbb{R}^+$, where $\mathbb{R}^+ = [0, +\infty)$ and $p: \mathcal{Y} \times \mathcal{Y} \to \mathbb{R}^+$ be given by $p(\rho, \sigma) = \max\{\rho, \sigma\}$ for all $\rho, \sigma \in \mathbb{R}^+$. Then $(\mathbb{R}^+, p)$ is a partial metric space.

Example 2.2. ([6]) Let I denote the set of all intervals $[v, w]$ for any real numbers $v \leq w$. Let $p: I \times I \to [0, \infty)$ be a function such that

$$p\Big([v, w], [r, s]\Big) = \max\{w, s\} - \min\{v, r\}.$$

Then (I, p) is a partial metric space.

Example 2.3. ([13]) Let $\mathcal{Y} = \mathbb{R}$ and $p: \mathcal{Y} \times \mathcal{Y} \to \mathbb{R}^+$ be given by $p(\rho, \sigma) = e^{\max\{\rho, \sigma\}}$ for all $\rho, \sigma \in \mathbb{R}$. Then $(\mathcal{Y}, p)$ is a partial metric space.

Many applications of this space has been extensively investigated by many authors (see, [30], [61] for details).

Note also that each partial metric p on $\mathcal{Y}$ generates a T_0 topology τ_p on $\mathcal{Y}$, whose base is a family of open p-balls $\{\mathcal{B}_p(\rho, \varepsilon) : \rho \in \mathcal{Y}, \varepsilon > 0\}$ where

$$\mathcal{B}_p(\rho, \varepsilon) = \{\sigma \in \mathcal{Y} : p(\rho, \sigma) < p(\rho, \rho) + \varepsilon\},$$

for all $\rho \in \mathcal{Y}$ and $\varepsilon > 0$.

Similarly, closed p-ball is defined as

$$\mathcal{B}_p[\rho, \varepsilon] = \{\sigma \in \mathcal{Y} : p(\rho, \sigma) \leq p(\rho, \rho) + \varepsilon\},$$

for all $\rho \in \mathcal{Y}$ and $\varepsilon > 0$.

Definition 2.2. ([35]) Let $(\mathcal{Y}, p)$ be a partial metric space. Then

(Γ_1) a sequence $\{\nu_n\}$ in $(\mathcal{Y}, p)$ is said to be convergent to a point $\nu \in \mathcal{Y}$ if and only if $p(\nu, \nu) = \lim_{n\to\infty} p(\nu_n, \nu)$;

(Γ_2) a sequence $\{\nu_n\}$ is called a Cauchy sequence if $\lim_{m,n\to\infty} p(\nu_m, \nu_n)$ exists and is finite;

(Γ_3) $(\mathcal{Y}, p)$ is said to be complete if every Cauchy sequence $\{\nu_n\}$ in $\mathcal{Y}$ converges to a point $\nu \in \mathcal{Y}$ with respect to τ_p. Furthermore,

$$\lim_{m,n\to\infty} p(\nu_m, \nu_n) = \lim_{n\to\infty} p(\nu_n, \nu) = p(\nu, \nu).$$

(Γ_4) A mapping $g: \mathcal{Y} \to \mathcal{Y}$ is said to be continuous at $\nu_0 \in \mathcal{Y}$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $g\Big(\mathcal{B}_p(\nu_0, \delta)\Big) \subset \mathcal{B}_p\Big(g(\nu_0), \varepsilon\Big)$.

Lemma 2.1. *([35, 36, 5]) Let $(\mathcal{Y}, p)$ be a partial metric space. Then*

(Δ_1) *a sequence $\{\nu_n\}$ in $(\mathcal{Y}, p)$ is a Cauchy sequence if and only if it is a Cauchy sequence in the metric space $(\mathcal{Y}, p^s)$,*

(Δ_2) *a partial metric space $(\mathcal{Y}, p)$ is complete if and only if the metric space $(\mathcal{Y}, p^s)$ is complete, furthermore, $\lim_{n\to\infty} p^s(\nu_n, \nu) = 0$ if and only if*

$$p(\nu, \nu) = \lim_{n\to\infty} p(\nu_n, \nu) = \lim_{n,m\to\infty} p(\nu_n, \nu_m). \tag{2.2}$$

Lemma 2.2. *(see [24]) Let $(\mathcal{Y}, p)$ be a partial metric space.*

(Θ_1) *If for all $\rho, \sigma \in \mathcal{Y}$, $p(\rho, \sigma) = 0$, then $\rho = \sigma$;*

(Θ_2) *If $\rho \neq \sigma$, then $p(\rho, \sigma) > 0$.*

Definition 2.3. ([5]) An element $(\rho, \sigma) \in \mathcal{Y} \times \mathcal{Y}$ is said to be a coupled fixed point of the mapping $F\colon \mathcal{Y} \times \mathcal{Y} \to \mathcal{Y}$ if $F(\rho, \sigma) = \rho$ and $F(\sigma, \rho) = \sigma$.

Example 2.4. Let $\mathcal{Y} = [0, +\infty)$ and $F\colon \mathcal{Y} \times \mathcal{Y} \to \mathcal{Y}$ defined by $F(\rho, \sigma) = \frac{\rho+\sigma}{6}$ for all $\rho, \sigma \in \mathcal{Y}$. One can easily see that F has a unique coupled fixed point $(0, 0)$.

Example 2.5. Let $X = [0, +\infty)$ and $F\colon \mathcal{Y} \times \mathcal{Y} \to \mathcal{Y}$ be defined by $F(\rho, \sigma) = \frac{\rho+\sigma}{2}$ for all $\rho, \sigma \in \mathcal{Y}$. Then we see that F has two coupled fixed point $(0, 0)$ and $(1, 1)$, that is, the coupled fixed point is not unique.

In 2011, *Aydi* [5] proved the following result.

Theorem 2.1. *Let $(\mathcal{Y}, p)$ be a complete partial metric space. Suppose that the mapping $F\colon \mathcal{Y} \times \mathcal{Y} \to \mathcal{Y}$ satisfies one of the following contractive conditions (N_1), (N_2), (N_3):*

(N_1) *for all $\rho, \sigma, \eta, \theta \in \mathcal{Y}$ and nonnegative constants h_1, h_2 with $h_1 + h_2 < 1$,*

$$p(F(\rho, \sigma), F(\eta, \theta)) \leq h_1 p(\rho, \eta) + h_2 p(\sigma, \theta), \tag{2.3}$$

(N_2) *for all $\rho, \sigma, \eta, \theta \in \mathcal{Y}$ and nonnegative constants h_1, h_2 with $h_1 + h_2 < 1$,*

$$p(F(\rho, \sigma), F(\eta, \theta)) \leq h_1 p(F(\rho, \sigma), \rho) + h_2 p(F(\eta, \theta), \eta), \tag{2.4}$$

(N_3) *for all $\rho, \sigma, \eta, \theta \in \mathcal{Y}$ and nonnegative constants h_1, h_2 with $h_1 + 2h_2 < 1$,*

$$p(F(\rho, \sigma), F(\eta, \theta)) \leq h_1 p(F(\rho, \sigma), \eta) + h_2 p(F(\eta, \theta), \rho). \tag{2.5}$$

Then F has a unique coupled fixed point.

3. Main Results

In this section, we shall prove a unique coupled fixed point theorem for a mapping satisfies generalized contractive condition in the framework of partial metric spaces.

Theorem 3.1. *Let $(\mathcal{Y}, p)$ be a complete partial metric space. Suppose that the mapping $F\colon \mathcal{Y}\times\mathcal{Y} \to \mathcal{Y}$ satisfying the following contractive condition for all $\rho, \sigma, \eta, \theta \in \mathcal{Y}$:*

$$\begin{aligned} p(F(\rho,\sigma), F(\eta,\theta)) \quad \leq \quad & q_1\,[p(\rho,\eta) + p(\sigma,\theta)] + q_2\,p(F(\rho,\sigma),\rho) + q_3\,p(F(\eta,\theta),\eta) \\ & +q_4\,p(F(\rho,\sigma),\eta) + q_5\,p(F(\eta,\theta),\rho) + q_6\,p(F(\sigma,\rho),\theta) \\ & +q_7\,p(F(\theta,\eta),\sigma), \end{aligned} \tag{3.1}$$

where $q_1, q_2, \ldots, q_7$ are nonnegative constants with $2q_1+q_2+q_3+q_4+2q_5+q_6+2q_7 < 1$. Then F has a unique coupled fixed point.

