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CURVATURE TENSORS AND PSEUDOTENSORS IN A GENERALIZED FINSLER SPACE*

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Abstract. We examine relations between the curvature tensor of the associated symmetric connection and curvature tensors, curvature pseudotensors and derived curvature tensors of non-symmetric affine connection in Rund's sense.

Keywords: curvature tensor; curvature pseudotensor; derived curvature tensor; generalized Finsler space; non-symmetric connection.

1. Introduction and preliminaries

Eisenhart [3] gave a generalization of Riemannian spaces by using (in general) the non-symmetric basic tensor. In the same way Shamihoke [11]–[14] gave a generalization of Finsler spaces. He introduced two kinds of generalized Finsler spaces. The first kind was a generalization of Finsler spaces in Rund's [10] sense and the second kind was a generalization of Cartan's [1] Finsler spaces. In the present paper, we follow Shamihoke's definition of generalized Finsler spaces of the first kind. We continue the results derived in several previously published papers [2, 7, 8, 9, 15].

A generalized Finsler space GF_N is a differentiable manifold equipped with a basic tensor $g_{ij}(x^1, \dots, x^N, \dot{x}^1, \dots, \dot{x}^N) \equiv g_{ij}(x, \dot{x})$, where

$$g_{ij}(x, \dot{x}) \neq g_{ji}(x, \dot{x}), \text{ in generally}, \qquad (g = det(g_{ij}) \neq 0,\ \dot{x} = dx/dt). \tag{1.1}$$

Based on (1.1), one can define the symmetric and the anti-symmetric part of g_{ij}, respectively, by

$$g_{\underline{ij}} = \frac{1}{2}(g_{ij} + g_{ji}), \qquad g_{\underset{\vee}{ij}} = \frac{1}{2}(g_{ij} - g_{ji}),$$

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where, following [13], it is

$$\text{(1.2)} \qquad a)\ g_{\underline{ij}}(x,\dot{x}) = \frac{1}{2}\frac{\partial^2 F^2(x,\dot{x})}{\partial \dot{x}^i \partial \dot{x}^j}, \qquad b)\ \frac{\partial g_{\underset{\vee}{ij}}}{\partial \dot{x}^k} = 0.$$

The function $F(x,\dot{x})$ is a metric function in GF_N, having the properties known from the theory of usual Finsler spaces (F_N). The following conditions are valid:

1. $F(x,\dot{x})$ is continuously differentiable at least four times in its $2N$ arguments.

2. $F(x,\dot{x}) > 0$ providing all dx^i are not 0.

3. $F(x,\dot{x})$ is positively homogeneous of the 1^{st} degree in $\dot{x}$, i.e.

$$F(x,k\dot{x}) = kF(x,\dot{x}), \quad k > 0.$$

4. $\frac{\partial^2 F^2(x,\dot{x})}{\partial \dot{x}^i \partial \dot{x}^j}\xi^i \xi^j > 0$ for any given $\dot{x}$, and $\sum_i (\xi^i)^2 > 0,\ \xi^i \in R$.

The lowering and the raising of indices are defined by the tensors $g_{\underline{ij}}$ and h^{ij}, respectively, where h^{ij} is defined as follows

$$g_{\underline{ij}} h^{jk} = \delta^k_i, \quad (\underline{g} = det(g_{\underline{ij}}) \neq 0).$$

Generalized Cristoffel symbols of the 1^{st} *and the* 2^{nd} *kind* are defined by

$$\gamma_{i.jk} = \frac{1}{2}(g_{ji,k} - g_{jk,i} + g_{ik,j}) \neq \gamma_{i.kj},$$
$$\gamma^i_{jk} = h^{ip}\gamma_{p.jk} = \frac{1}{2}h^{ip}(g_{jp,k} - g_{jk,p} + g_{pk,j}) \neq \gamma^i_{kj},$$

where, $g_{ji,k} = \partial g_{ji}/\partial x^k$.

Then we have

$$\gamma^p_{jk} g_{\underline{ip}} = \gamma_{s.jk} h^{ps} g_{\underline{ip}} = \gamma_{s.jk}\delta^s_i = \gamma_{i.jk}.$$

Introducing the tensor C_{ijk} analogously as in F_N, we have

$$\text{(1.3)} \qquad C_{ijk}(x,\dot{x}) \overset{def}{=} \frac{1}{2} g_{ij,\dot{x}^k} \overset{(1.2b)}{=} \frac{1}{2} g_{\underline{ij},\dot{x}^k} \overset{(1.2a)}{=} \frac{1}{4} F^2_{\dot{x}^i \dot{x}^j \dot{x}^k},$$

where "$\overset{(1.2b)}{=}$" signifies "equal based on (1.2*b*)".

From (1.3) we can conclude that C_{ijk} is symmetric with respect to each pair of indices. Also, we have

$$C^i_{jk} \overset{def}{=} h^{ip} C_{pjk} = h^{ip} C_{jpk} = h^{ip} C_{jkp}.$$

With the help of coefficients P^i_{jk}, given by

$$P^i_{jk} = \gamma^i_{jk} - C^i_{jp}\gamma^p_{sk}\dot{x}^s \neq P^i_{kj},$$

one obtains coefficients of non-symmetric affine connection in Rund's sense [14]:

$$\begin{aligned} P^{*i}_{jk} &= \gamma^i_{jk} - h^{iq}(C_{jqp}P^p_{ks} + C_{kqp}P^p_{js} - C_{jkp}P^p_{qs})\dot{x}^s \neq P^{*i}_{kj}, \\ P^*_{i.jk} &= P^{*r}_{jk}g_{\underline{ir}} = \gamma_{i.jk} - (C_{ijp}P^p_{ks} + C_{ikp}P^p_{js} - C_{jkp}P^p_{is})\dot{x}^s \neq P^*_{i.kj}. \end{aligned}$$

In GF_N we denote the double anti-symmetric and the double symmetric part of connection P^{*i}_{jk}, respectively, by:

$$a)\ T^{*i}_{jk}(x,\dot{x}) = P^{*i}_{[jk]} = P^{*i}_{jk} - P^{*i}_{kj}, \quad b)\ P^{*i}_{(jk)} = P^{*i}_{jk} + P^{*i}_{kj},$$

where T^{*i}_{jk} is the *torsion tensor*.

