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CHARACTERIZATION OF ϕ-SYMMETRIC LORENTZIAN PARA-KENMOTSU MANIFOLDS

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Abstract. The purpose of the present paper is to explore the characteristics of the Lorentzian ϕ-symmetric para-Kenmotsu manifold as an Einstein manifold. In this paper, we also study the parallel 2-form on the LP-Kenmotsu manifold (LP-Kenmotsu manifold is used in lieu of Lorentzian para-Kenmotsu manifold throughout the present research article). We explain that the conformally flat LP-Kenmotsu manifold is locally ϕ-symmetric iff, it has constant scalar curvature.
Keywords: Einstein manifold, ϕ-symmetric LP-Kenmotsu manifold, scalar curvature, Ricci tensor.

1. Introduction

A number of authors have examined the concept of weak local symmetry of Riemannian manifolds with different approaches in distinct areas. Takahashi [15] initiated the concept of locally ϕ-symmetry as a weaker form of local symmetry on Sasakian manifolds. De [4, 5] initiated the concept of ϕ-recurrent Sasakian manifolds by generalizing the concept of locally ϕ-symmetry. Haseeb, Pandey and Prasad studied solitons on Sasakian manifold [8]. The concept of ϕ-symmetry in reference to the contact geometry is initiated and examined by Vanhecke, Buecken and Boeckx [3]. Alternatively, Kenmotsu manifold has been established by Kenmotsu [10]. He explained Kenmotsu manifold as a category of contact metric manifold. Kenmotsu manifold is different from Sasakian manifold. Since div$\xi = 2n$, therefore, Kenmotsu

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manifold is not compact. A Kenmotsu manifold is said to be a locally warped product $I \times_f N$ of an interval I [10], which is Kähler manifold N together with warping function $f(t) = se^t$, here s is a non-zero constant.
We have organized this paper in the following manner:
We mention preliminaries in section-2. Section 3 establishes a result on LP-Kenmotsu manifold with parallel 2-form. Section 4 gives results on ϕ-symmetric LP-Kenmotsu manifold as an Einstein manifold. Section 5 explains that the conformally flat LP-Kenmotsu manifold is ϕ-symmetric, iff it has constant scalar curvature.
In the last section of this paper, examples on the ϕ-symmmetry together with locally ϕ-symmetric LP-Kenmotsu manifold are given.

2. Preliminaries

We assume that the M^n (ϕ, ξ, η, g) be a Lorentzian metric manifold. Here, ϕ is $(1, 1)$ tensor field, ξ is characteristic vector field, η is 1-form and g is the Lorentz metric. We are well acquainted with the results mentioned below:

$$\phi\xi = 0, \qquad \eta(\phi U) = 0, \qquad \eta(\xi) = -1, \tag{2.1}$$

$$\phi^2 U = U + \eta(U)\xi \tag{2.2}$$

$$g(U, \xi) = \eta(U), \tag{2.3}$$

$$g(\phi U, \phi V) = g(U, V) + \eta(U)\eta(V), \tag{2.4}$$

$\forall$ vector fields U, V on M [6],

$$(\nabla_U \phi)V = -g(\phi U, V)\xi - \eta(V)\phi U, \tag{2.5}$$

$\forall$ vector fields U, V on M,

$$\nabla_U \xi = -U - \eta(U)\xi, \tag{2.6}$$

here, ∇ represents the Levi-Civita connection of g, then M (ϕ, ξ, η, g) is said to be a LP-Kenmotsu manifold [6, 7]. Kenmotsu [10], De and Pathak [4], Jun, De and Pathak [9], Binh, Tamassy, De and Tarafdar [1], Özgür and De [13], Özgür [11, 12] and other mathematicians have explained the Kenmotsu manifolds.
In LP-Kenmotsu manifolds, the results given below hold:

$$(\nabla_U \eta)V = -g(U, V) - \eta(U)\eta(V), \tag{2.7}$$

$$\eta(R(U, V)Z) = g(V, Z)\eta(U) - g(U, Z)\eta(V), \tag{2.8}$$

$$R(U, V)\xi = \eta(V)U - \eta(U)V, \tag{2.9}$$

$$R(\xi, U)V = g(U, V)\xi - \eta(V)U, \tag{2.10}$$

$$R(\xi, U)\xi = U + \eta(U)\xi \tag{2.11}$$

$$S(U, \xi) = (n-1)\eta(U), \tag{2.12}$$

$$(\nabla_Z R)(U, V)\xi = g(U, Z)V - g(V, Z)U + R(U, V)Z, \tag{2.13}$$

$\forall$ vector fields U, V, Z on M, where R and S denote the Riemannian curvature tensor and the Ricci tensor respectively.

Definition 2.1. An LP-Kenmotsu manifold is called locally ϕ-symmetric if it satisfies the condition,

$$\phi^2((\nabla_W R)(U, V)Z) = 0, \tag{2.14}$$

$\forall$ vector fields U, V, Z, W orthogonal to ξ.

