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HYPERBOLIC KENMOTSU MANIFOLD ADMITTING A NEW TYPE OF SEMI-SYMMETRIC NON-METRIC CONNECTION

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Abstract. In this work, we study a new type of semi-symmetric non-metric connection on hyperbolic Kenmotsu manifold. Some Riemannian curvature's characteristics on hyperbolic Kenmotsu manifold are investigated. The properties of semi-symmetric, locally φ-symmetric and Ricci semi-symmetric hyperbolic Kenmotsu manifold endowed with a new type of semi-symmetric non-metric connection are evaluated. A semi-symmetric and Ricci semi-symmetric hyperbolic Kenmotsu manifold with a semi-symmetric non-metric connection is also demonstrated, the Ricci soliton of data $(\mathfrak{g}_1, \xi^\flat, \lambda)$ is shrinking. Finally, we demonstrate our results with a 3-dimensional example.
Keywords: Semi-symmetric non-metric, hyperbolic Kenmotsu manifold, Ricci soliton, Einstein manifold, Ricci semi-symmetric.

1. Introduction

A. Friedmann and A. Schouten [16] first established the concept of a semi-symmetric linear connection on differentiable manifold in 1924. E. Bartolotti [6] gave a geometrical meaning to such a connection. Further, H. A. Hayden [17] introduced the concept of metric connection with non zero torsion tensor on a Rieman-

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nian manifold. Agashe and Chafle [2] define a semi-symmetric non-metric connection in Riemannian manifold. This was further studied by Agashe and Chafle [3], S. K. Chaubey and A. C. Pandey [11] and many other geometers like [8, 14, 18]. Sengupta, De and Binh [21], De and Sengupta [13] define new type of semi-symmetric non-metric connection on Riemannian manifold. Inline with this S. K. Chaubey and A. Yildiz [9] define another new type of semi-symmetric non-metric connection and studied different geometrical properties. On Riemannian manifold $(\Omega_{2n+1}, \mathfrak{g}_1)$, a linear connection $\widetilde{\nabla}$ is semi-symmetric if $\widetilde{\mathcal{T}}(\mathfrak{J}_1, \mathfrak{J}_2) = \bar{\eta}(\mathfrak{J}_2)\mathfrak{J}_1 - \bar{\eta}(\mathfrak{J}_1)\mathfrak{J}_2$, $\forall\, \mathfrak{J}_1, \mathfrak{J}_2 \in \Gamma\Omega_{2n+1}$, where $\bar{\eta}$ is 1-form. Particularly, if $\mathfrak{J}_1 = \varphi\mathfrak{J}_1$ and $\mathfrak{J}_2 = \varphi\mathfrak{J}_2$, then the semi-symmetric connection reduces to the quarter-symmetric connection [15]. A semi-symmetric connection $\widetilde{\nabla}$ is metric if $\widetilde{\nabla}_{\mathfrak{g}_1} = 0$ & if $\widetilde{\nabla}_{\mathfrak{g}_1} \neq 0$, then it is non-metric. Since then, the properties of the semi-symmetric non-metric connection on different structures have been studied by many geometers [22, 12].

On the other hand, the almost hyperbolic (f, ξ, η, g)-structure was defined and studied by Upadhay and Dube [23]. Dube and Bhatt [7] studied CR-submanifold of trans-hyperbolic Sasakian manifold. Pankaj, S. K. Chaubey and Giilhanayar [20] studied Yamabe and gradient Yamabe soliton on 3-dimensional hyperbolic Kenmotsu manifolds. Mobin Ahmad and Kashif Ali [1] also studied CR-submanifold of a nearly hyperbolic Kenmotsu manifold admitting a quarter-symmetric non-metric connection. In the present article, it is initiated as follows: In section 2; contains some basic results of hyperbolic Kenmotsu manifolds. In section 3; we find some required results of the semi-symmetric non-metric connection. In section 4; we establish the relation between curvature tensor and semi-symmetric non-metric connection. The properties of semi-symmetric studied in section 5. Some results of locally φ-symmetric studied in section 6 and Ricci semi-symmetric hyperbolic Kenmatsu manifold equipped with semi-symmetric non-metric connection are investigated in section 7. We provided an example in section 8 and we also verified our results.

2. Hyperbolic Kenmotsu Manifold

Let $(\Omega_{2n+1}, \mathfrak{g}_1)$ be a contact manifold equipped with structure $(\varphi, \xi^\flat, \bar{\eta}, \mathfrak{g}_1)$, where φ is a $(1,1)$-tensor field, $\xi^\flat$ is a vector field, $\bar{\eta}$ is 1-form and $\mathfrak{g}_1$ is a Riemannian metric [20] such that-

$$\varphi^2\mathfrak{J}_1 = \mathfrak{J}_1 + \bar{\eta}(\mathfrak{J}_1)\xi^\flat, \qquad \bar{\eta}(\xi^\flat) = -1, \qquad \varphi\xi^\flat = 0, \qquad \bar{\eta}(\varphi\mathfrak{J}_1) = 0, \tag{2.1}$$

$$\mathfrak{g}_1(\varphi\mathfrak{J}_1, \varphi\mathfrak{J}_2) = -\mathfrak{g}_1(\mathfrak{J}_1, \mathfrak{J}_2) - \bar{\eta}(\mathfrak{J}_1)\bar{\eta}(\mathfrak{J}_2), \tag{2.2}$$

$$\mathfrak{g}_1(\varphi\mathfrak{J}_1, \mathfrak{J}_2) = -\mathfrak{g}_1(\mathfrak{J}_1, \varphi\mathfrak{J}_2), \qquad \mathfrak{g}_1(\mathfrak{J}_1, \xi^\flat) = \bar{\eta}(\mathfrak{J}_1), \tag{2.3}$$

for all $\mathfrak{J}_1, \mathfrak{J}_2 \in \Gamma\Omega_{2n+1}$. A contact manifold Ω_{2n+1} is hyperbolic Kenmotsu manifold iff

$$(\nabla_{\mathfrak{J}_1}\varphi)\mathfrak{J}_2 = \mathfrak{g}_1(\varphi\mathfrak{J}_1, \mathfrak{J}_2)\xi^\flat - \bar{\eta}(\mathfrak{J}_2)\varphi\mathfrak{J}_1, \tag{2.4}$$

where ∇ is Levi-Civita connection on Ω_{2n+1}. From (2.1), (2.2), (2.3) and (2.4), we find

$$d\bar{\eta} = 0, \qquad \nabla_{\mathfrak{J}_1}\xi^{\flat} = -\mathfrak{J}_1 - \bar{\eta}(\mathfrak{J}_1)\xi^{\flat}, \tag{2.5}$$

$$(\nabla_{\mathfrak{J}_1}\bar{\eta})\mathfrak{J}_2 = \mathfrak{g}_1(\varphi\mathfrak{J}_1, \varphi\mathfrak{J}_2) = -\mathfrak{g}_1(\mathfrak{J}_1, \mathfrak{J}_2) - \bar{\eta}(\mathfrak{J}_1)\bar{\eta}(\mathfrak{J}_2). \tag{2.6}$$

Also the hyperbolic Kenmotsu manifold hold the following relations:

$$\bar{\eta}(\mathcal{R}(\mathfrak{J}_1, \mathfrak{J}_2)\mathfrak{J}_3) = \mathfrak{g}_1(\mathfrak{J}_2, \mathfrak{J}_3)\bar{\eta}(\mathfrak{J}_1) - \mathfrak{g}_1(\mathfrak{J}_1, \mathfrak{J}_3)\bar{\eta}(\mathfrak{J}_2), \tag{2.7}$$

$$\mathcal{R}(\mathfrak{J}_1, \mathfrak{J}_2)\xi^{\flat} = \bar{\eta}(\mathfrak{J}_2)\mathfrak{J}_1 - \bar{\eta}(\mathfrak{J}_1)\mathfrak{J}_2, \tag{2.8}$$

$$\mathcal{R}(\xi^{\flat}, \mathfrak{J}_1)\mathfrak{J}_2 = \mathfrak{g}_1(\mathfrak{J}_1, \mathfrak{J}_2)\xi^{\flat} - \bar{\eta}(\mathfrak{J}_2)\mathfrak{J}_1, \tag{2.9}$$

$$\mathcal{R}(\xi^{\flat}, \mathfrak{J}_1)\xi^{\flat} = -\mathfrak{J}_1 - \bar{\eta}(\mathfrak{J}_1)\xi^{\flat}, \tag{2.10}$$

$$\mathcal{S}^{\flat}(\mathfrak{J}_1, \xi^{\flat}) = 2n\bar{\eta}(\mathfrak{J}_1), \tag{2.11}$$

$$\mathcal{S}^{\flat}(\xi^{\flat}, \xi^{\flat}) = -2n, \tag{2.12}$$

$$\mathcal{Q}^{\flat}(\xi^{\flat}) = -2n\xi^{\flat}, \tag{2.13}$$

$\mathcal{S}^{\flat}$ and $\mathcal{Q}^{\flat}$ are related by

$$\mathcal{S}^{\flat}(\mathfrak{J}_1, \mathfrak{J}_2) = \mathfrak{g}_1(\mathcal{Q}^{\flat}\mathfrak{J}_1, \mathfrak{J}_2). \tag{2.14}$$

Definition 2.1. An almost contact manifold Ω_{2n+1} is an η-Einstein manifold (η-EM) if Ricci-tensor $\mathcal{S}^{\flat}$ is of the form

$$\mathcal{S}^{\flat}(\mathfrak{J}_1, \mathfrak{J}_2) = a_1\mathfrak{g}_1(\mathfrak{J}_1, \mathfrak{J}_2) + a_2\bar{\eta}(\mathfrak{J}_1)\bar{\eta}(\mathfrak{J}_2), \tag{2.15}$$

where a_1 and a_2 are smooth functions on Ω_{2n+1}. If $a_2 = 0$, then manifold Ω_{2n+1} is an Einstein manifold (EM).