2. Curvature tensors and pseudotensors of non-symmetric connection in Rund's sense

In the generalized Finsler space GF_N based on the non-symmetry of connection coefficients P^{*i}_{jk} there exist four kinds of covariant derivative of a tensor (see, e.g.[7]–[9]). For a tensor $a^i_j(x,\dot{x})$ it is

$$a^i_{j\underset{\substack{1\\2\\3\\4}}{|}m}(x,\dot{x}) = \delta_m a^i_j + P^{*i}_{\substack{pm\\mp\\pm\\mp}} a^p_j - P^{*p}_{\substack{jm\\mj\\mj\\jm}} a^i_p, \tag{2.1}$$

where

$$\delta_m = \frac{\partial}{\partial x^m} + \frac{\partial \dot{x}^p}{\partial x^m}\frac{\partial}{\partial \dot{x}^p}.$$

By using the first and the second kind of covariant derivative, given in (2.1), we examine ten Ricci type identities [8]

$$
\begin{aligned}
&a^i_{j\underset{1}{|}mn} - a^i_{j\underset{1}{|}nm} = \underset{1}{R}{}^{*i}_{pmn} a^p_j - \underset{1}{R}{}^{*p}_{jmn} a^i_p - T^{*p}_{mn} a^i_{j\underset{1}{|}p},\\
&a^i_{j\underset{2}{|}mn} - a^i_{j\underset{2}{|}nm} = \underset{2}{R}{}^{*i}_{pmn} a^p_j - \underset{2}{R}{}^{*p}_{jmn} a^i_p + T^{*p}_{mn} a^i_{j\underset{2}{|}p},\\
&a^i_{j\underset{1}{|}m\underset{2}{|}n} - a^i_{j\underset{1}{|}n\underset{2}{|}m} = \underset{1}{A}{}^i_{pmn} a^p_j - \underset{2}{A}{}^p_{jmn} a^i_p + a^i_{j<[mn]>} + a^i_{j\leqslant[mn]\geqslant} + T^{*p}_{mn} a^i_{j\underset{1}{|}p},\\
&a^i_{j\underset{2}{|}m\underset{1}{|}n} - a^i_{j\underset{2}{|}n\underset{1}{|}m} = \underset{3}{A}{}^i_{pmn} a^p_j - \underset{4}{A}{}^p_{jmn} a^i_p - a^i_{j<[mn]>} - a^i_{j\leqslant[mn]\geqslant} - T^{*p}_{mn} a^i_{j\underset{2}{|}p},\\
&a^i_{j\underset{1}{|}mn} - a^i_{j\underset{2}{|}nm} = \underset{5}{A}{}^i_{pmn} a^p_j - \underset{6}{A}{}^p_{jmn} a^i_p + a^i_{j<(mn)>} + a^i_{j\leqslant(mn)\geqslant} - P^{*p}_{mn}(a^i_{j\underset{1}{|}p} - a^i_{j\underset{2}{|}p}),\\
&a^i_{j\underset{1}{|}mn} - a^i_{j\underset{1}{|}n\underset{2}{|}m} = \underset{7}{A}{}^i_{pmn} a^p_j - \underset{8}{A}{}^p_{jmn} a^i_p + a^i_{j<mn>} + a^i_{j\leqslant mn\geqslant},\\
&a^i_{j\underset{1}{|}mn} - a^i_{j\underset{2}{|}n\underset{1}{|}m} = \underset{9}{A}{}^i_{pmn} a^p_j - \underset{10}{A}{}^p_{jmn} a^i_p + a^i_{j<nm>} + a^i_{j\leqslant nm\geqslant} - (P^{*p}_{mn} a^i_{j\underset{1}{|}p} - P^{*p}_{nm} a^i_{j\underset{2}{|}p}),\\
&a^i_{j\underset{2}{|}mn} - a^i_{j\underset{1}{|}n\underset{2}{|}m} = \underset{11}{A}{}^i_{pmn} a^p_j - \underset{12}{A}{}^p_{jmn} a^i_p - a^i_{j<nm>} + a^i_{j\lessdot mn\gtrdot} + P^{*p}_{mn} a^i_{j\underset{1}{|}p} - P^{*p}_{nm} a^i_{j\underset{2}{|}p},\\
&a^i_{j\underset{2}{|}mn} - a^i_{j\underset{2}{|}n\underset{1}{|}m} = \underset{13}{A}{}^i_{pmn} a^p_j - \underset{14}{A}{}^p_{jmn} a^i_p - a^i_{j<mn>} + a^i_{j\lessdot nm\gtrdot},\\
&a^i_{j\underset{1}{|}m\underset{2}{|}n} - a^i_{j\underset{2}{|}n\underset{1}{|}m} = \underset{15}{A}{}^i_{pmn} a^p_j - \underset{15}{A}{}^p_{jmn} a^i_p - P^{*p}_{mn}(a^i_{j\underset{1}{|}p} - a^i_{j\underset{2}{|}p}), \quad \text{i.e.}
\end{aligned}
$$