Takahashi initiated the above concept for a Sasakian manifold [15]. We extend this concept for LP-Kenmotsu manifold in the above definition.

Definition 2.2. An LP-Kenmotsu manifold is called the ϕ-symmetric LP-Kenmotsu manifold if

$$\phi^2((\nabla_W R)(U, V)Z) = 0, \tag{2.15}$$

$\forall$ vector fields U, V, Z, W on M.

Definition 2.3. A second order tensor α is called the parallel tensor, if $\nabla\alpha = 0$, where, ∇ represents the Levi-Civita connection in the direction of metric g.

3. Parallel 2-form in the LP-Kenmotsu manifolds

Theorem 3.1. *There is no non-zero parallel 2-form on a LP-Kenmotsu manifold.*

Proof. We assume α to be a $(0, 2)$ type skew symmetric tensor. By definition, α is parallel tensor, if $\nabla\alpha = 0$. This provides the following relation,

$$\alpha(R(W, U)V, Z) + \alpha(V, R(W, U)Z) = 0, \tag{3.1}$$

$\forall$ vector fields U, V, Z, W on M.
Putting $W = V = \xi$ in the equation (3.1), we obtain,

$$\alpha(R(\xi, U)\xi, Z) + \alpha(\xi, R(\xi, U)Z) = 0.$$

Using the equations (2.10) and (2.11), we obtain,

$$\alpha(U,Z) = \eta(Z)\alpha(\xi,U) - \eta(U)\alpha(\xi,Z) - g(U,Z)\alpha(\xi,\xi). \tag{3.2}$$

Since, α is $(0,2)$ skew-symmetric tensor, which implies that $\alpha(\xi,\xi) = 0$, therefore equation (3.2) reduces to,

$$\alpha(U,Z) = \eta(Z)\alpha(\xi,U) - \eta(U)\alpha(\xi,Z). \tag{3.3}$$

Now, let A be $(1,1)$ tensor field, which is metrically equivalent to α, i.e., $\alpha(U,V) = g(AU,V)$, then the equation (3.3) becomes,

$$g(AU,Z) = \eta(Z)g(A\xi,U) - \eta(U)g(A\xi,Z),$$

which implies that,

$$AU = g(A\xi,U)\xi - \eta(U)A\xi. \tag{3.4}$$

Now, we have the relation,

$$\nabla_U(A\xi) = (\nabla_U A)\xi + A(\nabla_U \xi).$$

As, α is parallel, so A is parallel, therefore $\nabla_U A = 0$. Applying this relation together with $\nabla_U \xi = -U - \eta(U)\xi$ in the above equation, we get,

$$\nabla_U(A\xi) = A(-U - \eta(U)\xi),$$

or

$$\nabla_U(A\xi) = -AU - \eta(U)A\xi.$$

With the help of the equation (3.4), the above equation is reduced to

$$\nabla_U(A\xi) = -g(A\xi,U)\xi.$$

By calculation,

$$g(\nabla_U(A\xi), A\xi) = 0,$$

for any vector field U on M. Consequently $\|A\xi\|$ = constant on M.
From the above equation,

$$g((\nabla_U A)\xi + A(\nabla_U \xi), A\xi) = 0.$$

Because A is parallel, the first term in the above equation vanishes, and the above equation simplifies to become.

$$g(A(\nabla_U \xi), A\xi) = 0,$$

or,

$$\alpha(\nabla_U \xi, A\xi) = 0.$$

Since, $\alpha(U,V) = -\alpha(V,U)$, so the above equation becomes,

$$-\alpha(A\xi, \nabla_U \xi) = 0,$$

or,

$$-g(A^2\xi, \nabla_U \xi) = 0,$$

or,

$$-g(\nabla_U \xi, A^2\xi) = 0.$$

As, $\nabla_U \xi = -U - \eta(U)\xi$, the above equation implies,

$$-g(-U - \eta(U)\xi, A^2\xi) = 0.$$

or,

$$g(U, A^2\xi) + \eta(U)g(\xi, A^2\xi) = 0,$$

or

$$g(U, A^2\xi) = -g(\xi, A^2\xi)g(\xi, U).$$

Since, $-g(\xi, A^2\xi) = -\alpha(A\xi, \xi) = \alpha(\xi, A\xi) = g(A\xi, A\xi) = \|A\xi\|^2$, the above equation becomes,

$$g(U, A^2\xi) = \|A\xi\|^2 g(U, \xi),$$

or,

$$g(U, A^2\xi) = g(U, \|A\xi\|^2\xi),$$

or,

$$A^2\xi = \|A\xi\|^2\xi. \tag{3.5}$$

Differentiating covariantly the equation (3.5) along U, we obtain.

$$\nabla_U(A^2\xi) = (\nabla_U A^2)\xi + A^2(\nabla_U \xi) = \|A\xi\|^2 \nabla_U \xi.$$