3. A new type of semi-symmetric non-metric connection

Let Ω_{2n+1} be hyperbolic Kenmotsu manifold. A linear connection $\widetilde{\nabla}$ on Ω_{2n+1} is given as

$$\widetilde{\nabla}_{\mathfrak{J}_1}\mathfrak{J}_2 = \nabla_{\mathfrak{J}_1}\mathfrak{J}_2 + \frac{1}{2}[\bar{\eta}(\mathfrak{J}_2)\mathfrak{J}_1 - \bar{\eta}(\mathfrak{J}_1)\mathfrak{J}_2] \tag{3.1}$$

is known as a semi-symmetric non-metric connection $\widetilde{\nabla}$ if it satisfies

$$\widetilde{\mathcal{T}}(\mathfrak{J}_1, \mathfrak{J}_2) = \bar{\eta}(\mathfrak{J}_2)\mathfrak{J}_1 - \bar{\eta}(\mathfrak{J}_1)\mathfrak{J}_2 \tag{3.2}$$

and

(3.3)
$$(\widetilde{\nabla}_{\mathfrak{J}_1}\mathfrak{g}_1)(\mathfrak{J}_2,\mathfrak{J}_3)=\frac{1}{2}\left[2\bar{\eta}(\mathfrak{J}_1)\,\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_3)-\bar{\eta}(\mathfrak{J}_2)\,\mathfrak{g}_1(\mathfrak{J}_1,\mathfrak{J}_3)-\bar{\eta}(\mathfrak{J}_3)\,\mathfrak{g}_1(\mathfrak{J}_1,\mathfrak{J}_2)\right].$$

Now,

$$(\widetilde{\nabla}_{\mathfrak{J}_1}\varphi)(\mathfrak{J}_2)=\frac{1}{2}\left[2\left(\nabla_{\mathfrak{J}_1}\varphi\right)\mathfrak{J}_2-\bar{\eta}(\mathfrak{J}_2)\left(\varphi\mathfrak{J}_1\right)\right],\tag{3.4}$$

$$(\widetilde{\nabla}_{\mathfrak{J}_1}\bar{\eta})(\mathfrak{J}_2)=(\nabla_{\mathfrak{J}_1}\bar{\eta})(\mathfrak{J}_2),\tag{3.5}$$

$$(\widetilde{\nabla}_{\mathfrak{J}_1}\mathfrak{g}_1)(\varphi\mathfrak{J}_2,\mathfrak{J}_3)=\frac{1}{2}\left[2\bar{\eta}(\mathfrak{J}_1)\,\mathfrak{g}_1(\varphi\mathfrak{J}_2,\mathfrak{J}_3)-\bar{\eta}(\mathfrak{J}_3)\,\mathfrak{g}_1(\mathfrak{J}_1,\varphi\mathfrak{J}_2)\right].\tag{3.6}$$

Changing $\mathfrak{J}_2$ by $\xi^\flat$ in (3.1), we have

$$\widetilde{\nabla}_{\mathfrak{J}_1}\xi^\flat=\nabla_{\mathfrak{J}_1}\xi^\flat-\frac{1}{2}\varphi^2\mathfrak{J}_1.\tag{3.7}$$

Replacing $\mathfrak{J}_1$ by $\xi^\flat$ in (3.3), we get

$$(\widetilde{\nabla}_{\xi^\flat}\mathfrak{g}_1)(\mathfrak{J}_2,\mathfrak{J}_3)=\mathfrak{g}_1(\varphi\mathfrak{J}_2,\varphi\mathfrak{J}_3)=-\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_3)-\bar{\eta}(\mathfrak{J}_2)\,\bar{\eta}(\mathfrak{J}_3).\tag{3.8}$$

4. Curvature tensor of a hyperbolic Kenmotsu manifold endowed with semi-symmetric non-metric connection

The curvature tensor $\widetilde{\mathcal{R}}$ with $\widetilde{\nabla}$ defined as follows:

$$\widetilde{\mathcal{R}}(\mathfrak{J}_1,\mathfrak{J}_2)\,\mathfrak{J}_3=\widetilde{\nabla}_{\mathfrak{J}_1}\widetilde{\nabla}_{\mathfrak{J}_2}\mathfrak{J}_3-\widetilde{\nabla}_{\mathfrak{J}_2}\widetilde{\nabla}_{\mathfrak{J}_1}\mathfrak{J}_3-\widetilde{\nabla}_{[\mathfrak{J}_1,\mathfrak{J}_2]}\mathfrak{J}_3,\tag{4.1}$$

Using (3.1) in (4.1), we obtain

$$\begin{aligned}\widetilde{\mathcal{R}}(\mathfrak{J}_1,\mathfrak{J}_2)\,\mathfrak{J}_3 \;=\;& \mathcal{R}(\mathfrak{J}_1,\mathfrak{J}_2)\,\mathfrak{J}_3+\frac{1}{2}[(\nabla_{\mathfrak{J}_1}\bar{\eta})(\mathfrak{J}_3)\mathfrak{J}_2-(\nabla_{\mathfrak{J}_1}\bar{\eta})(\mathfrak{J}_2)\mathfrak{J}_3\\ &-(\nabla_{\mathfrak{J}_2}\bar{\eta})(\mathfrak{J}_3)\mathfrak{J}_1+(\nabla_{\mathfrak{J}_2}\bar{\eta})(\mathfrak{J}_1)\mathfrak{J}_3]\\ &+\frac{1}{4}\left[\bar{\eta}(\mathfrak{J}_2)\bar{\eta}(\mathfrak{J}_3)\mathfrak{J}_1-\bar{\eta}(\mathfrak{J}_1)\bar{\eta}(\mathfrak{J}_3)\mathfrak{J}_2\right],\end{aligned}\tag{4.2}$$

where,

$$\mathcal{R}(\mathfrak{J}_1,\mathfrak{J}_2)\,\mathfrak{J}_3=\nabla_{\mathfrak{J}_1}\nabla_{\mathfrak{J}_2}\mathfrak{J}_3-\nabla_{\mathfrak{J}_2}\nabla_{\mathfrak{J}_1}\mathfrak{J}_3-\nabla_{[\mathfrak{J}_1,\mathfrak{J}_2]}\mathfrak{J}_3.\tag{4.3}$$

Now, using (2.6) in (4.2), we find

$$\begin{aligned}\widetilde{\mathcal{R}}(\mathfrak{J}_1,\mathfrak{J}_2)\,\mathfrak{J}_3 \;=\;& \mathcal{R}(\mathfrak{J}_1,\mathfrak{J}_2)\,\mathfrak{J}_3+\frac{1}{2}\left[\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_3)\,\mathfrak{J}_1-\mathfrak{g}_1(\mathfrak{J}_1,\mathfrak{J}_3)\,\mathfrak{J}_2\right]\\ &+\frac{3}{4}\left[\bar{\eta}(\mathfrak{J}_2)\,\bar{\eta}(\mathfrak{J}_3)\,\mathfrak{J}_1-\bar{\eta}(\mathfrak{J}_1)\,\bar{\eta}(\mathfrak{J}_3)\,\mathfrak{J}_2\right].\end{aligned}\tag{4.4}$$

Contracting equation (4.4) along $\mathfrak{J}_1$, we get

$$\widetilde{\mathcal{S}}^\flat(\mathfrak{J}_2,\mathfrak{J}_3)=\mathcal{S}^\flat(\mathfrak{J}_2,\mathfrak{J}_3)+n\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_3)+\frac{3n}{2}\bar{\eta}(\mathfrak{J}_2)\bar{\eta}(\mathfrak{J}_3). \tag{4.5}$$

By virtue of (2.14) and (4.5) gives

$$\widetilde{\mathcal{Q}}^\flat(\mathfrak{J}_2)=\mathcal{Q}^\flat(\mathfrak{J}_2)+n(\mathfrak{J}_2)+\frac{3n}{2}\bar{\eta}(\mathfrak{J}_2)\xi^\flat. \tag{4.6}$$

Again, contracting (4.5), we get

$$\widetilde{\tau}=\tau+n(2n-\frac{1}{2}). \tag{4.7}$$

Where $\widetilde{\mathcal{R}};\mathcal{R}$, $\widetilde{\mathcal{S}}^\flat;\mathcal{S}^\flat$, $\widetilde{\mathcal{Q}}^\flat$; $\mathcal{Q}^\flat$ and $\widetilde{\tau}$; τ are curvature tensor, Ricci tensor, Ricci operators and scalar curvature respectively equipped with $\widetilde{\nabla}$ and Levi-Civita connection ∇.

Replacing $\mathfrak{J}_1=\xi^\flat$ in (4.4) and using (2.1), (2.3), we get

$$\begin{aligned}\widetilde{\mathcal{R}}(\xi^\flat,\mathfrak{J}_2)\mathfrak{J}_3 &= \mathcal{R}(\xi^\flat,\mathfrak{J}_2)\mathfrak{J}_3+\frac{1}{2}[\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_3)\xi^\flat-\bar{\eta}(\mathfrak{J}_3)\mathfrak{J}_2]\\ &\quad+\frac{3}{4}[\bar{\eta}(\mathfrak{J}_2)\bar{\eta}(\mathfrak{J}_3)\xi^\flat+\bar{\eta}(\mathfrak{J}_3)\mathfrak{J}_2].\end{aligned} \tag{4.8}$$

Using (2.9) in above equation (4.8), we get

$$\widetilde{\mathcal{R}}(\xi^\flat,\mathfrak{J}_2)\mathfrak{J}_3=\frac{3}{2}\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_3)\xi^\flat+\frac{3}{4}[\bar{\eta}(\mathfrak{J}_2)\bar{\eta}(\mathfrak{J}_3)\xi^\flat-\bar{\eta}(\mathfrak{J}_3)\mathfrak{J}_2]. \tag{4.9}$$

Fix $\mathfrak{J}_3=\xi^\flat$ in (4.4) and using (2.1), (2.3), (2.8), we get

$$\begin{aligned}\widetilde{\mathcal{R}}(\mathfrak{J}_1,\mathfrak{J}_2)\xi^\flat &= \frac{3}{4}\mathcal{R}(\mathfrak{J}_1,\mathfrak{J}_2)\xi^\flat\\ &= \frac{3}{4}(\bar{\eta}(\mathfrak{J}_2)\mathfrak{J}_1-\bar{\eta}(\mathfrak{J}_1)\mathfrak{J}_2).\end{aligned} \tag{4.10}$$

Remark 4.1. Equation (4.10) shows that the manifold endowed with $\widetilde{\nabla}$ is regular.