$$a^i_{j\underset{1}{|}m\underset{2}{|}n} - a^i_{j\underset{2}{|}n\underset{1}{|}m} = \underset{3}{R}{}^{*i}_{pmn} a^p_j - \underset{3}{R}{}^{*p}_{jmn} a^i_p,$$

where

$$
\begin{aligned}
a^i_{j<mn>} &= T^{*i}_{pm}\delta_n a^p_j - T^{*p}_{jm}\delta_n a^i_p,\\
a^i_{j\leqslant mn\geqslant} &= (P^{*i}_{mp}P^{*s}_{jn} - P^{*i}_{pm}P^{*s}_{nj})a^p_s,\\
a^i_{j\lessdot mn\gtrdot} &= (P^{*i}_{mp}P^{*s}_{nj} - P^{*i}_{pm}P^{*s}_{jn})a^p_s,\\
a^i_{j\leqslant mn\geqslant} &= (T^{*i}_{mp}P^{*s}_{jn} - P^{*i}_{pn}T^{*s}_{mj})a^p_s,\\
a^i_{j\lessdot mn\gtrdot} &= (P^{*i}_{mp}T^{*s}_{jn} - T^{*i}_{pn}P^{*s}_{mj})a^p_s,
\end{aligned}
$$

the magnitudes $\underset{t}{R}{}^{*i}_{jmn}$, $t = 1, 2, 3$, are curvature tensors and the magnitudes $\underset{t}{A}{}^{*i}_{jmn}$, $t = 1, \dots, 15$, are not tensors (we will verify this fact in what follows), we call them curvature pseudotensors of the first, the second,…, and the fifteenth kind, respectively.

Using the third and the fourth kind of the covariant derivative given in (2.1), one gets ten new Ricci type identities (see, [7]). In these identities appear the same quantities $\underset{t}{R}{}^{*i}_{jmn}$, $t = 1, 2, 3$, $\underset{s}{A}{}^{*i}_{jmn}$, $s = 1, \dots, 15$, but in different distribution. Only in the last case appears a new curvature tensor, the curvature tensor of the fourth kind $\underset{4}{R}{}^{*i}_{jmn}$ [7]:

$$a^i_{j\underset{3}{|}m\underset{4}{|}n} - a^i_{j\underset{4}{|}n\underset{3}{|}m} = \underset{4}{R}{}^{*i}_{pmn} a^p_j + \underset{3}{R}{}^{*p}_{jnm} a^i_p\ .$$

The curvature tensors $\underset{t}{R}{}^{*i}_{jmn}$, $t = 1, \dots, 4$, are given by [7]–[9]

$$\underset{1}{R}{}^{*i}_{jmn} = \delta_n P^{*i}_{jm} - \delta_m P^{*i}_{jn} + P^{*p}_{jm} P^{*i}_{pn} - P^{*p}_{jn} P^{*i}_{pm} = (\delta_n P^{*i}_{jm} + P^{*p}_{jm} P^{*i}_{pn})_{[mn]}, \tag{2.2}$$

$$\underset{2}{R}{}^{*i}_{jmn} = (\delta_n P^{*i}_{mj} + P^{*p}_{mj} P^{*i}_{np})_{[mn]}, \tag{2.3}$$

$$\underset{3}{R}{}^{*i}_{jmn} = \delta_n P^{*i}_{jm} - \delta_m P^{*i}_{nj} + P^{*p}_{jm} P^{*i}_{np} - P^{*p}_{nj} P^{*i}_{pm} + P^{*p}_{nm} P^{*i}_{[pj]}, \tag{2.4}$$

$$\underset{4}{R}{}^{*i}_{jmn} = \delta_n P^{*i}_{jm} - \delta_m P^{*i}_{nj} + P^{*p}_{jm} P^{*i}_{np} - P^{*p}_{nj} P^{*i}_{pm} + P^{*p}_{mn} P^{*i}_{[pj]}. \tag{2.5}$$

In [15] are given some properties of the mentioned curvature tensors (the antisymmetry with respect to two indices, the cyclic symmetry, the symmetry with respect to pairs of indices).

The curvature pseudotensors $\underset{t}{A}{}^{*i}_{jmn}$, $t = 1, \dots, 15$, are given by [8]

$$\underset{1}{A}{}^{*i}_{jmn} = (\delta_n P^{*i}_{jm} + P^{*p}_{jm} P^{*i}_{np})_{[mn]}, \tag{2.6}$$

$$\underset{2}{A}{}^{*i}_{jmn} = (\delta_n P^{*i}_{jm} + P^{*p}_{mj} P^{*i}_{pn})_{[mn]}, \tag{2.7}$$

$$\underset{3}{A}{}^{*i}_{jmn} = (\delta_n P^{*i}_{mj} + P^{*p}_{mj} P^{*i}_{pn})_{[mn]}, \tag{2.8}$$

$$\underset{4}{A}{}^{*i}_{jmn} = (\delta_n P^{*i}_{mj} + P^{*p}_{jm} P^{*i}_{np})_{[mn]}, \tag{2.9}$$

$$\underset{5}{A}{}^{*i}_{jmn} = \delta_n P^{*i}_{jm} - \delta_m P^{*i}_{nj} + P^{*p}_{jm} P^{*i}_{pn} - P^{*p}_{nj} P^{*i}_{mp}, \tag{2.10}$$

$$\underset{6}{A}{}^{*i}_{jmn} = \delta_n P^{*i}_{jm} - \delta_m P^{*i}_{nj} + P^{*p}_{mj} P^{*i}_{np} - P^{*p}_{jn} P^{*i}_{pm}, \tag{2.11}$$