Proof. Choose $\rho_0, \sigma_0 \in \mathcal{Y}$. Set $\rho_1 = F(\rho_0, \sigma_0)$ and $\sigma_1 = F(\sigma_0, \rho_0)$. Repeating this process, we obtain two sequences $\{\rho_n\}$ and $\{\sigma_n\}$ in $\mathcal{Y}$ such that $\rho_{n+1} = F(\rho_n, \sigma_n)$ and $\sigma_{n+1} = F(\sigma_n, \rho_n)$. Let $U_n = p(\rho_n, \rho_{n+1})$, $V_n = p(\sigma_n, \sigma_{n+1})$ and $S_n = U_n + V_n$. Then, from the equation (3.1) and using $(\mathcal{P}2)$, $(\mathcal{P}3)$, $(\mathcal{P}4)$, we have

$$\begin{aligned} U_n \quad = \quad & p(\rho_n, \rho_{n+1}) = p(F(\rho_{n-1}, \sigma_{n-1}), F(\rho_n, \sigma_n)) \\ \leq \quad & q_1\,[p(\rho_{n-1},\rho_n) + p(\sigma_{n-1},\sigma_n)] + q_2\,p(F(\rho_{n-1},\sigma_{n-1}),\rho_{n-1}) \\ & +q_3\,p(F(\rho_n,\sigma_n),\rho_n) + q_4\,p(F(\rho_{n-1},\sigma_{n-1}),\rho_n) \\ & +q_5\,p(F(\rho_n,\sigma_n),\rho_{n-1}) + q_6\,p(F(\sigma_{n-1},\rho_{n-1}),\sigma_n) \\ & +q_7\,p(F(\sigma_n,\rho_n),\sigma_{n-1}) \\ = \quad & q_1\,[p(\rho_{n-1},\rho_n) + p(\sigma_{n-1},\sigma_n)] + q_2\,p(\rho_n,\rho_{n-1}) \\ & +q_3\,p(\rho_{n+1},\rho_n) + q_4\,p(\rho_n,\rho_n) \\ & +q_5\,p(\rho_{n+1},\rho_{n-1}) + q_6\,p(\sigma_n,\sigma_n) \\ & +q_7\,p(\sigma_{n+1},\sigma_{n-1}) \\ \leq \quad & q_1\,[p(\rho_{n-1},\rho_n) + p(\sigma_{n-1},\sigma_n)] + q_2\,p(\rho_n,\rho_{n-1}) \\ & +q_3\,p(\rho_{n+1},\rho_n) + q_4\,p(\rho_n,\rho_{n+1}) \\ & +q_5\,[p(\rho_{n+1},\rho_n) + p(\rho_n,\rho_{n-1}) - p(\rho_n,\rho_n)] \\ & +q_6\,p(\sigma_n,\sigma_{n+1}) + q_7\,[p(\sigma_{n+1},\sigma_n) \\ & +p(\sigma_n,\sigma_{n-1}) - p(\sigma_n,\sigma_n)] \\ = \quad & q_1\,[p(\rho_{n-1},\rho_n) + p(\sigma_{n-1},\sigma_n)] + q_2\,p(\rho_n,\rho_{n-1}) \\ & +q_3\,p(\rho_{n+1},\rho_n) + q_4\,p(\rho_n,\rho_{n+1}) \\ & +q_5\,[p(\rho_{n+1},\rho_n) + p(\rho_n,\rho_{n-1})] \\ & +q_6\,p(\sigma_n,\sigma_{n+1}) + q_7\,[p(\sigma_{n+1},\sigma_n) \\ & +p(\sigma_n,\sigma_{n-1})] \\ = \quad & (q_1+q_2+q_5)U_{n-1} + (q_3+q_4+q_5)U_n + (q_1+q_7)V_{n-1} \\ & +(q_6+q_7)V_n. \end{aligned} \tag{3.2}$$

Likewise, we obtain

$$\begin{aligned} V_n &= p(\sigma_n, \sigma_{n+1}) = p(F(\sigma_{n-1}, \rho_{n-1}), F(\sigma_n, \rho_n)) \\ &\le (q_1+q_2+q_5)V_{n-1} + (q_3+q_4+q_5)V_n + (q_1+q_7)U_{n-1} \\ &\quad +(q_6+q_7)U_n. \end{aligned} \tag{3.3}$$

Hence from equations (3.2) and (3.3), we obtain

$$\begin{aligned} S_n &= U_n + V_n \\ &\le (q_1+q_2+q_5)(U_{n-1}+V_{n-1}) + (q_3+q_4+q_5)(U_n+V_n) \\ &\quad +(q_1+q_7)(U_{n-1}+V_{n-1}) + (q_6+q_7)(U_n+V_n) \\ &= (2q_1+q_2+q_5+q_7)(U_{n-1}+V_{n-1}) \\ &\quad +(q_3+q_4+q_5+q_6+q_7)(U_n+V_n) \\ &= (2q_1+q_2+q_5+q_7)S_{n-1} + (q_3+q_4+q_5+q_6+q_7)S_n, \end{aligned} \tag{3.4}$$

which implies

$$\begin{aligned} S_n &\le \Big(\frac{2q_1+q_2+q_5+q_7}{1-q_3-q_4-q_5-q_6-q_7}\Big)S_{n-1} \\ &= \gamma\, S_{n-1}, \end{aligned} \tag{3.5}$$

where $\gamma = \Big(\frac{2q_1+q_2+q_5+q_7}{1-q_3-q_4-q_5-q_6-q_7}\Big) < 1$, since $2q_1+q_2+q_3+q_4+2q_5+q_6+2q_7 < 1$.

Then for each $n \in \mathbb{N}$, we have

$$S_n \le \gamma S_{n-1} \le \gamma^2 S_{n-2} \le \dots \le \gamma^n S_0. \tag{3.6}$$

If $S_0 = 0$, then $p(\rho_0, \rho_1) + p(\sigma_0, \sigma_1) = 0$. Hence, from Remark 2.1, we get $\rho_0 = \rho_1 = F(\rho_0, \sigma_0)$ and $\sigma_0 = \sigma_1 = F(\sigma_0, \rho_0)$, means that (ρ_0, σ_0) is a coupled fixed point of F. Now, we assume that $S_0 > 0$. For each $n \ge m$, where $n, m \in \mathbb{N}$, we have, by using condition $(\mathcal{P}4)$

$$\begin{aligned} p(\rho_n, \rho_m) &\le p(\rho_n, \rho_{n-1}) + p(\rho_{n-1}, \rho_{n-2}) + \dots \\ &\quad + p(\rho_{m+1}, \rho_m) - p(\rho_{n-1}, \rho_{n-1}) - p(\rho_{n-2}, \rho_{n-2}) \\ &\quad - \dots - p(\rho_{m+1}, \rho_{m+1}) \\ &\le p(\rho_n, \rho_{n-1}) + p(\rho_{n-1}, \rho_{n-2}) + \dots + p(\rho_{m+1}, \rho_m). \end{aligned} \tag{3.7}$$

Likewise, we have

$$\begin{aligned} p(\sigma_n, \sigma_m) &\le p(\sigma_n, \sigma_{n-1}) + p(\sigma_{n-1}, \sigma_{n-2}) + \dots \\ &\quad + p(\sigma_{m+1}, \sigma_m) - p(\sigma_{n-1}, \sigma_{n-1}) - p(\sigma_{n-2}, \sigma_{n-2}) \\ &\quad - \dots - p(\sigma_{m+1}, \sigma_{m+1}) \\ &\le p(\sigma_n, \sigma_{n-1}) + p(\sigma_{n-1}, \sigma_{n-2}) + \dots + p(\sigma_{m+1}, \sigma_m). \end{aligned} \tag{3.8}$$