In view of (2.3), (2.8), (4.4) and $\mathfrak{g}_1(\mathcal{R}(\mathfrak{J}_1,\mathfrak{J}_2,\mathfrak{J}_3),\mathfrak{J}_4)=-\mathfrak{g}_1(\mathcal{R}(\mathfrak{J}_1,\mathfrak{J}_2,\mathfrak{J}_4),\mathfrak{J}_3)$, we have

$$\bar{\eta}(\widetilde{\mathcal{R}}(\mathfrak{J}_1,\mathfrak{J}_2)\mathfrak{J}_3) = \frac{3}{2}[\bar{\eta}(\mathfrak{J}_1)\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_3)-\bar{\eta}(\mathfrak{J}_2)\mathfrak{g}_1(\mathfrak{J}_1,\mathfrak{J}_3)]. \tag{4.11}$$

Contracting (4.10) with $\mathfrak{J}_1$, we find

$$\widetilde{\mathcal{S}}^\flat(\mathfrak{J}_2,\xi^\flat)=\frac{3n}{2}\bar{\eta}(\mathfrak{J}_2). \tag{4.12}$$

Taking $\widetilde{\mathcal{R}}(\mathfrak{J}_1,\mathfrak{J}_2)\mathfrak{J}_3 = 0$ in equation (4.4), we get

$$\begin{aligned}\mathcal{R}(\mathfrak{J}_1,\mathfrak{J}_2)\mathfrak{J}_3 &= \frac{1}{2}\left[\mathfrak{g}_1(\mathfrak{J}_1,\mathfrak{J}_3)\mathfrak{J}_2 - \mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_3)\mathfrak{J}_1\right] \\ &+\frac{3}{4}\left[\bar{\eta}(\mathfrak{J}_1)\bar{\eta}(\mathfrak{J}_3)\mathfrak{J}_2 - \bar{\eta}(\mathfrak{J}_2)\bar{\eta}(\mathfrak{J}_3)\mathfrak{J}_1\right].\end{aligned} \tag{4.13}$$

In view of $'\mathcal{R}(\mathfrak{J}_1,\mathfrak{J}_2,\mathfrak{J}_3,\mathfrak{J}_4) = \mathfrak{g}_1(\mathcal{R}(\mathfrak{J}_1,\mathfrak{J}_2)\mathfrak{J}_3,\mathfrak{J}_4)$ and (4.13), we yields

$$\begin{aligned}'\mathcal{R}(\mathfrak{J}_1,\mathfrak{J}_2,\mathfrak{J}_3,\mathfrak{J}_4) &= \frac{1}{2}\left[\mathfrak{g}_1(\mathfrak{J}_1,\mathfrak{J}_3)\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_4) - \mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_3)\mathfrak{g}_1(\mathfrak{J}_1,\mathfrak{J}_4)\right] \\ &+\frac{3}{4}\left[\bar{\eta}(\mathfrak{J}_1)\bar{\eta}(\mathfrak{J}_3)\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_4) - \bar{\eta}(\mathfrak{J}_2)\bar{\eta}(\mathfrak{J}_3)\mathfrak{g}_1(\mathfrak{J}_1,\mathfrak{J}_4)\right].\end{aligned} \tag{4.14}$$

Contracting above equation along $\mathfrak{J}_1$ and $\mathfrak{J}_4$, we get

$$\mathcal{S}^\flat(\mathfrak{J}_2,\mathfrak{J}_3) = -n\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_3) - \frac{3n}{2}\bar{\eta}(\mathfrak{J}_2)\bar{\eta}(\mathfrak{J}_3). \tag{4.15}$$

Theorem 4.1. *A hyperbolic Kenmotsu manifold Ω_{2n+1} is an η-EM, if Riemannian curvature tensor endowed with $\widetilde{\nabla}$ is vanished.*

5. Semi-symmetric hyperbolic Kenmotsu manifold equipped with connection $\widetilde{\nabla}$

A contact manifold Ω_{2n+1} with connection $\widetilde{\nabla}$ is semi-symmetric if

$$(\widetilde{\mathcal{R}}(\mathfrak{J}_1,\mathfrak{J}_2).\widetilde{\mathcal{R}})(\mathfrak{J}_3,\mathfrak{J}_4)\mathfrak{J}_5 = 0.$$

Then, we have

$$\begin{aligned}&\widetilde{\mathcal{R}}(\mathfrak{J}_1,\mathfrak{J}_2)\widetilde{\mathcal{R}}(\mathfrak{J}_3,\mathfrak{J}_4)\mathfrak{J}_5 - \widetilde{\mathcal{R}}(\widetilde{\mathcal{R}}(\mathfrak{J}_1,\mathfrak{J}_2)\mathfrak{J}_3,\mathfrak{J}_4)\mathfrak{J}_5 \\ &-\widetilde{\mathcal{R}}(\mathfrak{J}_3,\widetilde{\mathcal{R}}(\mathfrak{J}_1,\mathfrak{J}_2)\mathfrak{J}_4)\mathfrak{J}_5 - \widetilde{\mathcal{R}}(\mathfrak{J}_3,\mathfrak{J}_4)\widetilde{\mathcal{R}}(\mathfrak{J}_1,\mathfrak{J}_2)\mathfrak{J}_5 = 0.\end{aligned} \tag{5.1}$$

On changing $\mathfrak{J}_1 = \xi^\flat$ in (5.1), we get

$$\begin{aligned}&\widetilde{\mathcal{R}}(\xi^\flat,\mathfrak{J}_2)\widetilde{\mathcal{R}}(\mathfrak{J}_3,\mathfrak{J}_4)\mathfrak{J}_5 - \widetilde{\mathcal{R}}(\widetilde{\mathcal{R}}(\xi^\flat,\mathfrak{J}_2)\mathfrak{J}_3,\mathfrak{J}_4)\mathfrak{J}_5 \\ &-\widetilde{\mathcal{R}}(\mathfrak{J}_3,\widetilde{\mathcal{R}}(\xi^\flat,\mathfrak{J}_2)\mathfrak{J}_4)\mathfrak{J}_5 - \widetilde{\mathcal{R}}(\mathfrak{J}_3,\mathfrak{J}_4)\widetilde{\mathcal{R}}(\xi^\flat,\mathfrak{J}_2)\mathfrak{J}_5 = 0.\end{aligned} \tag{5.2}$$

In view of (4.9), we obtain

$$\begin{aligned}2\mathfrak{g}_1(\mathfrak{J}_2,\widetilde{\mathcal{R}}(\mathfrak{J}_3,\mathfrak{J}_4)\mathfrak{J}_5) &= -\bar{\eta}(\mathfrak{J}_2)\bar{\eta}(\widetilde{\mathcal{R}}(\mathfrak{J}_3,\mathfrak{J}_4)\mathfrak{J}_5) - \bar{\eta}(\mathfrak{J}_2)\bar{\eta}(\widetilde{\mathcal{R}}(\mathfrak{J}_3,\mathfrak{J}_4)\mathfrak{J}_5) \\ &-2\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_3)\bar{\eta}(\widetilde{\mathcal{R}}(\xi^\flat,\mathfrak{J}_4)\mathfrak{J}_5) - \bar{\eta}(\mathfrak{J}_2)\bar{\eta}(\mathfrak{J}_3)\bar{\eta}(\widetilde{\mathcal{R}}(\xi^\flat,\mathfrak{J}_4)\mathfrak{J}_5) \\ &+\bar{\eta}(\mathfrak{J}_3)\bar{\eta}(\widetilde{\mathcal{R}}(\mathfrak{J}_2,\mathfrak{J}_4)\mathfrak{J}_5) - 2\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_4)\bar{\eta}(\widetilde{\mathcal{R}}(\mathfrak{J}_3,\xi^\flat)\mathfrak{J}_5) \\ &-\bar{\eta}(\mathfrak{J}_2)\bar{\eta}(\mathfrak{J}_4)\bar{\eta}(\widetilde{\mathcal{R}}(\mathfrak{J}_3,\xi^\flat)\mathfrak{J}_5) + \bar{\eta}(\mathfrak{J}_4)\bar{\eta}(\widetilde{\mathcal{R}}(\mathfrak{J}_3,\mathfrak{J}_2)\mathfrak{J}_5) \\ &-2\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_5)\bar{\eta}(\widetilde{\mathcal{R}}(\mathfrak{J}_3,\mathfrak{J}_4)\xi^\flat) - \bar{\eta}(\mathfrak{J}_2)\bar{\eta}(\mathfrak{J}_5)\bar{\eta}(\widetilde{\mathcal{R}}(\mathfrak{J}_3,\mathfrak{J}_4)\xi^\flat)\end{aligned}$$

$$+\bar{\eta}(\mathfrak{J}_5)\,\bar{\eta}(\widetilde{\mathcal{R}}(\mathfrak{J}_3,\mathfrak{J}_4)\,\mathfrak{J}_2). \tag{5.3}$$

Using (2.1), (2.3), (4.9), (4.10) and (4.11) in (5.3), we get

$$\begin{aligned} {}'\widetilde{\mathcal{R}}(\mathfrak{J}_3,\mathfrak{J}_4,\mathfrak{J}_5,\mathfrak{J}_2) &= \frac{3}{2}[\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_3)\,\mathfrak{g}_1(\mathfrak{J}_4,\mathfrak{J}_5) - \mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_4)\,\mathfrak{g}_1(\mathfrak{J}_3,\mathfrak{J}_5)] \\ &\quad + \frac{3}{4}[\bar{\eta}(\mathfrak{J}_4)\,\bar{\eta}(\mathfrak{J}_5)\,\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_3) - \bar{\eta}(\mathfrak{J}_3)\,\bar{\eta}(\mathfrak{J}_5)\,\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_4)]. \end{aligned} \tag{5.4}$$

Hence, we have

$$\begin{aligned} \widetilde{\mathcal{R}}(\mathfrak{J}_3,\mathfrak{J}_4)\,\mathfrak{J}_5 &= \frac{3}{2}\,[\mathfrak{g}_1(\mathfrak{J}_4,\mathfrak{J}_5)\,\mathfrak{J}_3 - \mathfrak{g}_1(\mathfrak{J}_3,\mathfrak{J}_5)\,\mathfrak{J}_4] \\ &\quad + \frac{3}{4}\,[\bar{\eta}(\mathfrak{J}_4)\,\bar{\eta}(\mathfrak{J}_5)\,\mathfrak{J}_3 - \bar{\eta}(\mathfrak{J}_3)\,\bar{\eta}(\mathfrak{J}_5)\,\mathfrak{J}_4]. \end{aligned} \tag{5.5}$$