$$\underset{7}{A}{}^{*i}_{jmn} = (\delta_n P^{*i}_{jm})_{[mn]} + P^{*p}_{jm} P^{*i}_{pn} - P^{*p}_{jn} P^{*i}_{mp}, \tag{2.12}$$

$$\underset{8}{A}{}^{*i}_{jmn} = (\delta_n P^{*i}_{jm})_{[mn]} + P^{*p}_{mj} P^{*i}_{pn} - P^{*p}_{jn} P^{*i}_{pm}, \tag{2.13}$$

$$\underset{9}{A}{}^{*i}_{jmn} = \delta_n P^{*i}_{jm} - \delta_m P^{*i}_{nj} + P^{*p}_{jm} P^{*i}_{pn} - P^{*p}_{nj} P^{*i}_{pm}, \tag{2.14}$$

$$\underset{10}{A}{}^{*i}_{jmn} = \delta_n P^{*i}_{jm} - \delta_m P^{*i}_{nj} + P^{*p}_{jm} P^{*i}_{np} - P^{*p}_{jn} P^{*i}_{pm}, \tag{2.15}$$

$$\underset{11}{A}{}^{*i}_{jmn} = \delta_n P^{*i}_{mj} - \delta_m P^{*i}_{jn} + P^{*p}_{mj} P^{*i}_{np} - P^{*p}_{jn} P^{*i}_{mp}, \tag{2.16}$$

$$\underset{12}{A}{}^{*i}_{jmn} = \delta_n P^{*i}_{mj} - \delta_m P^{*i}_{jn} + P^{*p}_{mj} P^{*i}_{pn} - P^{*p}_{nj} P^{*i}_{mp}, \tag{2.17}$$

$$\underset{13}{A}{}^{*i}_{jmn} = \delta_n P^{*i}_{mj} - \delta_m P^{*i}_{nj} + P^{*p}_{mj} P^{*i}_{np} - P^{*p}_{nj} P^{*i}_{pm}, \tag{2.18}$$

$$\underset{14}{A}{}^{*i}_{jmn} = \delta_n P^{*i}_{mj} - \delta_m P^{*i}_{nj} + P^{*p}_{jm} P^{*i}_{np} - P^{*p}_{nj} P^{*i}_{mp}, \tag{2.19}$$

$$\underset{15}{A}{}^{*i}_{jmn} = \delta_n P^{*i}_{jm} - \delta_m P^{*i}_{nj} + P^{*p}_{jm} P^{*i}_{np} - P^{*p}_{nj} P^{*i}_{pm}. \tag{2.20}$$

The signature $(\ )_{(mn)}$ denotes an antisymmetrization without division with respect to m, n, of the expression in the bracket. Analogously, the signature $(\ )_{[mn]}$ signifies a symmetrization without division with respect to m, n.

We should mention that Ricci type identities, curvature tensors and pseudotensors, given in [7, 8] are analogous to Ricci type identities, curvature tensors and pseudotensors in the space of non-symmetric affine connection, given in [4, 6].

The non-symmetric connection P^{*i}_{jk} can be represented as follows:

$$P^{*i}_{jk} = \frac{1}{2}P^{*i}_{(jk)} + \frac{1}{2}T^{*i}_{jk}. \tag{2.21}$$

Let us denote by R^{*i}_{jmn} the curvature tensor of symmetric connection $\frac{1}{2}P^{*i}_{(jm)}$ i.e.

$$R^{*i}_{jmn}(x,\dot{x}) = \frac{1}{2}\Big(\delta_n P^{*i}_{(jm)} + \frac{1}{2}P^{*p}_{(jm)}P^{*i}_{(pn)}\Big)_{[mn]}. \tag{2.22}$$

According to (2.21) the curvature tensor of the first kind can be represented in the following manner:

$$\begin{aligned} \underset{1}{R}{}^{*i}_{jmn} &= \frac{1}{2}\Big\{\delta_n P^{*i}_{jm} + \delta_n T^{*i}_{jm} + \frac{1}{2}\Big(P^{*p}_{(jm)}P^{*i}_{(pn)} + T^{*p}_{jm}P^{*i}_{(pn)} \\ &+ P^{*p}_{(jm)}T^{*i}_{pn} + T^{*p}_{jm}T^{*i}_{pn}\Big)\Big\}_{[mn]}. \end{aligned} \tag{2.23}$$

If one denotes by semicolon (;) the covariant derivative with respect to symmetric connection $\frac{1}{2}P^{*i}_{(jk)}$, equation (2.23) becomes

$$\underset{1}{R}{}^{*i}_{jmn} = R^{*i}_{jmn} + \frac{1}{2}\Big(T^{*i}_{jm;n} + \frac{1}{2}T^{*p}_{jm}T^{*i}_{pn}\Big)_{[mn]}, \tag{2.24}$$

that is, we obtained the relation between $\underset{1}{R}{}^{*i}_{jmn}$, R^{*i}_{jmn} and the torsion tensor T^{*i}_{jk}.

Following this procedure for the curvature tensors $\underset{t}{R}{}^{*i}_{jmn}$, $t = 2,\dots,4$, in GF_N, we get:

$$\underset{2}{R}{}^{*i}_{jmn} = R^{*i}_{jmn} + \frac{1}{2}\Big(T^{*i}_{mj;n} + \frac{1}{2}T^{*p}_{mj}T^{*i}_{np}\Big)_{[mn]}, \tag{2.25}$$

$$\underset{3}{R}{}^{*i}_{jmn} = R^{*i}_{jmn} + \frac{1}{2}\Big((T^{*i}_{jm;n})_{(mn)} + \frac{1}{2}(T^{*p}_{jm}T^{*i}_{np})_{[mn]} + T^{p}_{mn}T^{i}_{jp}\Big), \tag{2.26}$$

$$\underset{4}{R}{}^{*i}_{jmn} = R^{*i}_{jmn} + \frac{1}{2}\Big((T^{*i}_{jm;n})_{(mn)} + \frac{1}{2}(T^{*p}_{jm}T^{*i}_{np})_{[mn]} - T^{*p}_{mn}T^{*i}_{jp}\Big). \tag{2.27}$$

Based on the discussion given above, we can formulate the next theorem.