Thus,

$$\begin{aligned} p(\rho_n, \rho_m) + p(\sigma_n, \sigma_m) &\leq S_{n-1} + S_{n-2} + \ldots + S_m \\ &\leq (\gamma^{n-1} + \gamma^{n-2} + \ldots + \gamma^m) S_0 \\ &\leq \Big(\frac{\gamma^m}{1-\gamma}\Big) S_0. \end{aligned} \tag{3.9}$$

By definition of metric p^s, we have $p^s(\rho, \sigma) \leq 2p(\rho, \sigma)$, therefore for any $n \geq m$

$$\begin{aligned} p^s(\rho_n, \rho_m) + p^s(\sigma_n, \sigma_m) &\leq 2p(\rho_n, \rho_m) + 2p(\sigma_n, \sigma_m) \\ &\leq \Big(\frac{2\gamma^m}{1-\gamma}\Big) S_0, \end{aligned} \tag{3.10}$$

which implies that $\{\rho_n\}$ and $\{\sigma_n\}$ are Cauchy sequences in $(\mathcal{Y}, p^s)$ because $0 \leq \gamma < 1$, where $\gamma = 2q_1 + q_2 + q_3 + q_4 + 2q_5 + q_6 + 2q_7 < 1$. Since the partial metric space $(\mathcal{Y}, p)$ is complete, by Lemma 2.1, the metric space $(\mathcal{Y}, p^s)$ is complete, so there exist $u_1, u_2 \in \mathcal{Y}$ such that

$$\lim_{n\to\infty} p^s(\rho_n, u_1) = \lim_{n\to\infty} p^s(\sigma_n, u_2) = 0. \tag{3.11}$$

From Lemma 2.1, we obtain

$$p(u_1, u_1) = \lim_{n\to\infty} p(\rho_n, u_1) = \lim_{n\to\infty} p(\rho_n, \rho_n), \tag{3.12}$$

and

$$p(u_2, u_2) = \lim_{n\to\infty} p(\sigma_n, u_2) = \lim_{n\to\infty} p(\sigma_n, \sigma_n). \tag{3.13}$$

But, from condition $(\mathcal{P}2)$ and equation (3.6), we have

$$p(\rho_n, \rho_n) \leq p(\rho_n, \rho_{n+1}) \leq S_n \leq \gamma^n S_0, \tag{3.14}$$

and since $0 \leq \gamma < 1$, hence letting $n \to \infty$, we get $\lim_{n\to\infty} p(\rho_n, \rho_n) = 0$. It follows that

$$p(u_1, u_1) = \lim_{n\to\infty} p(\rho_n, u_1) = \lim_{n\to\infty} p(\rho_n, \rho_n) = 0. \tag{3.15}$$

Likewise, we obtain

$$p(u_2, u_2) = \lim_{n\to\infty} p(\sigma_n, u_2) = \lim_{n\to\infty} p(\sigma_n, \sigma_n) = 0. \tag{3.16}$$

Now, using equation (3.1), the conditions $(\mathcal{P}3)$ and $(\mathcal{P}4)$, we have

$$\begin{aligned} p(F(u_1, u_2), u_1) &\leq p(F(u_1, u_2), \rho_{n+1}) + p(\rho_{n+1}, u_1) - p(\rho_{n+1}, \rho_{n+1}) \\ &\leq p(F(u_1, u_2), \rho_{n+1}) + p(\rho_{n+1}, u_1) \\ &= p(F(u_1, u_2), F(\rho_n, \sigma_n)) + p(\rho_{n+1}, u_1) \end{aligned}$$

$$\begin{aligned}
&= p(F(\rho_n,\sigma_n),F(u_1,u_2))+p(\rho_{n+1},u_1)\\
&\leq q_1\,[p(\rho_n,u_1)+p(\sigma_n,u_2)]+q_2\,p(F(\rho_n,\sigma_n),\rho_n)\\
&\quad +q_3\,p(F(u_1,u_2),u_1)+q_4\,p(F(\rho_n,\sigma_n),u_1)+q_5\,p(F(u_1,u_2),\rho_n)\\
&\quad +q_6\,p(F(\sigma_n,\rho_n),u_2)+q_7\,p(F(u_2,u_1),\sigma_n)+p(\rho_{n+1},u_1)\\
&= q_1\,[p(\rho_n,u_1)+p(\sigma_n,u_2)]+q_2\,p(\rho_{n+1},\rho_n)+q_3\,p(F(u_1,u_2),u_1)\\
&\quad +q_4\,p(\rho_{n+1},u_1)+q_5\,p(F(u_1,u_2),\rho_n)+q_6\,p(\sigma_{n+1},u_2)\\
&\quad +q_7\,p(F(u_2,u_1),\sigma_n)+p(\rho_{n+1},u_1). \qquad (3.17)
\end{aligned}$$

Passing to the limit as $n \to \infty$ in equation (3.17) and using equations (3.15), (3.16), we obtain

$$p(F(u_1,u_2),u_1) \leq (q_3+q_5)\,p(F(u_1,u_2),u_1)+q_7\,p(F(u_2,u_1),u_2). \tag{3.18}$$

Likewise, we have

$$p(F(u_2,u_1),u_2) \leq (q_3+q_5)\,p(F(u_2,u_1),u_2)+q_7\,p(F(u_1,u_2),u_1). \tag{3.19}$$

Set

$$\chi_1 = p(F(u_1,u_2),u_1) \quad \textit{and} \quad \chi_2 = p(F(u_2,u_1),u_2). \tag{3.20}$$

Hence from equations (3.18)-(3.20), we obtain

$$\begin{aligned}
\chi_1+\chi_2 &\leq (q_3+q_5)\,(\chi_1+\chi_2)+q_7\,(\chi_1+\chi_2)\\
&= (q_3+q_5+q_7)\,(\chi_1+\chi_2)\\
&\leq (2q_1+q_2+q_3+q_4+2q_5+q_6+2q_7)\,(\chi_1+\chi_2)\\
&< \chi_1+\chi_2,
\end{aligned}$$

which is a contradiction, since $2q_1+q_2+q_3+q_4+2q_5+q_6+2q_7 < 1$. Hence, we conclude that $\chi_1+\chi_2 = 0$, that is, $p(F(u_1,u_2),u_1)+p(F(u_2,u_1),u_2) = 0$ and hence $p(F(u_1,u_2),u_1) = 0$ and $p(F(u_2,u_1),u_2) = 0$. Thus, $F(u_1,u_2) = u_1$ and $F(u_2,u_1) = u_2$. This shows that (u_1,u_2) is a coupled fixed point of F.

Now, we show the uniqueness. Suppose that (v_1,v_2) is another coupled fixed point of F such that $(u_1,u_2) \neq (v_1,v_2)$, then from equation (3.1) and using (3.15), (3.16) and $(\mathcal{P}3)$, we have

$$\begin{aligned}
p(u_1,v_1) &= p(F(u_1,u_2),F(v_1,v_2))\\
&\leq q_1\,[p(u_1,v_1)+p(u_2,v_2)]+q_2\,p(F(u_1,u_2),u_1)\\
&\quad +q_3\,p(F(v_1,v_2),v_1)+q_4\,p(F(u_1,u_2),v_1)+q_5\,p(F(v_1,v_2),u_1)\\
&\quad +q_6\,p(F(u_2,u_1),v_2)+q_7\,p(F(v_2,v_1),u_2)\\
&= q_1\,[p(u_1,v_1)+p(u_2,v_2)]+q_2\,p(u_1,u_1)\\
&\quad +q_3\,p(v_1,v_1)+q_4\,p(u_1,v_1)+q_5\,p(v_1,u_1)\\
&\quad +q_6\,p(u_2,v_2)+q_7\,p(v_2,u_2)\\
&= (q_1+q_4+q_5)\,p(u_1,v_1)+(q_1+q_6+q_7)\,p(u_2,v_2). \qquad (3.21)
\end{aligned}$$

Similarly, we have

$$p(u_2, v_2) = p(F(u_2, u_1), F(v_2, v_1)) \le (q_1 + q_4 + q_5)\, p(u_2, v_2) + (q_1 + q_6 + q_7)\, p(u_1, v_1). \tag{3.22}$$