Contracting (5.5) with $\mathfrak{J}_3$, we get

$$\widetilde{\mathcal{S}}^\flat(\mathfrak{J}_4,\mathfrak{J}_5) = 3n\mathfrak{g}_1(\mathfrak{J}_4,\mathfrak{J}_5) + \frac{3n}{2}\bar{\eta}(\mathfrak{J}_4)\,\bar{\eta}(\mathfrak{J}_5) \tag{5.6}$$

and

$$\widetilde{\mathcal{Q}}^\flat(\mathfrak{J}_4) = 3n\mathfrak{J}_4 + \frac{3n}{2}\bar{\eta}(\mathfrak{J}_4)\,\xi^\flat. \tag{5.7}$$

Again contracting (5.6), we have

$$\widetilde{\tau} = \frac{3n}{2}[4n+1]. \tag{5.8}$$

By virtue (2.15) and equation (5.6), we state:

Theorem 5.1. *A semi-symmetric hyperbolic Kenmotsu manifold Ω_{2n+1} equipped with $\widetilde{\nabla}$, then Ω_{2n+1} is an η-EM.*

Now, using (4.5), (4.6), (4.7) in (5.6), (5.7) and (5.8), we obtain

$$\mathcal{S}^\flat(\mathfrak{J}_4,\mathfrak{J}_5) = 2n\mathfrak{g}_1(\mathfrak{J}_4,\mathfrak{J}_5), \tag{5.9}$$

$$\mathcal{Q}^\flat\mathfrak{J}_4 = 2n(\mathfrak{J}_4) \tag{5.10}$$

and

$$\tau = 2n(2n+1). \tag{5.11}$$

Corollary 5.1. *A semi-symmetric hyperbolic Kenmotsu manifold Ω_{2n+1} endowed with $\widetilde{\widetilde{\nabla}}$ is an EM with semi-symmetric non-metric connection $\widetilde{\widetilde{\nabla}}$.*

The conformal curvature tensor $\widetilde{\mathcal{L}}^{\dagger}$ endowed with $\widetilde{\nabla}$ is defined as

$$\begin{aligned}\widetilde{\mathcal{L}}^{\dagger}(\mathfrak{J}_1,\mathfrak{J}_2)\mathfrak{J}_3 &= \widetilde{\mathcal{R}}(\mathfrak{J}_1,\mathfrak{J}_2)\mathfrak{J}_3 - \frac{1}{2n-1}[\widetilde{\mathcal{S}}^{\flat}(\mathfrak{J}_2,\mathfrak{J}_3)\mathfrak{J}_1 - \widetilde{\mathcal{S}}^{\flat}(\mathfrak{J}_1,\mathfrak{J}_3)\mathfrak{J}_2 \\ &+\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_3)\widetilde{\mathcal{Q}}^{\flat}\mathfrak{J}_1 - \mathfrak{g}_1(\mathfrak{J}_1,\mathfrak{J}_3)\widetilde{\mathcal{Q}}^{\flat}\mathfrak{J}_2] \\ &+\frac{\widetilde{\tau}}{2n(2n-1)}[\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_3)\mathfrak{J}_1 - \mathfrak{g}_1(\mathfrak{J}_1,\mathfrak{J}_3)\mathfrak{J}_2]. \end{aligned} \tag{5.12}$$

Using (5.5), (5.6), (5.7) and (5.8) in (5.12), we find

$$\begin{aligned}\widetilde{\mathcal{L}}^{\dagger}(\mathfrak{J}_1,\mathfrak{J}_2)\mathfrak{J}_3 &= \frac{3}{4(2n-1)}[\mathfrak{g}_1(\mathfrak{J}_1,\mathfrak{J}_3)\mathfrak{J}_2 - \mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_3)\mathfrak{J}_1 \\ &+\bar{\eta}(\mathfrak{J}_1)\bar{\eta}(\mathfrak{J}_3)\mathfrak{J}_2 - \bar{\eta}(\mathfrak{J}_2)\bar{\eta}(\mathfrak{J}_3)\mathfrak{J}_1] \\ &-\frac{3}{2(2n-1)}[\bar{\eta}(\mathfrak{J}_1)\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_3)\xi^{\flat} - \bar{\eta}(\mathfrak{J}_2)\mathfrak{g}_1(\mathfrak{J}_1,\mathfrak{J}_3)\xi^{\flat}]. \end{aligned} \tag{5.13}$$

Taking $\mathfrak{J}_3 = \xi^{\flat}$ in (5.13), we obtain

$$\widetilde{\mathcal{L}}^{\dagger}(\mathfrak{J}_1,\mathfrak{J}_2)\xi^{\flat} = 0. \tag{5.14}$$

Then, we have following result

Theorem 5.2. *A semi-symmetric hyperbolic Kenmotsu manifold Ω_{2n+1} admitting connection $\widetilde{\nabla}$ is $\xi^{\flat}$-conformally flat with $\widetilde{\nabla}$.*

6. Locally φ-symmetric hyperbolic Kenmotsu manifold admitting a connection $\widetilde{\nabla}$

Definition 6.1. A manifold Ω_{2n+1} endowed with $\widetilde{\nabla}$ is locally φ-symmetric [4] if

$$\varphi^2((\widetilde{\nabla}_{\mathfrak{J}_4}\widetilde{\mathcal{R}})(\mathfrak{J}_1,\mathfrak{J}_2)\mathfrak{J}_3) = 0.$$

All vector fields $\mathfrak{J}_1,\mathfrak{J}_2,\mathfrak{J}_3,\mathfrak{J}_4$ orthogonal to $\xi^{\flat}$.

We know that

$$\begin{aligned}(\widetilde{\nabla}_{\mathfrak{J}_4}\mathcal{R})(\mathfrak{J}_1,\mathfrak{J}_2)\mathfrak{J}_3 &= \widetilde{\nabla}_{\mathfrak{J}_4}\mathcal{R}(\mathfrak{J}_1,\mathfrak{J}_2)\mathfrak{J}_3 - \mathcal{R}(\widetilde{\nabla}_{\mathfrak{J}_4}\mathfrak{J}_1,\mathfrak{J}_2)\mathfrak{J}_3 \\ &-\mathcal{R}(\mathfrak{J}_1,\widetilde{\nabla}_{\mathfrak{J}_4}\mathfrak{J}_2)\mathfrak{J}_3 - \mathcal{R}(\mathfrak{J}_1,\mathfrak{J}_2)(\widetilde{\nabla}_{\mathfrak{J}_4}\mathfrak{J}_3). \end{aligned} \tag{6.1}$$

Using (3.1) and (2.7) in (6.1), we get

$$\begin{aligned}(\widetilde{\nabla}_{\mathfrak{J}_4}\mathcal{R})(\mathfrak{J}_1,\mathfrak{J}_2)\mathfrak{J}_3 &= (\nabla_{\mathfrak{J}_4}\mathcal{R})(\mathfrak{J}_1,\mathfrak{J}_2)\mathfrak{J}_3 + \frac{1}{2}[2\bar{\eta}(\mathfrak{J}_4)\mathcal{R}(\mathfrak{J}_1,\mathfrak{J}_2)\mathfrak{J}_3 \\ &-\bar{\eta}(\mathfrak{J}_1)\mathcal{R}(\mathfrak{J}_4,\mathfrak{J}_2)\mathfrak{J}_3 - \bar{\eta}(\mathfrak{J}_2)\mathcal{R}(\mathfrak{J}_1,\mathfrak{J}_4)\mathfrak{J}_3 \\ &-\bar{\eta}(\mathfrak{J}_3)\mathcal{R}(\mathfrak{J}_1,\mathfrak{J}_2)\mathfrak{J}_4 + \bar{\eta}(\mathfrak{J}_1)\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_3)\mathfrak{J}_4 \end{aligned}$$

$$-\bar{\eta}(\mathfrak{J}_2)\,\mathfrak{g}_1(\mathfrak{J}_1,\mathfrak{J}_3)\,\mathfrak{J}_4]. \tag{6.2}$$

Covariant differentiation of (4.4) with respect to $\widetilde{\nabla}$ along $\mathfrak{J}_4$ and using (2.6), (3.5), (6.2), we obtain

$$\begin{aligned}(\widetilde{\nabla}_{\mathfrak{J}_4}\widetilde{\mathcal{R}})(\mathfrak{J}_1,\mathfrak{J}_2)\mathfrak{J}_3 &= (\nabla_{\mathfrak{J}_4}\mathcal{R})(\mathfrak{J}_1,\mathfrak{J}_2)\mathfrak{J}_3+\frac{1}{2}[2\bar{\eta}(\mathfrak{J}_4)\mathcal{R}(\mathfrak{J}_1,\mathfrak{J}_2)\mathfrak{J}_3\\ &-\bar{\eta}(\mathfrak{J}_1)\mathcal{R}(\mathfrak{J}_4,\mathfrak{J}_2)\mathfrak{J}_3-\bar{\eta}(\mathfrak{J}_2)\mathcal{R}(\mathfrak{J}_1,\mathfrak{J}_4)\mathfrak{J}_3\\ &-\bar{\eta}(\mathfrak{J}_3)\mathcal{R}(\mathfrak{J}_1,\mathfrak{J}_2)\mathfrak{J}_4+\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_3)\bar{\eta}(\mathfrak{J}_1)\mathfrak{J}_4\\ &-\mathfrak{g}_1(\mathfrak{J}_1,\mathfrak{J}_3)\bar{\eta}(\mathfrak{J}_2)\mathfrak{J}_4+\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_3)\bar{\eta}(\mathfrak{J}_4)\mathfrak{J}_1\\ &-\bar{\eta}(\mathfrak{J}_4)\mathfrak{g}_1(\mathfrak{J}_1,\mathfrak{J}_3)\mathfrak{J}_2+2\bar{\eta}(\mathfrak{J}_1)\mathfrak{g}_1(\mathfrak{J}_4,\mathfrak{J}_3)\mathfrak{J}_2\\ &-2\bar{\eta}(\mathfrak{J}_2)\mathfrak{g}_1(\mathfrak{J}_4,\mathfrak{J}_3)\mathfrak{J}_1+2\bar{\eta}(\mathfrak{J}_3)\mathfrak{g}_1(\mathfrak{J}_4,\mathfrak{J}_1)\mathfrak{J}_2\\ &-2\bar{\eta}(\mathfrak{J}_3)\mathfrak{g}_1(\mathfrak{J}_4,\mathfrak{J}_2)\mathfrak{J}_1-3\bar{\eta}(\mathfrak{J}_2)\bar{\eta}(\mathfrak{J}_3)\bar{\eta}(\mathfrak{J}_4)\mathfrak{J}_1\\ &+3\bar{\eta}(\mathfrak{J}_1)\bar{\eta}(\mathfrak{J}_3)\bar{\eta}(\mathfrak{J}_4)\mathfrak{J}_2].\end{aligned} \tag{6.3}$$