Theorem 2.1. *The curvature tensors* $\underset{\theta}{R}{}^{*i}_{jmn}$ $(\theta = 1,\dots,4)$, *corresponding to the non-symmetric connection* P^{*i}_{jk}, *given by* (2.2)–(2.5), *and the curvature tensor* R^{*i}_{jmn}, *corresponding to the symmetric connection* $\frac{1}{2}P^{*i}_{(jk)} = \frac{1}{2}(P^{*i}_{jk} + P^{*i}_{kj})$, *given by* (2.22), *satisfy relations* (2.24)–(2.27), *where* T^{*i}_{jk} *is the torsion tensor of connection* P^{*i}_{jk}.

In a similar manner, we can prove the following theorem.

Theorem 2.2. *The curvature pseudotensors* $\underset{t}{A}{}^{*i}_{jmn}$, $t = 1, \ldots, 15$, *given by equations* (2.6)–(2.20) *and the curvature tensor* R^{*i}_{jmn}, *given by* Eq. (2.22), *satisfy the following relations*

$$\begin{aligned}
\underset{1}{A}{}^{*i}_{jmn} &= R^{*i}_{jmn} + \Big(\frac{1}{2}\delta_n T^{*i}_{jm} + \frac{1}{4}T^{*p}_{jm}P^{*i}_{(np)} + \frac{1}{4}T^{*p}_{jm}T^{*i}_{np} + \frac{1}{4}P^{*p}_{(jm)}T^{*i}_{np}\Big)_{[mn]},\\
\underset{2}{A}{}^{*i}_{jmn} &= R^{*i}_{jmn} + \Big(\frac{1}{2}\delta_n T^{*i}_{jm} + \frac{1}{4}T^{*p}_{mj}P^{*i}_{(pn)} + \frac{1}{4}T^{*p}_{mj}T^{*i}_{pn} + \frac{1}{4}P^{*p}_{(mj)}T^{*i}_{pn}\Big)_{[mn]},\\
\underset{3}{A}{}^{*i}_{jmn} &= R^{*i}_{jmn} + \Big(\frac{1}{2}\delta_n T^{*i}_{mj} + \frac{1}{4}T^{*p}_{mj}P^{*i}_{(pn)} + \frac{1}{4}T^{*p}_{mj}T^{*i}_{pn} + \frac{1}{4}P^{*p}_{(mj)}T^{*i}_{pn}\Big)_{[mn]},\\
\underset{4}{A}{}^{*i}_{jmn} &= R^{*i}_{jmn} + \Big(\frac{1}{2}\delta_n T^{*i}_{mj} + \frac{1}{4}T^{*p}_{jm}P^{*i}_{(np)} + \frac{1}{4}T^{*p}_{jm}T^{*i}_{np} + \frac{1}{4}P^{*p}_{(jm)}T^{*i}_{np}\Big)_{[mn]},
\end{aligned}$$

$$\begin{aligned}
\underset{5}{A}{}^{*i}_{jmn} &= R^{*i}_{jmn} + \frac{1}{2}\delta_n T^{*i}_{jm} - \frac{1}{2}\delta_m T^{*i}_{nj} + \frac{1}{4}P^{*p}_{(jm)}T^{*i}_{pn} + \frac{1}{4}T^{*p}_{jm}P^{*i}_{(pn)}\\
&\quad + \frac{1}{4}T^{*p}_{jm}T^{*i}_{pn} - \frac{1}{4}P^{*p}_{(nj)}T^{*i}_{mp} - \frac{1}{4}T^{*p}_{nj}P^{*i}_{(mp)} - \frac{1}{4}T^{*p}_{nj}T^{*i}_{mp},\\
\underset{6}{A}{}^{*i}_{jmn} &= R^{*i}_{jmn} + \frac{1}{2}\delta_n T^{*i}_{jm} - \frac{1}{2}\delta_m T^{*i}_{nj} + \frac{1}{4}P^{*i}_{(mj)}T^{*i}_{np} + \frac{1}{4}T^{*p}_{mj}P^{*i}_{(np)}\\
&\quad + \frac{1}{4}T^{*p}_{mj}T^{*i}_{np} - \frac{1}{4}P^{*p}_{(jn)}T^{*i}_{pm} - \frac{1}{4}T^{*p}_{jn}P^{*i}_{(pm)} - \frac{1}{4}T^{*p}_{jn}T^{*i}_{pm},
\end{aligned}$$