Set

$$\zeta_1 = p(u_1, v_1) \quad and \quad \zeta_2 = p(u_2, v_2) \tag{3.23}$$

Then from equations (3.21)-(3.3), we get

$$\begin{aligned} \zeta_1 + \zeta_2 &\le (q_1 + q_4 + q_5)\,(\zeta_1 + \zeta_2) + (q_1 + q_6 + q_7)\,(\zeta_1 + \zeta_2) \\ &= (2q_1 + q_4 + q_5 + q_6 + q_7)\,(\zeta_1 + \zeta_2) \\ &\le (2q_1 + q_2 + q_3 + q_4 + 2q_5 + q_6 + 2q_7)\,(\zeta_1 + \zeta_2) \\ &< (\zeta_1 + \zeta_2), \end{aligned}$$

which is a contradiction, since $2q_1 + q_2 + q_3 + q_4 + 2q_5 + q_6 + 2q_7 < 1$. Hence, we conclude that $\zeta_1 + \zeta_2 = 0$, that is, $p(u_1, v_1) + p(u_2, v_2) = 0$ and hence $p(u_1, v_1) = 0$ and $p(u_2, v_2) = 0$. Thus, $u_1 = v_1$ and $u_2 = v_2$. This shows that the coupled fixed point of F is unique in $\mathcal{Y}$. This completes the proof of Theorem 3.1. □

4. Consequences of Theorem 3.1

By taking $q_1 = \frac{k}{2}$ and $q_2 = q_3 = \ldots = q_7 = 0$ in Theorem 3.1, then we obtain the following result.

Corollary 4.1. *([5], Corollary 2.2) Let $(\mathcal{Y}, p)$ be a complete partial metric space. Suppose that the mapping $F: \mathcal{Y} \times \mathcal{Y} \to \mathcal{Y}$ satisfying the following contractive condition for all $\rho, \sigma, \eta, \theta \in \mathcal{Y}$:*

$$p(F(\rho, \sigma), F(\eta, \theta)) \le \frac{k}{2}\,[p(\rho, \eta) + p(\sigma, \theta)], \tag{4.1}$$

where $k \in [0, 1)$ is a constant. Then F has a unique coupled fixed point.

By taking $q_2 = k$, $q_3 = l$ and $q_1 = q_4 = \ldots = q_7 = 0$ in Theorem 3.1, then we obtain the following result.

Corollary 4.2. *([5], Theorem 2.4) Let $(\mathcal{Y}, p)$ be a complete partial metric space. Suppose that the mapping $F: \mathcal{Y} \times \mathcal{Y} \to \mathcal{Y}$ satisfying the following contractive condition for all $\rho, \sigma, \eta, \theta \in \mathcal{Y}$:*

$$p(F(\rho, \sigma), F(\eta, \theta)) \le k\, p(F(\rho, \sigma), \rho) + l\, p(F(\eta, \theta), \eta), \tag{4.2}$$

where k, l are nonnegative constant with $k + l < 1$. Then F has a unique coupled fixed point.

By taking $q_4 = k$, $q_5 = l$ and $q_1 = q_2 = \ldots = q_7 = 0$ in Theorem 3.1, then we obtain the following result.

Corollary 4.3. *([5], Theorem 2.5) Let $(\mathcal{Y}, p)$ be a complete partial metric space. Suppose that the mapping $F: \mathcal{Y} \times \mathcal{Y} \to \mathcal{Y}$ satisfying the following contractive condition for all $\rho, \sigma, \eta, \theta \in \mathcal{Y}$:*

$$p(F(\rho,\sigma), F(\eta,\theta)) \leq k\, p(F(\rho,\sigma),\eta) + l\, p(F(\eta,\theta),\rho), \tag{4.3}$$

where k, l are nonnegative constant with $k + 2l < 1$. Then F has a unique coupled fixed point.

By taking $q_6 = k$, $q_7 = l$ and $q_1 = q_2 = \ldots = q_5 = 0$ in Theorem 3.1, then we obtain the following result.

Corollary 4.4. *Let $(\mathcal{Y}, p)$ be a complete partial metric space. Suppose that the mapping $F: \mathcal{Y} \times \mathcal{Y} \to \mathcal{Y}$ satisfying the following contractive condition for all $\rho, \sigma, \eta, \theta \in \mathcal{Y}$:*

$$p(F(\rho,\sigma), F(\eta,\theta)) \leq k\, p(F(\sigma,\rho),\theta) + l\, p(F(\theta,\eta),\sigma), \tag{4.4}$$

where k, l are nonnegative constant with $k + 2l < 1$. Then F has a unique coupled fixed point.

Remark 4.1. Theorem 3.1 extends the results of Aydi [5].

Example 4.1. Let $\mathcal{Y} = [0, +\infty)$ endowed with the usual partial metric p defined by $p: \mathcal{Y} \times \mathcal{Y} \to [0, +\infty)$ with $p(\rho, \sigma) = \max\{\rho, \sigma\}$. The partial metric space $(\mathcal{Y}, p)$ is complete because $(\mathcal{Y}, p^s)$ is complete. Indeed, for any $\rho, \sigma \in \mathcal{Y}$,

$$\begin{aligned} p^s(\rho,\sigma) &= 2p(\rho,\sigma) - p(\rho,\rho) - p(\sigma,\sigma) \\ &= 2\max\{\rho,\sigma\} - (\rho+\sigma) = |\rho - \sigma|. \end{aligned}$$

Thus, $(\mathcal{Y}, p^s)$ is the Euclidean metric space which is complete. Consider the mapping $F: \mathcal{Y} \times \mathcal{Y} \to \mathcal{Y}$ defined by $F(\rho, \sigma) = \frac{\rho+\sigma}{6}$. Now, for any $\rho, \sigma, \eta, \theta \in \mathcal{Y}$, we have

(1)

$$\begin{aligned} p(F(\rho,\sigma), F(\eta,\theta)) &= \frac{1}{6}\max\{\rho+\sigma, \eta+\theta\} \\ &\leq \frac{1}{6}[\max\{\rho,\eta\} + \max\{\sigma,\theta\}] \\ &= \frac{1}{6}[p(\rho,\eta) + p(\sigma,\theta)], \end{aligned}$$

which is the contractive condition of Corollary 4.1 for $k = 1/3 < 1$. Therefore, by Corollary 4.1, F has a unique coupled fixed point, which is $(0, 0)$.

(2)

$$\begin{aligned} p(F(\rho,\sigma),F(\eta,\theta)) &= \frac{1}{6}\max\{\rho+\sigma,\eta+\theta\} \\ &\le \frac{1}{6}[\max\{\rho+\sigma,\rho\}+\max\{\eta+\theta,\eta\}] \\ &= \frac{1}{6}[p(F(\rho,\sigma),\rho)+p(F(\eta,\theta),\eta)], \end{aligned}$$

which is the contractive condition of Corollary 4.2 for $k = 1/3 < 1$ (if $k = l$). Therefore, by Corollary 4.2, F has a unique coupled fixed point, which is $(0, 0)$.

(3)

$$\begin{aligned} p(F(\rho,\sigma),F(\eta,\theta)) &= \frac{1}{6}\max\{\rho+\sigma,\eta+\theta\} \\ &\le \frac{1}{6}[\max\{\rho+\sigma,\eta\}+\max\{\eta+\theta,\rho\}] \\ &= \frac{1}{6}[p(F(\rho,\sigma),\eta)+p(F(\eta,\theta),\rho)], \end{aligned}$$

which is the contractive condition of Corollary 4.3 for $k = 1/3 < 2/3 < 1$ (if $k = l$). Therefore, by Corollary 4.3, F has a unique coupled fixed point, which is $(0, 0)$.

Note that if the mapping $F\colon \mathcal{Y}\times\mathcal{Y}\to\mathcal{Y}$ is given by $F(\rho,\sigma) = \frac{\rho+\sigma}{2}$, then F satisfies contractive condition of Corollary 4.1, 4.2, 4.3 for $k = 1$ (if $k = l$),

$$\begin{aligned} p(F(\rho,\sigma),F(\eta,\theta)) &= \frac{1}{2}\max\{\rho+\sigma,\eta+\theta\} \\ &\le \frac{1}{2}[\max\{\rho,\eta\}+\max\{\sigma,\theta\}] \\ &= \frac{1}{2}[p(\rho,\eta)+p(\sigma,\theta)]. \end{aligned}$$

In this case $(0, 0)$ and $(1, 1)$ are both coupled fixed points of F, and hence, the coupled fixed point of F is not unique. This shows that the condition $k < 1$ in Corollary 4.1, 4.2 and hence $h_1 + h_2 < 1$ in Theorem 2.1 (N_1), (N_2) and the condition $k < 2/3$ in Corollary 4.3 and hence $h_1 + 2h_2 < 1$ in Theorem 2.1 (N_3) cannot be omitted in the statement of the aforesaid results.