Applying φ^2 on both side of equation (6.3) and using (2.1), (2.2), (2.3), we obtain

$$\begin{aligned}\varphi^2((\widetilde{\nabla}_{\mathfrak{J}_4}\widetilde{\mathcal{R}})(\mathfrak{J}_1,\mathfrak{J}_2)\mathfrak{J}_3) &= \varphi^2((\nabla_{\mathfrak{J}_4}\mathcal{R})(\mathfrak{J}_1,\mathfrak{J}_2)\mathfrak{J}_3)+\frac{1}{2}[2\bar{\eta}(\mathfrak{J}_4)\mathcal{R}(\mathfrak{J}_1,\mathfrak{J}_2)\mathfrak{J}_3\\ &+2\bar{\eta}(\mathfrak{J}_4)\bar{\eta}(\mathcal{R}(\mathfrak{J}_1,\mathfrak{J}_2)\mathfrak{J}_3)\xi^\flat-\bar{\eta}(\mathfrak{J}_1)\mathcal{R}(\mathfrak{J}_4,\mathfrak{J}_2)\mathfrak{J}_3\\ &-\bar{\eta}(\mathfrak{J}_1)\bar{\eta}(\mathcal{R}(\mathfrak{J}_4,\mathfrak{J}_2)\mathfrak{J}_3)\xi^\flat-\bar{\eta}(\mathfrak{J}_2)\mathcal{R}(\mathfrak{J}_1,\mathfrak{J}_4)\mathfrak{J}_3\\ &-\bar{\eta}(\mathfrak{J}_2)\bar{\eta}(\mathcal{R}(\mathfrak{J}_1,\mathfrak{J}_4)\mathfrak{J}_3)\xi^\flat-\bar{\eta}(\mathfrak{J}_3)\mathcal{R}(\mathfrak{J}_1,\mathfrak{J}_2)\mathfrak{J}_4\\ &-\bar{\eta}(\mathfrak{J}_3)\bar{\eta}(\mathcal{R}(\mathfrak{J}_1,\mathfrak{J}_2)\mathfrak{J}_4)\xi^\flat+\bar{\eta}(\mathfrak{J}_1)\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_3)\mathfrak{J}_4\\ &+2\bar{\eta}(\mathfrak{J}_1)\bar{\eta}(\mathfrak{J}_4)\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_3)\xi^\flat-\bar{\eta}(\mathfrak{J}_2)\mathfrak{g}_1(\mathfrak{J}_1,\mathfrak{J}_3)\mathfrak{J}_4\\ &-2\bar{\eta}(\mathfrak{J}_2)\bar{\eta}(\mathfrak{J}_4)\mathfrak{g}_1(\mathfrak{J}_1,\mathfrak{J}_3)\xi^\flat+\bar{\eta}(\mathfrak{J}_4)\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_3)\mathfrak{J}_1\\ &-\bar{\eta}(\mathfrak{J}_4)\mathfrak{g}_1(\mathfrak{J}_1,\mathfrak{J}_3)\mathfrak{J}_2-2\bar{\eta}(\mathfrak{J}_2)\mathfrak{g}_1(\mathfrak{J}_4,\mathfrak{J}_3)\mathfrak{J}_1\\ &-2\bar{\eta}(\mathfrak{J}_3)\bar{\eta}(\mathfrak{J}_1)\mathfrak{g}_1(\mathfrak{J}_4,\mathfrak{J}_2)\xi^\flat-2\bar{\eta}(\mathfrak{J}_3)\mathfrak{g}_1(\mathfrak{J}_4,\mathfrak{J}_2)\mathfrak{J}_1\\ &+2\bar{\eta}(\mathfrak{J}_1)\mathfrak{g}_1(\mathfrak{J}_4,\mathfrak{J}_3)\mathfrak{J}_2+2\bar{\eta}(\mathfrak{J}_3)\mathfrak{g}_1(\mathfrak{J}_4,\mathfrak{J}_1)\mathfrak{J}_2\\ &+2\bar{\eta}(\mathfrak{J}_3)\bar{\eta}(\mathfrak{J}_2)\mathfrak{g}_1(\mathfrak{J}_4,\mathfrak{J}_1)\xi^\flat+3\bar{\eta}(\mathfrak{J}_1)\bar{\eta}(\mathfrak{J}_3)\bar{\eta}(\mathfrak{J}_4)\mathfrak{J}_2\\ &-3\bar{\eta}(\mathfrak{J}_2)\bar{\eta}(\mathfrak{J}_3)\bar{\eta}(\mathfrak{J}_4)\mathfrak{J}_1].\end{aligned} \tag{6.4}$$

Taking $\mathfrak{J}_1,\mathfrak{J}_2,\mathfrak{J}_3$ and $\mathfrak{J}_4$ orthogonal to $\xi^\flat$, then (6.4) yields

$$\varphi^2((\widetilde{\nabla}_{\mathfrak{J}_4}\widetilde{\mathcal{R}})(\mathfrak{J}_1,\mathfrak{J}_2)\mathfrak{J}_3)=\varphi^2((\nabla_{\mathfrak{J}_4}\mathcal{R})(\mathfrak{J}_1,\mathfrak{J}_2)\mathfrak{J}_3). \tag{6.5}$$

Hence, the following theorem

Theorem 6.1. *The necessary and sufficient condition for manifold Ω_{2n+1} to be locally φ-symmetric equipped with ∇ is that it is also locally φ-symmetric endowed with $\widetilde{\nabla}$.*

7. Ricci semi-symmetric hyperbolic Kenmotsu manifold admitting a connection $\widetilde{\nabla}$

A contact metric manifold Ω_{2n+1} endowed with $\widetilde{\nabla}$ is Ricci semi-symmetric if $(\widetilde{\mathcal{R}}(\mathfrak{J}_1,\mathfrak{J}_2).\widetilde{\mathcal{S}}^\flat)(\mathfrak{J}_3,\mathfrak{J}_4)=0$, then we have

$$\widetilde{\mathcal{S}}^\flat(\widetilde{\mathcal{R}}(\mathfrak{J}_1,\mathfrak{J}_2)\mathfrak{J}_3,\mathfrak{J}_4)+\widetilde{\mathcal{S}}^\flat(\mathfrak{J}_3,\widetilde{\mathcal{R}}(\mathfrak{J}_1,\mathfrak{J}_2)\mathfrak{J}_4)=0. \tag{7.1}$$

Replacing $\mathfrak{J}_1$ by $\xi^\flat$ and using (4.9), we have

$$\frac{3}{2}\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_3)\widetilde{\mathcal{S}}^\flat(\xi^\flat,\mathfrak{J}_4)+\frac{3}{4}\bar{\eta}(\mathfrak{J}_2)\bar{\eta}(\mathfrak{J}_3)\widetilde{\mathcal{S}}^\flat(\xi^\flat,\mathfrak{J}_4)-\frac{3}{4}\bar{\eta}(\mathfrak{J}_3)\widetilde{\mathcal{S}}^\flat(\mathfrak{J}_2,\mathfrak{J}_4)$$
$$+\frac{3}{2}\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_4)\widetilde{\mathcal{S}}^\flat(\mathfrak{J}_3,\xi^\flat)+\frac{3}{4}\bar{\eta}(\mathfrak{J}_2)\bar{\eta}(\mathfrak{J}_4)\widetilde{\mathcal{S}}^\flat(\mathfrak{J}_3,\xi^\flat)$$
$$-\frac{3}{4}\bar{\eta}(\mathfrak{J}_4)\widetilde{\mathcal{S}}^\flat(\mathfrak{J}_3,\mathfrak{J}_2)=0. \tag{7.2}$$

Equations (4.12) and (7.2) reduce to

$$\frac{9n}{4}\bar{\eta}(\mathfrak{J}_4)\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_3)+\frac{9n}{8}\bar{\eta}(\mathfrak{J}_2)\bar{\eta}(\mathfrak{J}_3)\bar{\eta}(\mathfrak{J}_4)-\frac{3}{4}\bar{\eta}(\mathfrak{J}_3)\widetilde{\mathcal{S}}^\flat(\mathfrak{J}_2,\mathfrak{J}_4)$$
$$+\frac{9n}{4}\bar{\eta}(\mathfrak{J}_3)\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_4)+\frac{9n}{8}\bar{\eta}(\mathfrak{J}_2)\bar{\eta}(\mathfrak{J}_3)\bar{\eta}(\mathfrak{J}_4)$$
$$-\frac{3}{4}\bar{\eta}(\mathfrak{J}_4)\widetilde{\mathcal{S}}^\flat(\mathfrak{J}_3,\mathfrak{J}_2)=0. \tag{7.3}$$

Taking $\mathfrak{J}_4=\xi^\flat$ and using (4.12), we have

$$\widetilde{\mathcal{S}}^\flat(\mathfrak{J}_2,\mathfrak{J}_3)=3n\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_3)+\frac{3n}{2}\bar{\eta}(\mathfrak{J}_2)\bar{\eta}(\mathfrak{J}_3). \tag{7.4}$$

Using (4.5) in (7.4), we have

$$\mathcal{S}^\flat(\mathfrak{J}_2,\mathfrak{J}_3)=2n\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_3). \tag{7.5}$$

Hence, we have the following theorem

Theorem 7.1. *A Ricci semi-symmetric hyperbolic Kenmotsu manifold Ω_{2n+1} equipped with $\widetilde{\nabla}$ is an η-EM.*

Now, we have

$$(\widetilde{\mathcal{R}}(\mathfrak{J}_1,\mathfrak{J}_2).\widetilde{\mathcal{S}}^\flat)(\mathfrak{J}_3,\mathfrak{J}_4)=-\widetilde{\mathcal{S}}^\flat(\widetilde{\mathcal{R}}(\mathfrak{J}_1,\mathfrak{J}_2)\mathfrak{J}_3,\mathfrak{J}_4)$$
$$-\widetilde{\mathcal{S}}^\flat(\mathfrak{J}_3,\widetilde{\mathcal{R}}(\mathfrak{J}_1,\mathfrak{J}_2)\mathfrak{J}_4). \tag{7.6}$$