$$\begin{aligned}
\underset{7}{A}{}^{*i}_{jmn} &= R^{*i}_{jmn} + \frac{1}{2}\delta_n T^{*i}_{jm} - \frac{1}{2}\delta_m T^{*i}_{jn} + \frac{1}{4}P^{*p}_{(jm)}T^{*i}_{pn} + \frac{1}{4}T^{*p}_{jm}P^{*i}_{(pn)} \\
&\quad + \frac{1}{4}T^{*p}_{jm}T^{*i}_{pn} - \frac{1}{4}P^{*p}_{(jn)}T^{*i}_{mp} - \frac{1}{4}T^{*p}_{jn}P^{*i}_{(mp)} - \frac{1}{4}T^{*p}_{jn}T^{*i}_{mp}, \\
\underset{8}{A}{}^{*i}_{jmn} &= R^{*i}_{jmn} + \frac{1}{2}\delta_n T^{*i}_{jm} - \frac{1}{2}\delta_m T^{*i}_{jn} + \frac{1}{4}P^{*p}_{(mj)}T^{*i}_{pn} + \frac{1}{4}T^{*p}_{mj}P^{*i}_{(pn)} \\
&\quad + \frac{1}{4}T^{*p}_{mj}T^{*i}_{pn} - \frac{1}{4}P^{*p}_{(jn)}T^{*i}_{pm} - \frac{1}{4}T^{*p}_{jn}P^{*i}_{(pm)} - \frac{1}{4}T^{*p}_{jn}T^{*i}_{pm}, \\
\underset{9}{A}{}^{*i}_{jmn} &= R^{*i}_{jmn} + \frac{1}{2}\delta_n T^{*i}_{jm} - \frac{1}{2}\delta_m T^{*i}_{nj} + \frac{1}{4}P^{*p}_{(jm)}T^{*i}_{pn} + \frac{1}{4}T^{*p}_{jm}P^{*i}_{(pn)} \\
&\quad + \frac{1}{4}T^{*p}_{jm}T^{*i}_{pn} - \frac{1}{4}P^{*p}_{(nj)}T^{*i}_{pm} - \frac{1}{4}T^{*p}_{nj}P^{*i}_{(pm)} - \frac{1}{4}T^{*p}_{nj}T^{*i}_{pm}, \\
\underset{10}{A}{}^{*i}_{jmn} &= R^{*i}_{jmn} + \frac{1}{2}\delta_n T^{*i}_{jm} - \frac{1}{2}\delta_m T^{*i}_{nj} + \frac{1}{4}P^{*p}_{(jm)}T^{*i}_{np} + \frac{1}{4}T^{*p}_{jm}P^{*i}_{(np)} \\
&\quad + \frac{1}{4}T^{*p}_{jm}T^{*i}_{np} - \frac{1}{4}P^{*p}_{(jn)}T^{*i}_{pm} - \frac{1}{4}T^{*p}_{jn}P^{*i}_{(pm)} - \frac{1}{4}T^{*p}_{jn}T^{*i}_{pm}, \\
\underset{11}{A}{}^{*i}_{jmn} &= R^{*i}_{jmn} + \frac{1}{2}\delta_n T^{*i}_{mj} - \frac{1}{2}\delta_m T^{*i}_{jn} + \frac{1}{4}P^{*p}_{(mj)}T^{*i}_{np} + \frac{1}{4}T^{*p}_{mj}P^{*i}_{(np)} \\
&\quad + \frac{1}{4}T^{*p}_{jm}T^{*i}_{np} - \frac{1}{4}P^{*p}_{(jn)}T^{*i}_{mp} - \frac{1}{4}T^{*p}_{jn}P^{*i}_{(mp)} - \frac{1}{4}T^{*p}_{jn}T^{*i}_{mp}, \\
\underset{12}{A}{}^{*i}_{jmn} &= R^{*i}_{jmn} + \frac{1}{2}\delta_n T^{*i}_{mj} - \frac{1}{2}\delta_m T^{*i}_{jn} + \frac{1}{4}P^{*p}_{(mj)}T^{*i}_{pn} + \frac{1}{4}T^{*p}_{mj}P^{*i}_{(pn)} \\
&\quad + \frac{1}{4}T^{*p}_{mj}T^{*i}_{pn} - \frac{1}{4}P^{*p}_{(nj)}T^{*i}_{mp} - \frac{1}{4}T^{*p}_{nj}P^{*i}_{(mp)} - \frac{1}{4}T^{*p}_{nj}T^{*i}_{mp}, \\
\underset{13}{A}{}^{*i}_{jmn} &= R^{*i}_{jmn} + \frac{1}{2}\delta_n T^{*i}_{mj} - \frac{1}{2}\delta_m T^{*i}_{nj} + \frac{1}{4}P^{*p}_{(mj)}T^{*i}_{np} + \frac{1}{4}T^{*p}_{mj}P^{*i}_{(np)} \\
&\quad + \frac{1}{4}T^{*p}_{mj}T^{*i}_{np} - \frac{1}{4}P^{*p}_{(nj)}T^{*i}_{pm} - \frac{1}{4}T^{*p}_{nj}P^{*i}_{(pm)} - \frac{1}{4}T^{*p}_{nj}T^{*i}_{pm}, \\
\underset{14}{A}{}^{*i}_{jmn} &= R^{*i}_{jmn} + \frac{1}{2}\delta_n T^{*i}_{mj} - \frac{1}{2}\delta_m T^{*i}_{nj} + \frac{1}{4}P^{*p}_{(jm)}T^{*i}_{np} + \frac{1}{4}T^{*p}_{jm}P^{*i}_{(np)} \\
&\quad + \frac{1}{4}T^{*p}_{jm}T^{*i}_{np} - \frac{1}{4}P^{*p}_{(nj)}T^{*i}_{mp} - \frac{1}{4}T^{*p}_{nj}P^{*i}_{(mp)} - \frac{1}{4}T^{*p}_{nj}T^{*i}_{mp}, \\
\underset{15}{A}{}^{*i}_{jmn} &= R^{*i}_{jmn} + \frac{1}{2}\delta_n T^{*i}_{jm} - \frac{1}{2}\delta_m T^{*i}_{nj} + \frac{1}{4}P^{*p}_{(jm)}T^{*i}_{np} + \frac{1}{4}T^{*p}_{jm}P^{*i}_{(np)} \\
&\quad + \frac{1}{4}T^{*p}_{jm}T^{*i}_{np} - \frac{1}{4}P^{*p}_{(nj)}T^{*i}_{pm} - \frac{1}{4}T^{*p}_{nj}P^{*i}_{(pm)} - \frac{1}{4}T^{*p}_{nj}T^{*i}_{pm},
\end{aligned}$$

where T^{*i}_{jk} is the torsion tensor of connection P^{*i}_{jk}.