5. Well-Posedness Theorem

In this section, we prove well-posedness of coupled fixed point problem of mapping in Theorem 3.1.

Definition 5.1. ([50]) Let $(\mathcal{Y}, d)$ be a metric space and let $\mathcal{T}\colon\mathcal{Y}\to\mathcal{Y}$ be a mapping. The fixed point problem of $\mathcal{T}$ is said to be well posed if:

(1) $\mathcal{T}$ has a unique fixed point ρ_0,

(2) for any sequence $\{\rho_n\} \in \mathcal{Y}$ with $\lim_{n\to\infty} d(\mathcal{T}\rho_n, \rho_n) = 0$, we have

$$\lim_{n\to\infty} d(\rho_n, \rho_0) = 0.$$

Now, we define well-posedness of coupled fixed point in partial metric spaces.

Let $\mathcal{C}_0\mathcal{FP}(F, \mathcal{Y} \times \mathcal{Y})$ denote a coupled fixed point problem of mapping F and $\mathcal{C}_0\mathcal{F}(F)$ denote the set of all coupled fixed points of F.

Definition 5.2. Let $(\mathcal{Y}, p)$ be a partial metric space and let $F\colon \mathcal{Y} \times \mathcal{Y} \to \mathcal{Y}$ be a mapping. $\mathcal{C}_0\mathcal{FP}(F, \mathcal{Y} \times \mathcal{Y})$ is called well posed if:

(1) $\mathcal{C}_0\mathcal{F}(F)$ is unique,

(2) for any sequences $\{\rho_n\}$, $\{\sigma_n\}$ in $\mathcal{Y}$ with $(\bar{\rho}, \bar{\sigma}) \in \mathcal{C}_0\mathcal{F}(F)$ and

$$\lim_{n\to\infty} p(F(\rho_n, \sigma_n), \rho_n) = 0 = \lim_{n\to\infty} p(F(\sigma_n, \rho_n), \sigma_n)$$

implies

$$\bar{\rho} = \lim_{n\to\infty} \rho_n, \;\; \bar{\sigma} = \lim_{n\to\infty} \sigma_n.$$

Theorem 5.1. *Let $(\mathcal{Y}, p)$ be a complete partial metric space and $F\colon \mathcal{Y} \times \mathcal{Y} \to \mathcal{Y}$ be a mapping as in Theorem 3.1. For any sequences $\{\rho_n\}$, $\{\sigma_n\}$ in $\mathcal{Y}$ and $(\rho, \sigma) \in \mathcal{C}_0\mathcal{F}(F)$, if*

$$\lim_{n\to\infty} p(\rho, F(\rho_n, \sigma_n)) = 0 = \lim_{n\to\infty} p(\sigma, F(\sigma_n, \rho_n)),$$

then the coupled fixed point problem of F is well-posed with $p(w, w) = 0$ for some $w \in \mathcal{Y}$.

Proof. From Theorem 3.1, the mapping F has a unique coupled fixed point, $(\rho_0, \sigma_0) \in \mathcal{Y} \times \mathcal{Y}$. Let $\{\rho_n\}$, $\{\sigma_n\}$ in $\mathcal{Y}$ and

$$\lim_{n\to\infty} p(F(\rho_n, \sigma_n), \rho_n) = 0 = \lim_{n\to\infty} p(F(\sigma_n, \rho_n), \sigma_n).$$

Without loss of generality, we assume that $(\rho_0, \sigma_0) \neq (\rho_n, \sigma_n)$ for any non-negative integer n. Using $F(\rho_0, \sigma_0) = \rho_0$ and $F(\sigma_0, \rho_0) = \sigma_0$, we obtain

$$\begin{aligned} p(\rho_0, \rho_n) &= p(F(\rho_0, \sigma_0), F(\rho_n, \sigma_n)) + p(F(\rho_n, \sigma_n), \rho_n) \\ &\quad - P(F(\rho_n, \sigma_n), F(\rho_n, \sigma_n)) \\ &\leq p(F(\rho_0, \sigma_0), F(\rho_n, \sigma_n)) + p(F(\rho_n, \sigma_n), \rho_n) \\ &\leq q_1 \left[p(\rho_0, \rho_n) + p(\sigma_0, \sigma_n)\right] + q_2\, p(F(\rho_0, \sigma_0), \rho_0) + q_3\, p(F(\rho_n, \sigma_n), \rho_n) \\ &\quad + q_4\, p(F(\rho_0, \sigma_0), \rho_n) + q_5\, p(F(\rho_n, \sigma_n), \rho_0) + q_6\, p(F(\sigma_0, \rho_0), \sigma_n) \\ &\quad + q_7\, p(F(\sigma_n, \rho_n), \sigma_0) + p(F(\rho_n, \sigma_n), \rho_n), \end{aligned} \tag{5.1}$$

and

$$\begin{aligned} p(\sigma_0,\sigma_n) &\leq p(F(\sigma_0,\rho_0),F(\sigma_n,\rho_n))+p(F(\sigma_n,\rho_n),\sigma_n)\\ &\quad -P(F(\sigma_n,\rho_n),F(\sigma_n,\rho_n))\\ &\leq p(F(\sigma_0,\rho_0),F(\sigma_n,\rho_n))+p(F(\sigma_n,\rho_n),\sigma_n)\\ &\leq q_1\,[p(\sigma_0,\sigma_n)+p(\rho_0,\rho_n)]+q_2\,p(F(\sigma_0,\rho_0),\sigma_0)+q_3\,p(F(\sigma_n,\rho_n),\sigma_n)\\ &\quad +q_4\,p(F(\sigma_0,\rho_0),\sigma_n)+q_5\,p(F(\sigma_n,\rho_n),\sigma_0)+q_6\,p(F(\rho_0,\sigma_0),\rho_n)\\ &\quad +q_7\,p(F(\rho_n,\sigma_n),\rho_0)+p(F(\sigma_n,\rho_n),\sigma_n). \end{aligned} \tag{5.2}$$

Since

$$\lim_{n\to\infty} p(\rho_0,F(\rho_n,\sigma_n))=0=\lim_{n\to\infty} p(\sigma_0,F(\sigma_n,\rho_n)),$$

for $(\rho_0,\sigma_0)\in\mathcal{C}_0\mathcal{F}(F)$, using $(\mathcal{P}3)$, we obtain

$$\begin{aligned} \lim_{n\to\infty} p(\rho_0,\rho_n) &\leq (q_1+q_4)\lim_{n\to\infty} p(\rho_0,\rho_n)\\ &\quad +(q_1+q_6)\lim_{n\to\infty} p(\sigma_0,\sigma_n), \end{aligned} \tag{5.3}$$

and

$$\begin{aligned} \lim_{n\to\infty} p(\sigma_0,\sigma_n) &\leq (q_1+q_4)\lim_{n\to\infty} p(\sigma_0,\sigma_n)\\ &\quad +(q_1+q_6)\lim_{n\to\infty} p(\rho_0,\rho_n). \end{aligned} \tag{5.4}$$

Set

$$\psi_1=p(\rho_0,\rho_n)\quad \textit{and}\quad \psi_2=p(\sigma_0,\sigma_n). \tag{5.5}$$

Now from equations (5.3)-(5.5), we obtain

$$\begin{aligned} \lim_{n\to\infty}\psi_1+\lim_{n\to\infty}\psi_2 &\leq (q_1+q_4)\,[\lim_{n\to\infty}\psi_1+\lim_{n\to\infty}\psi_2]\\ &\quad +(q_1+q_6)\,[\lim_{n\to\infty}\psi_1+\lim_{n\to\infty}\psi_2]\\ &= (2q_1+q_4+q_6)\,[\lim_{n\to\infty}\psi_1+\lim_{n\to\infty}\psi_2]\\ &\leq (2q_1+q_2+q_3+q_4+2q_5+q_6+2q_7)\times\\ &\quad [\lim_{n\to\infty}\psi_1+\lim_{n\to\infty}\psi_2]\\ &< \lim_{n\to\infty}\psi_1+\lim_{n\to\infty}\psi_2, \end{aligned}$$

which is a contradiction, since $2q_1+q_2+q_3+q_4+2q_5+q_6+2q_7<1$. Hence, we conclude that $\lim_{n\to\infty}\psi_1+\lim_{n\to\infty}\psi_2=0$, that is, $\lim_{n\to\infty}p(\rho_0,\rho_n)+\lim_{n\to\infty}p(\sigma_0,\sigma_n)=0$ and hence

$$\lim_{n\to\infty}\rho_n=\rho_0\quad \textit{and}\quad \lim_{n\to\infty}\sigma_n=\sigma_0.$$

Thus the coupled fixed point problem of the mapping is well-posed. This completes the proof. □

6. Applications

In this section, we state some applications to the main result of a self mapping which is involved in an integral type contraction.