Using (4.4), (4.5) in (7.6), we have

$$(\widetilde{\mathcal{R}}(\mathfrak{J}_1,\mathfrak{J}_2).\widetilde{\mathcal{S}}^\flat)(\mathfrak{J}_3,\mathfrak{J}_4) = (\mathcal{R}(\mathfrak{J}_1,\mathfrak{J}_2).\mathcal{S}^\flat)(\mathfrak{J}_3,\mathfrak{J}_4)-\frac{1}{2}\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_3)\mathcal{S}^\flat(\mathfrak{J}_1,\mathfrak{J}_4)$$

$$+\frac{1}{2}\mathfrak{g}_1(\mathfrak{J}_1,\mathfrak{J}_3)\mathcal{S}^\flat(\mathfrak{J}_2,\mathfrak{J}_4)-\frac{3}{4}\bar{\eta}(\mathfrak{J}_2)\bar{\eta}(\mathfrak{J}_3)\mathcal{S}^\flat(\mathfrak{J}_1,\mathfrak{J}_4)$$
$$+\frac{3}{4}\bar{\eta}(\mathfrak{J}_1)\bar{\eta}(\mathfrak{J}_3)\mathcal{S}^\flat(\mathfrak{J}_2,\mathfrak{J}_4)-\frac{1}{2}\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_4)\mathcal{S}^\flat(\mathfrak{J}_3,\mathfrak{J}_1)$$
$$+\frac{1}{2}\mathfrak{g}_1(\mathfrak{J}_1,\mathfrak{J}_4)\mathcal{S}^\flat(\mathfrak{J}_3,\mathfrak{J}_2)-\frac{3}{4}\bar{\eta}(\mathfrak{J}_2)\bar{\eta}(\mathfrak{J}_4)\mathcal{S}^\flat(\mathfrak{J}_3,\mathfrak{J}_1)$$
$$+\frac{3}{4}\bar{\eta}(\mathfrak{J}_1)\bar{\eta}(\mathfrak{J}_4)\mathcal{S}^\flat(\mathfrak{J}_3,\mathfrak{J}_2)+\frac{3n}{2}\bar{\eta}(\mathfrak{J}_2)\bar{\eta}(\mathfrak{J}_3)\mathfrak{g}_1(\mathfrak{J}_1,\mathfrak{J}_4)$$
$$+\frac{3n}{2}\bar{\eta}(\mathfrak{J}_1)\bar{\eta}(\mathfrak{J}_3)\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_4)-\frac{3n}{2}\bar{\eta}(\mathfrak{J}_1)\bar{\eta}(\mathfrak{J}_4)\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_3)$$
$$-\frac{3n}{2}\bar{\eta}(\mathfrak{J}_2)\bar{\eta}(\mathfrak{J}_4)\mathfrak{g}_1(\mathfrak{J}_1,\mathfrak{J}_3), \tag{7.7}$$

We suppose that $\widetilde{\mathcal{R}}.\widetilde{\mathcal{S}}^\flat=\mathcal{R}.\mathcal{S}^\flat$, then (7.7) can be expressed as

$$-\frac{1}{2}\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_3)\mathcal{S}^\flat(\mathfrak{J}_1,\mathfrak{J}_4)+\frac{1}{2}\mathfrak{g}_1(\mathfrak{J}_1,\mathfrak{J}_3)\mathcal{S}^\flat(\mathfrak{J}_2,\mathfrak{J}_4)-\frac{3}{4}\bar{\eta}(\mathfrak{J}_2)\bar{\eta}(\mathfrak{J}_3)\mathcal{S}^\flat(\mathfrak{J}_1,\mathfrak{J}_4)$$
$$+\frac{3}{4}\bar{\eta}(\mathfrak{J}_1)\bar{\eta}(\mathfrak{J}_3)\mathcal{S}^\flat(\mathfrak{J}_2,\mathfrak{J}_4)-\frac{1}{2}\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_4)\mathcal{S}^\flat(\mathfrak{J}_3,\mathfrak{J}_1)+\frac{1}{2}\mathfrak{g}_1(\mathfrak{J}_1,\mathfrak{J}_4)\mathcal{S}^\flat(\mathfrak{J}_3,\mathfrak{J}_2)$$
$$-\frac{3}{4}\bar{\eta}(\mathfrak{J}_2)\bar{\eta}(\mathfrak{J}_4)\mathcal{S}^\flat(\mathfrak{J}_3,\mathfrak{J}_1)+\frac{3}{4}\bar{\eta}(\mathfrak{J}_1)\bar{\eta}(\mathfrak{J}_4)\mathcal{S}^\flat(\mathfrak{J}_3,\mathfrak{J}_2)+\frac{3n}{2}\bar{\eta}(\mathfrak{J}_2)\bar{\eta}(\mathfrak{J}_3)\mathfrak{g}_1(\mathfrak{J}_1,\mathfrak{J}_4)$$
$$+\frac{3n}{2}\bar{\eta}(\mathfrak{J}_1)\bar{\eta}(\mathfrak{J}_3)\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_4)-\frac{3n}{2}\bar{\eta}(\mathfrak{J}_1)\bar{\eta}(\mathfrak{J}_4)\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_3)$$
$$-\frac{3n}{2}\bar{\eta}(\mathfrak{J}_2)\bar{\eta}(\mathfrak{J}_4)\mathfrak{g}_1(\mathfrak{J}_1,\mathfrak{J}_3)=0. \tag{7.8}$$

Replacing $\mathfrak{J}_4$ by $\xi^\flat$ in the (7.8) and using (2.1), (2.2), (2.3) and (2.11), we obtain

$$\frac{n}{2}\bar{\eta}(\mathfrak{J}_1)\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_3)+\frac{5n}{2}\bar{\eta}(\mathfrak{J}_2)\mathfrak{g}_1(\mathfrak{J}_1,\mathfrak{J}_3)+\frac{1}{4}\bar{\eta}(\mathfrak{J}_2)\mathcal{S}^\flat(\mathfrak{J}_3,\mathfrak{J}_1)$$
$$-\frac{1}{4}\bar{\eta}(\mathfrak{J}_1)\mathcal{S}^\flat(\mathfrak{J}_3,\mathfrak{J}_2)+\frac{3n}{2}\bar{\eta}(\mathfrak{J}_1)\bar{\eta}(\mathfrak{J}_2)\bar{\eta}(\mathfrak{J}_3)=0. \tag{7.9}$$

Putting $\mathfrak{J}_1=\xi^\flat$ in (7.9), we find

$$\mathcal{S}^\flat(\mathfrak{J}_2,\mathfrak{J}_3)=2n\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_3)-6n\bar{\eta}(\mathfrak{J}_2)\bar{\eta}(\mathfrak{J}_3)\Rightarrow r=4n(n+2). \tag{7.10}$$

Hence, we conclude the following theorem

Theorem 7.2. *A hyperbolic Kenmotsu manifold Ω_{2n+1} equipped with $\widetilde{\nabla}$ satisfies $\widetilde{\mathcal{R}}.\widetilde{\mathcal{S}}^\flat-\mathcal{R}.\mathcal{S}^\flat=0$, then manifold Ω_{2n+1} is an η-EM.*

Definition 7.1. A Ricci soliton $(\mathfrak{g}_1,V_\flat,\lambda)$ on a Riemannian manifold is defined as

$$(\pounds_{V_\flat}\mathfrak{g}_1)(\mathfrak{J}_1,\mathfrak{J}_2)+2\mathcal{S}^\flat(\mathfrak{J}_1,\mathfrak{J}_2)+2\lambda\mathfrak{g}_1(\mathfrak{J}_1,\mathfrak{J}_2)=0, \tag{7.11}$$

where $\pounds_{V_\flat}$ is a Lie-derivative along $V_\flat$ and λ is a constant. A triplet $(\mathfrak{g}_1,V_\flat,\lambda)$ is shrinking, steady and expanding according to $\lambda<0$, $\lambda=0$, $\lambda>0$ respectively [5].

We have two situations regarding the vector field $V_\flat : V_\flat \in Span\xi^\flat$ and $V_\flat \perp \xi^\flat$. We investigate only the case $V_\flat = \xi^\flat$. The Ricci soliton of data $(\mathfrak{g}_1, \xi^\flat, \lambda)$ on manifold Ω_{2n+1} equipped with $\widetilde{\nabla}$ can be defined by

$$(\widetilde{\mathscr{L}}_{\xi^\flat}\mathfrak{g}_1)(\mathfrak{J}_1, \mathfrak{J}_2) + 2\widetilde{\mathcal{S}}^\flat(\mathfrak{J}_1, \mathfrak{J}_2) + 2\lambda\mathfrak{g}_1(\mathfrak{J}_1, \mathfrak{J}_2) = 0, \tag{7.12}$$

A straightforward calculation gives

$$(\widetilde{\mathscr{L}}_{\xi^\flat}\mathfrak{g}_1)(\mathfrak{J}_1, \mathfrak{J}_2) = (\widetilde{\nabla}_{\xi^\flat}\mathfrak{g}_1)(\mathfrak{J}_1, \mathfrak{J}_2) - \mathfrak{g}_1(\widetilde{\nabla}_{\mathfrak{J}_1}\xi^\flat, \mathfrak{J}_2) - \mathfrak{g}_1(\mathfrak{J}_1, \widetilde{\nabla}_{\mathfrak{J}_2}\xi^\flat). \tag{7.13}$$

Now using (2.1), (2.5), (3.7) and (3.8) in (7.13), we have

$$(\widetilde{\mathscr{L}}_{\xi^\flat}\mathfrak{g}_1)(\mathfrak{J}_1, \mathfrak{J}_2) = 2[\mathfrak{g}_1(\mathfrak{J}_1, \mathfrak{J}_2) + \bar{\eta}(\mathfrak{J}_1)\bar{\eta}(\mathfrak{J}_2)]. \tag{7.14}$$

From (4.5), (5.9), (7.5) and (7.12), we yields

$$(1 + 3n + \lambda)\mathfrak{g}_1(\mathfrak{J}_1, \mathfrak{J}_2) + (1 + \frac{3n}{2})\bar{\eta}(\mathfrak{J}_1)\bar{\eta}(\mathfrak{J}_2) = 0. \tag{7.15}$$

Taking $\mathfrak{J}_1 = \mathfrak{J}_2 = \xi^\flat$ in (7.15), we get

$$\lambda = -\frac{3n}{2} < 0.$$

Thus, we state the following theorem

Theorem 7.3. *A triplet $(\mathfrak{g}_1, \xi^\flat, \lambda)$ on manifold Ω_{2n+1} endowed with $\widetilde{\widetilde{\nabla}}$ is always shrinking.*

8. Example of hyperbolic Kenmotsu Manifold

Example 8.1. Let $\Omega_3 = (x, y, z) \in R^3 : z \neq 0$ be a 3-dimensional manifold with the standard coordinates (x, y, z) of R^3 [20]. Let $\varsigma_1 = e^z\frac{\partial}{\partial x}, \varsigma_2 = e^z\frac{\partial}{\partial y}, \varsigma_3 = \frac{\partial}{\partial z} = \xi^\flat$ be linear independent vector fields.