From the obtained expressions of magnitudes $\underset{t}{A}{}^{*i}_{jmn}$, $t = 1, \dots, 15$, given in Theorem 2.2, we see that these magnitudes are not tensors because T^{*i}_{jk} is a torsion tensor and $P^{*i}_{(jk)}$ is not a tensor. Also, we can obtain the transformation laws for curvature pseudotensors. For example, for $\underset{7}{A}{}^{*i}_{jmn}$, we see that the last addend is not a tensor, in coordinates $x^{i'}$ it is

$$P^{*p'}_{(j'n')}T^{*i'}_{m'p'} = (P^{*p}_{(jn)}x^{p'}_{p}x^{j}_{j'}x^{n}_{n'} + x^{p}_{j'n'}x^{p'}_{p})T^{*i}_{mp}x^{i'}_{i}x^{m}_{m'}x^{q}_{p'} = P^{*p}_{(jn)}T^{*i}_{mp}x^{i'}_{i}x^{j}_{j'}x^{m}_{m'}x^{n}_{n'} + x^{p}_{j'n'}T^{*i}_{mp}x^{i'}_{i}x^{m}_{m'}, \tag{2.28}$$

because

$$T^{*i}_{mq}x^{p'}_{p}x^{q}_{p'} = T^{*i}_{mq}\delta^{q}_{p} = T^{*i}_{mp}.$$

Now, from (2.28) we obtain

$$\underset{7}{A}{}^{*i'}_{j'm'n'} = \underset{7}{A}{}^{*i}_{jmn}x^{i'}_{i}x^{j}_{j'}x^{m}_{m'}x^{n}_{n'} - 2x^{p}_{j'n'}T^{*i}_{mp}x^{i'}_{i}x^{m}_{m'}.$$

Remark 2.1. *If $g_{\underset{\vee}{ij}} = 0$ i.e. $g_{ij}(x,\dot{x}) = g_{\underline{ij}}(x,\dot{x})$, then the generalized Finsler space reduces to the usual Finsler space. In this case the curvature tensors $\underset{1}{R}{}^{*i}_{jmn},\dots,\underset{4}{R}{}^{*i}_{jmn}$ and the curvature pseudotensors $\underset{1}{A}{}^{*i}_{jmn},\dots,\underset{15}{A}{}^{*i}_{jmn}$ reduce to the curvature tensor R^{*i}_{jmn} of usual Finsler space.*

3. Derived curvature tensors of non-symmetric connection in Rund's sense

In the paper [9] we examined combined Ricci type identities and derived eight new curvature tensors $\underset{t}{\widetilde{R}}{}^{*i}_{jmn}$, $t = 1,\dots,8$, in the generalized Finsler space, it is analogous to related results in the non-symmetric affine connection space [5]. Derived curvature tensors $\underset{t}{\widetilde{R}}{}^{*i}_{jmn}$, $t = 1,\dots,8$, are given by

$$\underset{1}{\widetilde{R}}{}^{*i}_{jmn} = \frac{1}{2}(\underset{1}{A}{}^{*i}_{jmn} + \underset{3}{A}{}^{*i}_{jmn}) = \frac{1}{2}(\underset{2}{A}{}^{*i}_{jmn} + \underset{4}{A}{}^{*i}_{jmn}) = \frac{1}{2}\underset{15}{A}{}^{*i}_{j[mn]}, \tag{3.1}$$

$$\underset{2}{\widetilde{R}}{}^{*i}_{jmn} = \frac{1}{2}(\underset{7}{A}{}^{*i}_{jmn} + \underset{13}{A}{}^{*i}_{jmn}) = \frac{1}{2}(\underset{9}{A}{}^{*i}_{jmn} + \underset{11}{A}{}^{*i}_{jmn}), \tag{3.2}$$

$$\underset{3}{\widetilde{R}}{}^{*i}_{jmn} = \frac{1}{2}(\underset{8}{A}{}^{*i}_{jmn} + \underset{14}{A}{}^{*i}_{jmn}) = \frac{1}{2}(\underset{10}{A}{}^{*i}_{jmn} + \underset{12}{A}{}^{*i}_{jmn}), \tag{3.3}$$

$$\begin{aligned} \underset{4}{\widetilde{R}}{}^{*i}_{jmn} &= \frac{1}{6}(\underset{3}{R}{}^{*} + \underset{11}{A}{}^{*} + \underset{13}{A}{}^{*})^{i}_{j[mn]} - \underset{13}{A}{}^{*i}_{jnm} \\ &= \frac{1}{6}(\underset{3}{R}{}^{*} + \underset{12}{A}{}^{*} + \underset{14}{A}{}^{*})^{i}_{j[mn]} - \underset{13}{A}{}^{*i}_{jnm}, \end{aligned} \tag{3.4}$$

$$\underset{5}{\widetilde{R}}{}^{*i}_{jmn} = (\underset{1}{A}{}^{*} - \underset{7}{A}{}^{*})^{i}_{jmn} - \underset{13}{A}{}^{*i}_{jnm} = -(\underset{11}{A}{}^{*} + \underset{15}{A}{}^{*})^{i}_{jnm} - \underset{7}{A}{}^{*i}_{jmn}, \tag{3.5}$$