Let us denote a set τ of all of functions $\chi\colon[0,+\infty)\to[0,+\infty)$ satisfying the following properties:

(i) Each χ is a Lebesgue-integrable mapping on every compact subset of $[0,+\infty)$,

(ii) For any $\varepsilon>0$ we have $\int_0^{\varepsilon}\chi(t)dt>0$.

Theorem 6.1. *Let $(\mathcal{Y},p)$ be a complete partial metric space. Suppose that the mapping $F\colon\mathcal{Y}\times\mathcal{Y}\to\mathcal{Y}$ satisfying the following contractive condition for all $\rho,\sigma,\eta,\theta\in\mathcal{Y}$:*

$$
\begin{aligned}
\int_0^{p(F(\rho,\sigma),F(\eta,\theta))}\varphi(t)dt &\le q_1\int_0^{[p(\rho,\eta)+p(\sigma,\theta)]}\varphi(t)dt\\
&\quad +q_2\int_0^{p(F(\rho,\sigma),\rho)}\varphi(t)dt+q_3\int_0^{p(F(\eta,\theta),\eta)}\varphi(t)dt\\
&\quad +q_4\int_0^{p(F(\rho,\sigma),\eta)}\varphi(t)dt+q_5\int_0^{p(F(\eta,\theta),\rho)}\varphi(t)dt\\
&\quad +q_6\int_0^{p(F(\sigma,\rho),\theta)}\varphi(t)dt+q_7\int_0^{p(F(\theta,\eta),\sigma)}\varphi(t)dt,
\end{aligned}
\tag{6.1}
$$

where $q_1,q_2,\dots,q_7$ are nonnegative constants with $2q_1+q_2+q_3+q_4+2q_5+q_6+2q_7<1$ and $\varphi\in\tau$. Then F has a unique coupled fixed point.

Similarly, we can obtain the following coupled fixed point results by taking (i) $q_1=k$ and $q_2=q_3=\dots=q_7=0$ (ii) $q_2=k,q_3=l$ and $q_1=q_4=\dots=q_7=0$ (iii) $q_4=k,q_5=l$ and $q_1=q_2=q_3=q_6=q_7=0$ (iv) $q_6=k,q_7=l$ and $q_1=q_2=\dots=q_5=0$ and many more results.

Theorem 6.2. *Let $(\mathcal{Y},p)$ be a complete partial metric space. Suppose that the mapping $F\colon\mathcal{Y}\times\mathcal{Y}\to\mathcal{Y}$ satisfying the following contractive condition for all $\rho,\sigma,\eta,\theta\in\mathcal{Y}$:*

$$
\int_0^{p(F(\rho,\sigma),F(\eta,\theta))}\varphi(t)dt\le k\int_0^{[p(\rho,\eta)+p(\sigma,\theta)]}\varphi(t)dt,
\tag{6.2}
$$

where $k\in[0,1/2)$ is a constant and $\varphi\in\tau$. Then F has a unique coupled fixed point.

Remark 6.1. Theorem 6.2 extends Theorem 2.1 of Aydi [5] to the case of integral type contraction.

Theorem 6.3. *Let $(\mathcal{Y}, p)$ be a complete partial metric space. Suppose that the mapping $F\colon \mathcal{Y}\times\mathcal{Y} \to \mathcal{Y}$ satisfying the following contractive condition for all $\rho, \sigma, \eta, \theta \in \mathcal{Y}$:*

$$(6.3)\int_0^{p(F(\rho,\sigma),F(\eta,\theta))} \varphi(t)dt \leq k \int_0^{p(F(\rho,\sigma),\rho)} \varphi(t)dt + l \int_0^{p(F(\eta,\theta),\eta)} \varphi(t)dt,$$

where k, l are nonnegative constants with $k+l < 1$ and $\varphi \in \tau$. Then F has a unique coupled fixed point.

Remark 6.2. Theorem 6.3 extends Theorem 2.4 of Aydi [5] to the case of integral type contraction.

Theorem 6.4. *Let $(\mathcal{Y}, p)$ be a complete partial metric space. Suppose that the mapping $F\colon \mathcal{Y}\times\mathcal{Y} \to \mathcal{Y}$ satisfying the following contractive condition for all $\rho, \sigma, \eta, \theta \in \mathcal{Y}$:*

$$(6.4)\int_0^{p(F(\rho,\sigma),F(\eta,\theta))} \varphi(t)dt \leq k \int_0^{p(F(\rho,\sigma),\eta)} \varphi(t)dt + l \int_0^{p(F(\eta,\theta),\rho)} \varphi(t)dt,$$

where k, l are nonnegative constants with $k + 2l < 1$ and $\varphi \in \tau$. Then F has a unique coupled fixed point.

Remark 6.3. Theorem 6.3 extends Theorem 2.5 of Aydi [5] to the case of integral type contraction.

Theorem 6.5. *Let $(\mathcal{Y}, p)$ be a complete partial metric space. Suppose that the mapping $F\colon \mathcal{Y}\times\mathcal{Y} \to \mathcal{Y}$ satisfying the following contractive condition for all $\rho, \sigma, \eta, \theta \in \mathcal{Y}$:*

$$(6.5)\int_0^{p(F(\rho,\sigma),F(\eta,\theta))} \varphi(t)dt \leq k \int_0^{p(F(\sigma,\rho),\theta)} \varphi(t)dt + l \int_0^{p(F(\theta,\eta),\sigma)} \varphi(t)dt,$$

where k, l are nonnegative constants with $k + 2l < 1$ and $\varphi \in \tau$. Then F has a unique coupled fixed point.

7. Application to integral equation

As an application of Corollary 4.1, we find an existence and uniqueness result for a type of the following system of nonlinear integral equations:

$$\begin{aligned} \rho(t) &= \int_0^{\mathcal{M}} \kappa(t,\alpha)[\mathcal{H}_1(\alpha,\rho(\alpha)) + \mathcal{H}_2(\alpha,\sigma(\alpha))]d(\alpha) + \delta(t), \\ \sigma(t) &= \int_0^{\mathcal{M}} \kappa(t,\alpha)[\mathcal{H}_1(\alpha,\sigma(\alpha)) + \mathcal{H}_2(\alpha,\rho(\alpha))]d(\alpha) + \delta(t), \end{aligned} \tag{7.1}$$

$t \in [0, \mathcal{M}]$, $\mathcal{M} \geq 1$.

Let $\mathcal{Y} = C([0,\mathcal{M}],\mathbb{R})$ be the class of all real valued continuous functions on $[0,\mathcal{M}]$. Define $F\colon \mathcal{Y}\times\mathcal{Y}\to\mathcal{Y}$ by

$$F(\rho,\sigma)(t) = \int_0^{\mathcal{M}} \kappa(t,\alpha)[\mathcal{H}_1(\alpha,\rho(\alpha)) + \mathcal{H}_2(\alpha,\sigma(\alpha))]d(\alpha) + \delta(t).$$

Obviously, $(\rho(t),\sigma(t))$ is a solution of system of nonlinear integral equations (7.1) if and only if $(\rho(t),\sigma(t))$ is a coupled fixed point of F. Define $p\colon \mathcal{Y}\times\mathcal{Y}\to[0,+\infty)$ by

$$p(\rho,\sigma) = |\rho-\sigma|,$$

for all $\rho,\sigma\in\mathcal{Y}$. Now, we state and prove our result as follows.