Suppose $\mathfrak{g}_1$ be the Ω_3 Riemannian metric specified by

$$\begin{gathered}\mathfrak{g}_1(\varsigma_1, \varsigma_2) = \mathfrak{g}_1(\varsigma_2, \varsigma_3) = \mathfrak{g}_1(\varsigma_3, \varsigma_1) = 0,\\ \mathfrak{g}_1(\varsigma_1, \varsigma_1) = 1, \quad \mathfrak{g}_1(\varsigma_2, \varsigma_2) = \mathfrak{g}_1(\varsigma_3, \varsigma_3) = -1,\end{gathered} \tag{8.1}$$

where

$$\mathfrak{g}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

and φ is $(1,1)$-tensor field defined by

$$\varphi(\varsigma_1) = \varsigma_2, \varphi(\varsigma_2) = \varsigma_1, \varphi(\varsigma_3) = 0. \tag{8.2}$$

By using linearity of φ and $\mathfrak{g}_1$, we have

$$\bar{\eta}(\varsigma_3) = -1, \quad \varphi^2\mathfrak{J}_1 = \mathfrak{J}_1 + \bar{\eta}(\mathfrak{J}_1)\varsigma_3, \tag{8.3}$$
$$\mathfrak{g}_1(\varphi\mathfrak{J}_1, \varphi\mathfrak{J}_2) = -\mathfrak{g}_1(\mathfrak{J}_1, \mathfrak{J}_2) - \bar{\eta}(\mathfrak{J}_1)\bar{\eta}(\mathfrak{J}_2)$$

Here $\bar{\eta}(\mathfrak{J}_1) = \mathfrak{g}_1(\mathfrak{J}_1, \varsigma_3)$ defines a 1-form on Ω_3. Hence for $\xi^\flat = \varsigma_3$, the structure $(\varphi, \xi^\flat, \bar{\eta}, \mathfrak{g}_1)$ defined on Ω_3. By applying definition $[\mathfrak{J}_1, \mathfrak{J}_2] = \mathfrak{J}_1(\mathfrak{J}_2 f) - \mathfrak{J}_2(\mathfrak{J}_1 f)$, the Lie bracket can be computed

$$\begin{aligned} &[\varsigma_1, \varsigma_1] = 0, \quad [\varsigma_1, \varsigma_2] = 0, \quad [\varsigma_1, \varsigma_3] = -\varsigma_1, \\ &[\varsigma_2, \varsigma_1] = 0, \quad [\varsigma_2, \varsigma_2] = 0, \quad [\varsigma_2, \varsigma_3] = -\varsigma_2, \\ &[\varsigma_3, \varsigma_1] = \varsigma_1, \quad [\varsigma_3, \varsigma_2] = \varsigma_2, \quad [\varsigma_3, \varsigma_3] = 0. \end{aligned} \tag{8.4}$$

Koszul's formula is given as

$$\begin{aligned} 2\mathfrak{g}_1\left(\nabla_{\mathfrak{J}_1}\mathfrak{J}_2, \mathfrak{J}_3\right) &= \mathfrak{J}_1\mathfrak{g}_1\left(\mathfrak{J}_2, \mathfrak{J}_3\right) + \mathfrak{J}_2\mathfrak{g}_1\left(\mathfrak{J}_3, \mathfrak{J}_1\right) - \mathfrak{J}_3\mathfrak{g}_1\left(\mathfrak{J}_1, \mathfrak{J}_2\right) \\ &\quad + \mathfrak{g}_1\left(\left[\mathfrak{J}_1, \mathfrak{J}_2\right], \mathfrak{J}_3\right) - \mathfrak{g}_1\left(\left[\mathfrak{J}_2, \mathfrak{J}_3\right], \mathfrak{J}_1\right) + \mathfrak{g}_1\left(\left[\mathfrak{J}_3, \mathfrak{J}_1\right], \mathfrak{J}_2\right). \end{aligned} \tag{8.5}$$

Now utilizing the above equation, we can compute

$$\begin{aligned} &\nabla_{\varsigma_1}\varsigma_1 = -\varsigma_3, \quad \nabla_{\varsigma_1}\varsigma_2 = 0, \quad \nabla_{\varsigma_1}\varsigma_3 = -\varsigma_1, \\ &\nabla_{\varsigma_2}\varsigma_1 = 0, \quad \nabla_{\varsigma_2}\varsigma_2 = \varsigma_3, \quad \nabla_{\varsigma_2}\varsigma_3 = -\varsigma_2, \\ &\nabla_{\varsigma_3}\varsigma_1 = 0, \quad \nabla_{\varsigma_3}\varsigma_2 = 0, \quad \nabla_{\varsigma_3}\varsigma_3 = 0. \end{aligned} \tag{8.6}$$

Also $\mathfrak{J}_1 = \mathfrak{J}_1^1\varsigma_1 + \mathfrak{J}_1^2\varsigma_2 + \mathfrak{J}_1^3\varsigma_3$ and $\xi^\flat = \varsigma_3$, then we have

$$\begin{aligned} \nabla_{\mathfrak{J}_1}\xi^\flat &= \nabla_{\mathfrak{J}_1^1\varsigma_1 + \mathfrak{J}_1^2\varsigma_2 + \mathfrak{J}_1^3\varsigma_3}\varsigma_3 \\ &= \mathfrak{J}_1^1\nabla_{\varsigma_1}\varsigma_3 + \mathfrak{J}_1^2\nabla_{\varsigma_2}\varsigma_3 + \mathfrak{J}_1^3\nabla_{\varsigma_3}\varsigma_3 \\ &= -\mathfrak{J}_1^1\varsigma_1 - \mathfrak{J}_1^2\varsigma_2 \end{aligned} \tag{8.7}$$

and

$$\begin{aligned} -\mathfrak{J}_1 - \bar{\eta}(\mathfrak{J}_1)\xi^\flat &= -\left(\mathfrak{J}_1^1\varsigma_1 + \mathfrak{J}_1^2\varsigma_2 + \mathfrak{J}_1^3\varsigma_3\right) - \mathfrak{g}_1\left(\mathfrak{J}_1^1\varsigma_1 + \mathfrak{J}_1^2\varsigma_2 + \mathfrak{J}_1^3\varsigma_3, \varsigma_3\right)\varsigma_3 \\ &= -\mathfrak{J}_1^1\varsigma_1 - \mathfrak{J}_1^2\varsigma_2 - \mathfrak{J}_1^3\varsigma_3 + \mathfrak{J}_1^3\varsigma_3 \\ &= -\mathfrak{J}_1^1\varsigma_1 - \mathfrak{J}_1^2\varsigma_2, \end{aligned} \tag{8.8}$$

where $\mathfrak{J}_1^1, \mathfrak{J}_1^2, \mathfrak{J}_1^3$ are scalars. From (8.7) and (8.8), the structure $(\varphi, \xi^\flat, \bar{\eta}, \mathfrak{g}_1)$ is hyperbolic Kenmotsu structure. Therefore $\Omega_3(\varphi, \xi^\flat, \bar{\eta}, \mathfrak{g}_1)$ is hyperbolic Kenmotsu manifold. In reference of (2.1), (2.3), (3.1) and (8.6), we get

$$\widetilde{\nabla}_{\varsigma_1}\varsigma_1 = -\varsigma_3, \quad \widetilde{\nabla}_{\varsigma_1}\varsigma_2 = 0, \quad \widetilde{\nabla}_{\varsigma_1}\varsigma_3 = -\frac{3}{2}\varsigma_1,$$

$$\widetilde{\nabla}_{\varsigma_2}\varsigma_1 = 0, \quad \widetilde{\nabla}_{\varsigma_2}\varsigma_2 = \varsigma_3, \quad \widetilde{\nabla}_{\varsigma_2}\varsigma_3 = -\frac{3}{2}\varsigma_2, \tag{8.9}$$

$$\widetilde{\nabla}_{\varsigma_3}\varsigma_1 = \frac{1}{2}\varsigma_1, \quad \widetilde{\nabla}_{\varsigma_3}\varsigma_2 = \frac{1}{2}\varsigma_2, \quad \widetilde{\nabla}_{\varsigma_3}\varsigma_3 = 0.$$

From (3.2) and (3.3), we yields

$$\widetilde{\mathcal{T}}(\varsigma_1, \varsigma_3) = \bar{\eta}(\varsigma_3)\varsigma_1 - \bar{\eta}(\varsigma_1)\varsigma_3 = -\varsigma_1 \neq 0$$

and

$$\begin{aligned}(\widetilde{\nabla}_{\varsigma_1}\mathfrak{g}_1)(\varsigma_1, \varsigma_3) &= \frac{1}{2}\left[2\bar{\eta}(\varsigma_1)\mathfrak{g}_1(\varsigma_1, \varsigma_3) - \bar{\eta}(\varsigma_1)\mathfrak{g}_1(\varsigma_1, \varsigma_3) - \bar{\eta}(\varsigma_3)\mathfrak{g}_1(\varsigma_1, \varsigma_1)\right] \\ &= \frac{1}{2} \neq 0.\end{aligned}$$

Consequently, a new type of semi-symmetric non-metric connection defined in (3.1). Also,

$$\begin{aligned}\widetilde{\nabla}_{\mathfrak{J}_1}\xi^\flat &= \widetilde{\nabla}_{\mathfrak{J}_1^1\varsigma_1 + \mathfrak{J}_1^2\varsigma_2 + \mathfrak{J}_1^3\varsigma_3}\varsigma_3 \\ &= \mathfrak{J}_1^1\widetilde{\nabla}_{\varsigma_1}\varsigma_3 + \mathfrak{J}_1^2\widetilde{\nabla}_{\varsigma_2}\varsigma_3 + \mathfrak{J}_1^3\widetilde{\nabla}_{\varsigma_3}\varsigma_3 \\ &= -\frac{3}{2}\mathfrak{J}_1^1\varsigma_1 - \frac{3}{2}\mathfrak{J}_1^2\varsigma_2,\end{aligned} \tag{8.10}$$

Equation (3.7) can be verified by using (8.7) and (8.10).