$$\underset{6}{\widetilde{R}}{}^{*i}_{jmn} = (\underset{2}{A}{}^{*} - \underset{8}{A}{}^{*})^{i}_{jmn} - \underset{14}{A}{}^{*i}_{jnm} = -(\underset{12}{A}{}^{*} + \underset{15}{A}{}^{*})^{i}_{jnm} - \underset{8}{A}{}^{*i}_{jmn}, \tag{3.6}$$

$$\underset{7}{\widetilde{R}}{}^{*i}_{jmn} = (\underset{3}{A}{}^{*} + \underset{7}{A}{}^{*})^{i}_{jmn} + \underset{13}{A}{}^{*i}_{jnm} = (\underset{13}{A}{}^{*} - \underset{15}{A}{}^{*})^{i}_{jnm} + \underset{9}{A}{}^{*i}_{jmn}, \tag{3.7}$$

$$\underset{8}{\widetilde{R}}{}^{*i}_{jmn} = (\underset{4}{A}{}^{*} + \underset{8}{A}{}^{*})^{i}_{jmn} + \underset{14}{A}{}^{*i}_{jnm} = (\underset{14}{A}{}^{*} - \underset{15}{A}{}^{*})^{i}_{jnm} + \underset{10}{A}{}^{*i}_{jmn}, \tag{3.8}$$

where $\underset{3}{R}^*$ is the third curvature tensor and $\underset{t}{A}^*$, $t = 1, \ldots, 15$, are the curvature pseudotensors of non-symmetric connection in Rund's sense.

As an easy corollary of Theorem 2.1 and Theorem 2.2 we obtain relations between derived curvature tensors $\underset{t}{\widetilde{R}}{}^{*i}_{jmn}$, $t = 1, \ldots, 8$, and the curvature tensor R^{*i}_{jmn} of associated symmetric connection.

Corollary 3.1. *Derived curvature tensors* $\underset{\theta}{\widetilde{R}}{}^*$ $(\theta = 1, \ldots, 8)$, *corresponding to the non-symmetric connection* P^*, *given by* (3.1)–(3.8) *and the curvature tensor* R^{*i}_{jmn}, *corresponding to the symmetric connection* $\frac{1}{2}P^{*i}_{(jk)} = \frac{1}{2}(P^{*i}_{jk} + P^{*i}_{kj})$, *satisfy the following relations*

$$\begin{aligned}
\underset{1}{\widetilde{R}}{}^{*i}_{jmn} &= R^{*i}_{jmn} - \frac{1}{4}T^{*p}_{jm}T^{*i}_{pn} + \frac{1}{4}T^{*p}_{jn}T^{*i}_{pm},\\
\underset{2}{\widetilde{R}}{}^{*i}_{jmn} &= R^{*i}_{jmn} + \frac{1}{4}T^{*p}_{jm}T^{*i}_{pn} + \frac{1}{4}T^{*p}_{jn}T^{*i}_{pm},\\
\underset{3}{\widetilde{R}}{}^{*i}_{jmn} &= R^{*i}_{jmn} - \frac{1}{4}T^{*p}_{jm}T^{*i}_{pn} - \frac{1}{4}T^{*p}_{jn}T^{*i}_{pm},\\
\underset{4}{\widetilde{R}}{}^{*i}_{jmn} &= R^{*i}_{jmn} - \frac{1}{6}\Big(\Big(T^{*i}_{jm;n} + \frac{1}{4}T^{*p}_{jm}T^{*i}_{pn}\Big)_{[mn]} + T^{*p}_{mn}T^{*i}_{pj}\Big),\\
\underset{5}{\widetilde{R}}{}^{*i}_{jmn} &= R^{*i}_{jmn} - (T^{*i}_{jm;n})_{[mn]} - \frac{3}{4}T^{*p}_{jm}T^{*i}_{pn} - \frac{1}{4}T^{*p}_{jn}T^{*i}_{pm},\\
\underset{6}{\widetilde{R}}{}^{*i}_{jmn} &= R^{*i}_{jmn} - (T^{*i}_{jm;n})_{[mn]} + \frac{1}{4}T^{*p}_{jm}T^{*i}_{pn} + \frac{3}{4}T^{*p}_{jn}T^{*i}_{pm},\\
\underset{7}{\widetilde{R}}{}^{*i}_{jmn} &= R^{*i}_{jmn} + (T^{*i}_{jm;n})_{[mn]} + \frac{1}{4}T^{*p}_{jm}T^{*i}_{pn} + \frac{3}{4}T^{*p}_{jn}T^{*i}_{pm},\\
\underset{8}{\widetilde{R}}{}^{*i}_{jmn} &= R^{*i}_{jmn} + (T^{*i}_{jm;n})_{[mn]} - \frac{3}{4}T^{*p}_{jm}T^{*i}_{pn} - \frac{1}{4}T^{*p}_{jn}T^{*i}_{pm},
\end{aligned}$$

where T^{*i}_{jk} *is the torsion tensor of connection* P^{*i}_{jk}.

Remark 3.1. *If* $T^{*i}_{jk} = 0$, *then derived curvature tensors* $\underset{t}{\widetilde{R}}{}^{*i}_{jmn}$, $t = 1, \ldots, 8$, *reduce to the curvature tensor* R^{*i}_{jmn}. *This fact evidently follows from* Corollary 3.1.

4. Concluding remarks

The presented results are generalizations of related results in the generalized (non-symmetric) Riemannian spaces.

If $g_{ij}(x, \dot{x}) = g_{ij}(x) \neq g_{ji}(x)$, in general, GF_N reduces to GR_N (generalized Riemannian space), if $g_{ij}(x, \dot{x}) = g_{ij}(x) = g_{ji}(x)$ we obtain the usual Riemannian space R_N.

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