Theorem 7.1. *Suppose the following:*

1. *The mappings* $\mathcal{H}_1\colon[0,\mathcal{M}]\times\mathbb{R}\to\mathbb{R}$, $\mathcal{H}_1\colon[0,\mathcal{M}]\times\mathbb{R}\to\mathbb{R}$, $\delta\colon[0,\mathcal{M}]\to\mathbb{R}$ *and* $\kappa\colon[0,\mathcal{M}]\to\mathbb{R}$ *are continuous.*
2. *There exists* $\lambda>0$ *and* k *is a nonnegative constant with* $0\le k<1$, *such that*

$$\begin{aligned}\mathcal{H}_1(\alpha,\rho(\alpha)) - \mathcal{H}_1(\alpha,\sigma(\alpha)) &\le \lambda\frac{k}{2}(|\rho-\sigma|),\\ \mathcal{H}_2(\alpha,\sigma(\alpha)) - \mathcal{H}_2(\alpha,\rho(\alpha)) &\le \lambda\frac{k}{2}(|\sigma-\rho|).\end{aligned}$$

3. $$\int_0^{\mathcal{M}} \lambda|\kappa(t,\alpha)|d(\alpha)\le 1.$$

Then, the integral equation (7.1) has a unique solution in $\mathcal{Y}$.

Proof. Consider

$$p(F(\rho,\sigma),F(\eta,\theta)) = |F(\rho,\sigma)-F(\eta,\theta)|$$

$$\begin{aligned}&= \Big|\int_0^{\mathcal{M}} \kappa(t,\alpha)[\mathcal{H}_1(\alpha,\rho(\alpha)) + \mathcal{H}_2(\alpha,\sigma(\alpha))]d(\alpha)+\delta(t)\\ &\quad -\Big(\int_0^{\mathcal{M}} \kappa(t,\alpha)[\mathcal{H}_1(\alpha,\eta(\alpha)) + \mathcal{H}_2(\alpha,\theta(\alpha))]d(\alpha)+\delta(t)\Big)\Big|\\ &= \Big|\int_0^{\mathcal{M}} \kappa(t,\alpha)[\mathcal{H}_1(\alpha,\rho(\alpha)) - \mathcal{H}_1(\alpha,\eta(\alpha)) + \mathcal{H}_2(\alpha,\sigma(\alpha))\\ &\quad -\mathcal{H}_2(\alpha,\theta(\alpha))]d(\alpha)\Big|\\ &\le \int_0^{\mathcal{M}} |\kappa(t,\alpha)|\Big[\big|\mathcal{H}_1(\alpha,\rho(\alpha)) - \mathcal{H}_1(\alpha,\eta(\alpha))\big| + \big|\mathcal{H}_2(\alpha,\sigma(\alpha))\\ &\quad -\mathcal{H}_2(\alpha,\theta(\alpha))\big|\Big]d(\alpha)\\ &\le \int_0^{\mathcal{M}} |\kappa(t,\alpha)|d(\alpha)\Big([\lambda\frac{k}{2}(|\rho-\eta|)] + [\lambda\frac{k}{2}(|\sigma-\theta|)]\Big)\end{aligned}$$

$$
\begin{aligned}
&= \int_0^{\mathcal{M}} \lambda|\kappa(t,\alpha)|d(\alpha)\Big[\frac{k}{2}(|\rho-\eta|)+\frac{k}{2}(|\sigma-\theta|)\Big] \\
&\leq \frac{k}{2}\Big[(|\rho-\eta|)+(|\sigma-\theta|)\Big] \\
&= \frac{k}{2}[p(\rho,\eta)+p(\sigma,\theta)]
\end{aligned}
$$

for all $\rho, \sigma, \eta, \theta \in \mathcal{Y}$ and $0 \leq k < 1$. Hence, all the hypothesis of Corollary 4.1 are verified, and consequently, the integral equation (7.1) has a unique solution. □

8. Conclusion

In this paper, we prove a unique coupled fixed point theorem in the setting of partial metric spaces and give some corollaries of the established results. Also we give some examples to support these results. We also prove well-posedness of a coupled fixed point problem and give some applications of the main result. Furthermore, we provide an application to integral equation. Our results extend and generalize several results from the existing literature. In addition, our future plan is to investigate coupled and common coupled fixed point results on generalized metric spaces with applications to integral equations.

9. Acknowledgement

The author is grateful to the anonymous referees for their careful reading and useful suggestions, which helped us to improve this manuscript.

REFERENCES

1. M. Abbas, M. Ali Khan and S. Radenović: *Common coupled fixed point theorems in cone metric spaces for w-compatible mappings*, Appl. Math. Comput. **217** (2010), 195–202.
2. R. P. Agarwal, M. A. El-Gebeily and D. O'Regan: *Generalized contraction in partially ordered metric spaces*, Applicable Anal. **87(1)** (2008), 109–116.
3. J. Ahmad, M. Arshad and C. Vetro: *On a theorem of Khan in a generalized metric space*, Int. J. Anal. (2013)(2). DOI:10.1155/2013/852727.
4. I. Altun, F. Sola and H. Simsek: *Generalized contractions on partial metric spaces*, Topology and its Appl. **157** (2010), 2778–2785.
5. H. Aydi: *Some coupled fixed point results on partial metric spaces*, International J. Math. Math. Sci. 2011, Article ID 647091, 11 pages.
6. H. Aydi, M. Abbas and C. Vetro: *Partial Hausdorff metric and Nadler's fixed point theorem on partial metric spaces*, Topology and Its Appl. **159** (2012), No. 14, 3234–3242.
7. H. Aydi, M. Postolache and W. Shatanawi: *Coupled fixed point results for (ψ, ϕ)-weakly contractive mappings in ordered G-metric spaces*, Comput. Math. Appl. **63** (2012), 298–309.

8. S. Banach: *Sur les operation dans les ensembles abstraits et leur application aux equation integrals*, Fund. Math. **3**(1922), 133–181.
9. T. Gnana Bhaskar and V. Lakshmikantham: *Fixed point theorems in partially ordered metric spaces and applications*, Nonlinear Analysis: TMA, **65(7)** (2006), 1379–1393.
10. L. Ciric and V. Lakshmikantham: *Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces*, Nonlinear Analysis: TMA, **70(12)** (2009), 4341–4349.
11. S. Chandok, Z. Mustafa and M. Postolache: *Coupled common fixed point theorems for mixed g-monotone mappings in partially ordered G-metric spaces*, U. Politech. Buch.: Ser. A **75(4)** (2013), 13–26.
12. S. Chandok: *Coupled common fixed point theorems for a contractive condition of rational type in ordered metric spaces*, J. Appl. Math. Informatics **31** (2013), No. 5-6, 643–649.
13. S. Chandok, D. Kumar and M. S. Khan: *Some results in partial metric space using auxiliary functions*, Applied Math. E-Notes **15** (2015), 233–242.
14. S. Chandok, T. D. Narang and M. A. Taoudi: *Some coupled fixed point theorems for mappings satisfying a generalized contractive condition of rational type*, Palestine J. Math. **4(2)** (2015), 360–366.
15. S. Chandok: *Some fixed point theorems for (α, β)-admisible Geraghty type contractive mappings and related results*, Math. Sci. **9** (2015), 127–135.
16. B. S. Choudhury and A. Kundu: *A coupled coicidence point result in partially ordered metric spaces for compatible mappings*, Nonlinear Anal.: TMA, **73** (2010), 2524–2531.
17. B. S. Choudhury, N. Metiya and M. Postolache: *A generalized weak contraction principle with applications to coupled coincidence point problems*, Fixed Point Theory Appl. (2013), **2013:152**.
18. M. Edelstein: *On fixed points and periodic points under contraction mappings*, J. Lond. Math. Soc. **37** (1962), 74–79.
19. D. Guo and V. Lakshmikantham: *Coupled fixed point of nonlinear operator with application*, Nonlinear Anal. TMA., **11** (1987), 623–632.
20. G. C. Hardy and T. Rogers: *A generalization of fixed point theorem of S. Reich*, Can. Math. Bull. **16** (1973), 201–206.
21. R. Heckmann: *Approximation of metric spaces by partial metric spaces*, Appl. Categ. Structures, **7**, No. 1-2, (1999), 71–83.
22. S. Hong: *Fixed points of multivalued operators in ordered metric spaces with applications*, Nonlinear Anal., TMA., **72(11)** (2010), 3929–3942.
23. R. Kannan: *Some results on fixed points-II*, Amer. Math. Mon. **76** (1969), 71–76.
24. E. Karapinar, W. Shatanawi and K. Tas: *Fixed point theorems on partial metric spaces involving rational expressions*, Miskolc Math. Notes **14** (2013), 135–142.
25. B. Khomdaram and Y. Rohen: *Some common coupled fixed point theorems in S_b-metric spaces*, Fasciculi Math. **60** (2018), 79–92.
26. J. K. Kim and S. Chandok: *Coupled common fixed point theorems for generalized nonlinear contraction mappings with the mixed monotone property in partially ordered metric spaces*, Fixed Point Theory Appl. (2013), **2013:307**.