The components of $\mathcal{R}$ with connection ∇ are given as

$$\mathcal{R}(\varsigma_1, \varsigma_2)\varsigma_1 = -\varsigma_2, \quad \mathcal{R}(\varsigma_1, \varsigma_3)\varsigma_1 = -\varsigma_3, \quad \mathcal{R}(\varsigma_2, \varsigma_3)\varsigma_1 = 0,$$

$$\mathcal{R}(\varsigma_1, \varsigma_2)\varsigma_2 = -\varsigma_1, \quad \mathcal{R}(\varsigma_1, \varsigma_3)\varsigma_2 = 0, \quad \mathcal{R}(\varsigma_2, \varsigma_3)\varsigma_2 = \varsigma_3, \tag{8.11}$$

$$\mathcal{R}(\varsigma_1, \varsigma_2)\varsigma_3 = 0, \quad \mathcal{R}(\varsigma_1, \varsigma_3)\varsigma_3 = -\varsigma_1, \quad \mathcal{R}(\varsigma_2, \varsigma_3)\varsigma_3 = -\varsigma_2,$$

also $\mathcal{R}(\varsigma_i, \varsigma_i)\varsigma_i = 0; i = 1, 2, 3$ from simple calculations. We can verify (2.7), (2.8), (2.9), (2.10) and (2.11).

Similarly, the component of $\widetilde{\mathcal{R}}$ endowed with connection $\widetilde{\nabla}$ are as under:

$$\widetilde{\mathcal{R}}(\varsigma_1,\varsigma_2)\varsigma_1 = -\frac{3}{2}\varsigma_2, \quad \widetilde{\mathcal{R}}(\varsigma_1,\varsigma_3)\varsigma_1 = -\frac{3}{2}\varsigma_3, \quad \widetilde{\mathcal{R}}(\varsigma_2,\varsigma_3)\varsigma_1 = 0,$$

$$\widetilde{\mathcal{R}}(\varsigma_1,\varsigma_2)\varsigma_2 = -\frac{3}{2}\varsigma_1, \quad \widetilde{\mathcal{R}}(\varsigma_1,\varsigma_3)\varsigma_2 = 0, \quad \widetilde{\mathcal{R}}(\varsigma_2,\varsigma_3)\varsigma_2 = \frac{3}{2}\varsigma_3, \tag{8.12}$$

$$\widetilde{\mathcal{R}}(\varsigma_1,\varsigma_2)\varsigma_3 = 0, \quad \widetilde{\mathcal{R}}(\varsigma_1,\varsigma_3)\varsigma_3 = -\frac{3}{4}\varsigma_1, \quad \widetilde{\mathcal{R}}(\varsigma_2,\varsigma_3)\varsigma_3 = -\frac{3}{4}\varsigma_2,$$

along with $\widetilde{\mathcal{R}}(\varsigma_i,\varsigma_i)\varsigma_i = 0; i = 1,2,3$. Thus, we can verify (4.4), (4.8), (4.9), (4.10) and (4.11).

The Ricci tensor $\mathcal{S}^\flat(\mathfrak{J}_1,\mathfrak{J}_2)$ of connection ∇ can be derived by using (8.11) in $\mathcal{S}^\flat(\mathfrak{J}_1,\mathfrak{J}_2) = \sum_{i=1}^{3}\mathfrak{g}_1(\mathcal{R}(e_i,\mathfrak{J}_1)\mathfrak{J}_2,e_i)$. As follows:

$$\mathcal{S}^\flat(\varsigma_1,\varsigma_1) = 2, \quad \mathcal{S}^\flat(\varsigma_2,\varsigma_2) = -2, \quad \mathcal{S}^\flat(\varsigma_3,\varsigma_3) = -2. \tag{8.13}$$

The Ricci tensor $\widetilde{\mathcal{S}}^\flat(\mathfrak{J}_1,\mathfrak{J}_2)$ endowed with $\widetilde{\nabla}$ can be derived by using (8.12) in $\widetilde{\mathcal{S}}^\flat(\mathfrak{J}_1,\mathfrak{J}_2) = \sum_{i=1}^{3}\mathfrak{g}_1(\widetilde{\mathcal{R}}(e_i,\mathfrak{J}_1)\mathfrak{J}_2,e_i)$. It is as follows:

$$\widetilde{\mathcal{S}}^\flat(\varsigma_1,\varsigma_1) = 3, \quad \widetilde{\mathcal{S}}^\flat(\varsigma_2,\varsigma_2) = -3, \quad \widetilde{\mathcal{S}}^\flat(\varsigma_3,\varsigma_3) = -\frac{3}{2}. \tag{8.14}$$

In view of (8.13) and (8.14), the scalar curvature can be calculated as under:

$$\tau = \sum_{i=1}^{3}\mathcal{S}^\flat(e_i,e_i) = \mathcal{S}^\flat(\varsigma_1,\varsigma_1) - \mathcal{S}^\flat(\varsigma_2,\varsigma_2) - \mathcal{S}^\flat(\varsigma_3,\varsigma_3) = 6,$$

$$\widetilde{\tau} = \sum_{i=1}^{3}\widetilde{\mathcal{S}}^\flat(e_i,e_i) = \widetilde{\mathcal{S}}^\flat(\varsigma_1,\varsigma_1) - \widetilde{\mathcal{S}}^\flat(\varsigma_2,\varsigma_2) - \widetilde{\mathcal{S}}^\flat(\varsigma_3,\varsigma_3) = \frac{15}{2}.$$

Therefore, we can say that the example I provided completely correspond to our investigations.

REFERENCES

1. M. Ahmad and K. Ali: *CR-submanifolds of a nearly hyperbolic Kenmotsu manifold admitting a quarter-symmetric non-metric connection*, J. Math. Comput. Sci. **3**(2013), no. 3, 905-917.
2. N. S. Agashe and M. R. Chafle: *A semi-symmetric non-metric connection in a Riemannian manifold*, Indian J. Pure Appl. Math. **23**(1992), 399-409.

3. N. S. Agashe and M. R. Chafle: *On submanifolds of a Riemannian manifold with semi-symmetric non-metric connection*, Tensor N. S., **55(2)**(1994), 120-130.
4. A. Barman: *On P-Sasakian manifolds admitting semi-symmetric metric connection*, Publication de L' inst. Math., **95**(2014), 239-247.
5. C. S. Bagewadi and G. Ingalahalli: *Ricci Solitons in Lorentzian α-Sasakian manifolds*, Acta Math. Acad. P. N., **28**(2012), 59-68.
6. E. Bartolotti: *Sulla geometria della variata a connection affine*, Ann. diMat. **4(8)**(1930), 53-101.
7. L. Bhatt and K. K. Dube: *CR-submanifolds of trans-hyperbolic Sasakian manifold*, Acta Ciencialndica, **31**(2003), 91-96.
8. B. B. Chaturvedi and P. N. Pandey: *Semi-symmetric non-metric connection on a K̈ahler manifold*, Differential Geometry-Dynamical Systems, **10**(2008), 86-90.
9. S. K. Chaubey and A. Yildiz: *Riemannian manifold admitting a new type semi-symmetric non-metric connection*, Turkish Journal of Mathematics, **43**(2019), 1887-1904.
10. S. K. Chaubey: *On semi-symmetric non-metric connection*, Prog. of Math, **41-42**(2007), 11-20.
11. S. K. Chaubey and A. C. Pandey: *Some properties of a semi-symmetric non-metric connection on a Sasakian manifold*, Int. J. Contemp. Math. Sciences, **8**(2013), no. 16, 789-799.
12. U. C. De and D. Kamilya: *Hypersurfaces of a Riemannian manifold with semisymmetric non-metric connection*, J. Indian Inst. Sci., **75** (1995), 707-710.
13. U. C. De and J. Sengupta: *On a type of semi-symmetric non-metric connection*, Bull. Cal. Math. Soc., **92**(2000), no. 5, 375-384.
14. L. Das, M. Ahmad and A. Haseeb: *On semi-symmetric submanifolds of a nearly Sasakian manifold admitting a semi symmetric non-metric connection*, Journal of Applied Analysis, USA, **17**(2011), pp.1-12.
15. L. Das and M. Ahmad: *CR-submanifolds of a Lorenzian Para Sasakian Manifold enbowed with a quarter symmetric non-metric connection*, Math Science Research Journal, **13(7)**(2009), 161-169.
16. A. Friedmann and A. Schouten: *Über die Geometric der halbsymmetrischen Übertragung*, Math. Zeitschr., **21**(1924), 211-223.
17. H. A. Hayden: *Subspaces of a space with torsion*, Proc. London Math. Soc., **34**(1932), 27-50.
18. L. K. Pandey and R. H. Ojha: *Semi-symmetric metric and non-metric connections in Lorentzian paracontact manifold*, Bull. Cal. Math. Soc., **93**(2001), no. 6, 497-504.
19. Pankaj, S. K. Chaubey and R. Prasad: *Sasakian manifolds with admitting a non-symmetric non-metric connection*, Palestine Journal of Mathematics, **9(2)**(2020), 698-710.
20. Pankaj, S. K. Chaubey and G. Ayar: *Yamabe and gradient Yamabe solitons on 3-dimensional hyperbolic Kenmotsu manofolds*, Balkan Society of Geometers, **23**(2021), 175-188.
21. J. Sengupta, U. C. De and T. Q. Binh: *On a type of semi-symmetric non-metric connection on a Riemannian manifold*, Indian J. Pure Appl. Math., **31**(2000), no.(1-2), 1659-1670.

22. M. M. Tripathi and N. Kakkar: *On a semi-symmetric non-metric connection in a Kenmotsu manifold*, Bull. Cal. Math. Soc., **16**(2001), no. 4, 323-330.

23. M. D. Upadhyay and K. K. Dube: *Almost contact hyperbolic* (ϕ, ξ, η, g)*-structure*, Acta. Math. Acad. Scient. Hung. Tomus, **28**(1976), 1-4.