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A RESULT ON THE CHERMAK-DELGADO MEASURE OF A FINITE GROUP

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Abstract. In this short note, we describe finite groups all of whose non-trivial cyclic subgroups have the same Chermak-Delgado measure.
Keywords: Chermak-Delgado measure, Chermak-Delgado lattice, subgroup lattice, TH-p-group.

1. Introduction

Let G be a finite group and $L(G)$ be the subgroup lattice of G. The *Chermak-Delgado measure* of a subgroup H of G is defined by

$$m_G(H) = |H||C_G(H)|. \tag{1.1}$$

Let

$$m^*(G) = \max\{m_G(H) \mid H \leq G\} \text{ and } \mathcal{CD}(G) = \{H \leq G \mid m_G(H) = m^*(G)\}. \tag{1.2}$$

Then the set $\mathcal{CD}(G)$ forms a modular, self-dual sublattice of $L(G)$, which is called the *Chermak-Delgado lattice* of G. It was first introduced by Chermak and Delgado [3], and revisited by Isaacs [5]. In the last years there has been a growing interest in understanding this lattice, especially for p-groups (see e.g. [9]). We recall several important properties of the Chermak-Delgado measure:

- if $H \leq G$ then $m_G(H) \leq m_G(C_G(H))$, and if the measures are equal then $C_G(C_G(H)) = H$;

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- if $H \in \mathcal{CD}(G)$ then $C_G(H) \in \mathcal{CD}(G)$ and $C_G(C_G(H)) = H$;
- the maximum member M of $\mathcal{CD}(G)$ is characteristic and satisfies $\mathcal{CD}(M) = \mathcal{CD}(G)$, while the minimum member $M(G)$ of $\mathcal{CD}(G)$ (called the *Chermak-Delgado subgroup* of G) is characteristic, abelian and contains $Z(G)$.

In [8], the Chermak-Delgado measure of G has been seen as a function

$$m_G : L(G) \longrightarrow \mathbb{N}^*,\ H \mapsto m_G(H),\ \forall\, H \in L(G), \tag{1.3}$$

which has at least two distinct values if G is non-trivial. We studied finite groups G such that m_G has exactly k values, with an emphasis on the case $k = 2$. Also, in [4], finite groups G with $|\mathcal{CD}(G)| = |L(G)| - k$, $k = 1, 2$, have been determined. Note that a small $|\mathrm{Im}(m_G)|$ or a large $\mathcal{CD}(G)$ mean that many subgroups of G have the same Chermak-Delgado measure. This constitutes the starting point of our discussion.

Our main result is stated as follows. By a *TH-p-group* we will understand a p-group G all of whose elements of order p are central, that is $\Omega_1(G) \le Z(G)$ (see e.g. [1, 2]).

Theorem 1.1. *Let G be a finite group and $C(G)^*$ be the set of non-trivial cyclic subgroups of G. If $m_G(H_1) = m_G(H_2)$ for all $H_1, H_2 \in C(G)^*$, then G is a TH-p-group with $\Omega_1(G) = Z(G)$. Moreover, if $|G| = p^n$, $\exp(G) = p^m$ and $|Z(G)| = p^k$, then $k \le n - 2m + 2$.*

Obviously, a finite abelian group as in Theorem 1.1 is an elementary abelian p-group. The smallest non-abelian examples are Q_8, $Q_8 \times C_2$, $C_4 \rtimes C_4$ for $p = 2$ and $C_9 \rtimes C_9$ for p odd. More generally, we observe that all groups $Q_8 \times C_2^n$ with $n \in \mathbb{N}$ and all groups $C_{p^2} \rtimes C_{p^2}$ with p prime satisfy the hypothesis of Theorem 1.1.

Two particular cases of the above theorem are as follow.

Corollary 1.1. *Let G be a finite group. If any of the following two conditions holds*

a) *$m_G(H_1) = m_G(H_2) = m^*(G)$, for all $H_1, H_2 \in C(G)^*$,*

b) *$m_G(H_1) = m_G(H_2)$, for all non-trivial abelian subgroups H_1, H_2 of G,*

then either $G \cong C_p$ for some prime p or $G \cong Q_8$.

Most of our notation is standard and will usually not be repeated here. Elementary notions and results on groups can be found in [5, 7]. For subgroup lattice concepts we refer the reader to [6].

2. Proof of the main results

2.1. Proof of Theorem 1.1.

Let $|G| = p_1^{n_1} \cdots p_r^{n_r}$ and $G_i \in \mathrm{Syl}_{p_i}(G)$, $i = 1, ..., r$. If P_i is a cyclic subgroup of order p_i which is contained in $Z(G_i)$, then $G_i \subseteq C_G(P_i)$ and so $m_G(P_i)$ is divisible by $p_i^{n_i+1}$. Since for $j \neq i$ the maximal power of p_i in $m_G(P_j)$ is $p_i^{n_i}$, we infer that $m_G(P_i) \neq m_G(P_j)$. This shows that we must have $r = 1$, i.e. G is a p-group.

Next we observe that $Z(G)$ cannot contain elements of order p^s with $s \geq 2$. Indeed, if a is such an element, then $\langle a^p \rangle \neq 1$ and

$$m_G(\langle a \rangle) > m_G(\langle a^p \rangle), \tag{2.1}$$

contradicting our hypothesis. Thus the common value of $m_G(H)$, $H \in C(G)^*$, is p^{n+1}.

Assume now that there is $H \leq G$ with $|H| = p$ and $H \nsubseteq Z(G)$. Then $C_G(H) \neq G$ and so

$$m_G(H) \leq p^n < p^{n+1}, \tag{2.2}$$

a contradiction. Consequently, G is a TH-p-group with $\Omega_1(G) = Z(G)$.

Finally, let $b \in G$ with $o(b) = p^m$. Then both $\langle b \rangle$ and $Z(G)$ are contained in $C_G(\langle b \rangle)$, implying that $\langle b \rangle Z(G) \subseteq C_G(\langle b \rangle)$. This shows that

$$p^{m+k-1} = |\langle b \rangle Z(G)| \tag{2.3}$$

divides $|C_G(\langle b \rangle)|$ and therefore p^{2m+k-1} divides $p^{n+1} = m_G(\langle a \rangle)$. It follows that $2m + k - 1 \leq n + 1$, i.e.

$$k \leq n - 2m + 2, \tag{2.4}$$

as desired. □

2.2. Proof of Corollary 1.2.

Under the hypothesis of Theorem 1.1, we observe that if $k = 1$, then $Z(G)$ is the unique subgroup of order p of G. By (4.4) of [7], II, it follows that G is either cyclic or a generalized quaternion 2-group. Clearly, if G is cyclic we must have $n = 1$, that is $G \cong C_p$, where p is a prime. If $G \cong Q_{2^n}$ for some integer $n \geq 3$, then $m = n - 1$ and therefore the inequality $k \leq n - 2m + 2$ leads to $n \leq 3$. Thus $n = 3$ and $G \cong Q_8$.

The proof is completed by the remark that each of conditions a) and b) implies $k = 1$. □

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