27. J. K. Kim, G. A. Okeke and W. H. Lim: *Common coupled fixed point theorems for w-compatible mappings in partial metric spaces*, Global J. Pure Appl. Math. **13(2)** (2017), 519–536.

28. K. S. Kim: *Coupled fixed point theorems under new coupled implicit relation in Hilbert spaces*, Demonstratio Math. **55** (2022), 81–89.

29. P. Konar, S. Chandok, S. Dutta and M. De la sen: *Coupled optimal results with an application to integral equations*, Axioms **2021**, 10, 73. (https//:doi.org/10.3390/axioms 10020073)

30. H. P. A. Künzi: *Nonsymmetric distances and their associated topologies about the origins of basic ideas in the area of asymptotic topology*, Handbook of the History Gen. Topology (eds. C.E. Aull and R. Lowen), Kluwer Acad. Publ., **3** (2001), 853–868.

31. N. V. Luong and N. X. Thuan: *Coupled fixed points in partially ordered metric spaces and application*, Nonlinear Anal. **74** (2011), 983–992.

32. N. V. Luong and N. X. Thuan: *Fixed point theorems for generalized weak contractions satisfying rational expressions in ordered metric spaces*, Fixed Point Theory Appl., **2011**(46), (2011).

33. N. V. Luong and N. X. Thuan: *Coupled fixed points theorems for mixed monotone mappings and an application to integral equations*, Comput. Math. Appl. **62** (2011), 4238–4248.

34. H. P. Masiha, F. Sabetghadam and N. Shahzad: *Fixed point theorems in partial metric spaces with an application*, Filomat **27(4)**(2013), 617–624.

35. S. G. Matthews: *Partial metric topology*, Research report 2012, Dept. Computer Science, University of Warwick, 1992.

36. S. G. Matthews: *Partial metric topology*, Proceedings of the 8th summer conference on topology and its applications, Annals of the New York Academy of Sciences, **728** (1994), 183–197.

37. B. Monjardet: *Metrics on partially ordered sets: A survey*, Discrete Math. **35** (1981), 173–184.

38. H. K. Nashine and W. Shatanawi: *Coupled common fixed point theorems for pair of commuting in partially ordered complete metric spaces*, Comput. Math. Appl. **62** (2011), 1984–1993.

39. H. K. Nashine and Z. Kadelburg: *Partially ordered metric spaces, rational contractive expressios and couple fixed points*, Nonlinear Funct. Anal. Appl. **17(4)** (2012), 471–489.

40. H. K. Nashine: *Coupled common fixed point results in ordered G-metric spaces*, J. Nonlinear Sci. Appl. **1** (2012), 1–13.

41. H. K. Nashine, Z. Kadelburg and S. Radenović: *Common fixed point theorems for weakly isotone increasing mappings in ordered partial metric spaces*, Math. Comput. Model. **57(9-10)** (2013), 2355–2365.

42. H. K. Nashine, J. K. Kim, A. K. Sharma and G. S. Saluja: *Some coupled fixed point without mixed monotone mappings*, Nonlinear Funct. Anal. Appl. **21(2)** (2016), 235–247.

43. J. J. Nieto and R. R. Lopez: *Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equation*, Acta Math. Sinica, English Series, **23** (2007), No. 12, 2205–2212.

44. J. O. Olaleru, G. A. Okeke and H. Akewe: *Coupled fixed point theorems for generalized φ-mappings satisfying contractive condition of integral type on cone metric spaces*, Int. J. Math. Model. Comput. **2(2)** (2012), 87–98.

45. R. Pant, R. Shukla, H. K. Nashine and R. Panicker: *Some new fixed point theorems in partial metric spaces with applications*, J. Function spaces, **2017**, Article ID 1072750, 13 pages.

46. S. Radenović: *Bhaskar – Lakshmikantham type results for monotone mappings in partially ordered metric spaces*, Int. J. Nonlinear Anal. **5** (2014), 37–49.

47. A. C. M. Ran and M. C. B. Reurings: *A fixed point theorem in partially ordered sets and some application to matrix equations*, Proc. Amer. Math. Soc. **132** (2004), 1435–1443.

48. N. Seshagiri Rao and K. Kalyani: *Coupled fixed point theorems in partially ordered metric spaces*, Fasciculi Math. **64** (2020), 77–89.

49. S. Reich: *Some remarks concerning contraction mappings*, Can. Math. Bull. **14** (1971), 121–124.

50. S. Reich and A. J. Zaslavski: *Well posedness of fixed point problem*, Far East J. Math. special volume part III (2001), 393–401.

51. F. Sabetghadam, H. P. Mashiha and A. H. Sanatpour: *Some coupled fixed point theorems in cone metric spaces*, Fixed Point Theory Appl. (2009), Article ID 125426, 8 pages.

52. B. Samet: *Coupled fixed point theorems for a generalized Meir – Keeler contractions in a partially ordered metric spaces*, Nonlinear Anal. **72(12)** (2010), 4508–4517.

53. B. Samet and H. Yazidi: *Coupled fixed point theorems in a partial order ϵ-chainable metric spaces*, J. Math. Comput. Sci. **1(3)** (2010), 142–151.

54. B. Samet and H. Yazidi: *Coupled fixed point theorems for contraction involving rational expressions in partially ordered metric spaces*, priprint, arXiv:1005.3142v1 [math GN].

55. S. Sedghi and A. Gholidehneh: *Coupled fixed point theorems in S_b-metric spaces*, Nonlinear Funct. Anal. Appl. **22(2)** (2017), 217–228.

56. S. Sharma and S. Chandok: *Existence of best proximity point with an application to nonlinear intigral equations*, J. Math. (2021), Article ID 3886659, 7 pages. DOI: 10.1155/2021/3886659.

57. W. Shatanawi, B. Samet and M. Abbas: *Coupled fixed point theorems for mixed monotone mappings in ordered partial metric spaces*, Math. Comput. Modell. **55** (2012), Nos. 3-4, 680–687.

58. W. Shatanawi and M. Posolache: *Common fixed point results of mappings for nonlinear contractions of cyclic form in ordered metric spaces*, Fixed Point Theory Appl. (2013), **2013:60**. DOI: 10.1186/1687-1812-2013-60.

59. M. R. Singh and A. K. Chatterjee: *Fixed point theorems*, Commun. Fac. Sci. Univ. Ank., Series A1 **37** (1988), 1–4.

60. D. R. Smart: *Fixed point theorems*, Cambridge University Press, Cambridge, 1974.

61. O. Valero: *On Banach fixed point theorems for partial metric spaces*, Appl. Gen. Topl. **6(2)** (2005), 229–240.

62. E. S. Wolk: *Continuous convergence in partially ordered sets*, Gen. Topol. Appl. **5** (1975), 221–234.

63. C. S. Wong: *Common fixed points of two mappings*, Pac. J. Math. **48** (1973), 299–312.