Using $\nabla_U A = 0$ and $\nabla_U \xi = -U - \eta(U)\xi$, the above equation becomes,

$$\nabla_U(A^2\xi) = A^2(-U - \eta(U)\xi),$$

or,

$$\nabla_U(A^2\xi) = -A^2U - \eta(U)A^2\xi.$$

From equation (3.5), the above equation turns into,

$$\nabla_U(\|A\xi\|^2\xi) = -A^2U - \eta(U)\|A\xi\|^2\xi,$$

or,

$$\|A\xi\|^2\nabla_U \xi = -A^2U - \eta(U)\|A\xi\|^2\xi,$$

or,

$$-\|A\xi\|^2U - \eta(U)\|A\xi\|^2\xi = -A^2U - \eta(U)\|A\xi\|^2\xi.$$

On simplification, the above equation becomes,

$$A^2U = \|A\xi\|^2U. \tag{3.6}$$

If, $\|A\xi\| \neq 0$, then the equation (3.6) becomes,

$$(\frac{A}{\|A\xi\|})^2U = U.$$

Let F $= \dfrac{A}{\|A\xi\|}$, then we have,

$$F^2U = U. \tag{3.7}$$

Therefore on M, F defines the almost product structure. Then the fundamental 2-form is given by,

$$g(FU, V) = g(\frac{AU}{\|A\xi\|}, V) = \frac{1}{\|A\xi\|}g(AU, V).$$

Suppose $\lambda = \dfrac{1}{\|A\xi\|}$. Using the relation $\alpha(U, V) = g(AU, V)$ together with the above equation, we get

$$g(FU, V) = \lambda g(AU, V) = \lambda\alpha(U, V).$$

But the equation (3.3) shows that α is degenerate, which is a contradiction, this implies,

$$\|A\xi\| = 0$$

and

$$\alpha = 0.$$

This completes the proof of the theorem 3.1. □

4. ϕ-symmetric LP-Kenmotsu manifolds

Assuming M is a ϕ-symmetric LP-Kenmotsu manifold. With the help of equation (2.2) and (2.14), we get

$$(\nabla_W R)(U, V)Z + \eta((\nabla_W R)(U, V)Z)\xi = 0. \tag{4.1}$$

Let $\{e_i\}_{i=1}^n$ be the orthonormal basis of T_pM at any point p of M. Now, contracting the equation (4.1) along U, we obtain

$$\sum_{i=1}^{n} g((\nabla_W R)(e_i, V)Z, e_i) + \sum_{i=1}^{n} g((\nabla_W R)(e_i, V)Z, \xi)g(e_i, \xi) = 0. \tag{4.2}$$

Putting $Z = \xi$ in the above equation, we obtain,

$$(\nabla_W S)(V, \xi) + \sum_{i=1}^{n} g((\nabla_W R)(e_i, V)\xi, \xi)g(e_i, \xi) = 0. \tag{4.3}$$

Second term of the above equation,

$$g((\nabla_W R)(e_i), V)\xi, \xi) = g(\nabla_W (R(e_i, V)\xi, \xi) - g(R(\nabla_W e_i, V)\xi, \xi) - g(R(e_i, \nabla_W V)\xi, \xi) - g(R(e_i, V)\nabla_W \xi, \xi). \tag{4.4}$$

As, e_i is orthonormal basis at p, therefore, $\nabla_W e_i = 0$. On applying the relation $\nabla_W e_i = 0$ in the second term together with equation (2.9) in 3^{rd} term of the above equation, we obtain,

$$g(R(e_i, \nabla_W V)\xi, \xi) = g(\eta(\nabla_W V)e_i - \eta(e_i)\nabla_W V, \xi),$$

or,

$$g(R(e_i, \nabla_W V)\xi, \xi) = \eta(\nabla_W V)g(e_i, \xi) - \eta(e_i)g(\nabla_W V, \xi),$$

which again implies,

$$g(R(e_i, \nabla_W V)\xi, \xi) = \eta(e_i)\eta(\nabla_W V) - \eta(e_i)\eta(\nabla_W V),$$

or,

$$g(R(e_i, \nabla_W V)\xi, \xi) = 0. \tag{4.5}$$

Using the equation (4.5) into the equation (4.4), we get

$$g((\nabla_W R)(e_i), V)\xi, \xi) = g(\nabla_W (R(e_i, V)\xi, \xi) - g(R(e_i, V)\nabla_W \xi, \xi). \tag{4.6}$$

As,

$$g(R(e_i, V)\xi, \xi) = -g(R(\xi, \xi)V, e_i) = 0, \tag{4.7}$$

therefore,

$$g(R(e_i, V)\xi, \xi) = 0.$$

Differentiating covariantly the above equation with respect to W, we obtain,

$$(\nabla_W g)(R(e_i, V)\xi, \xi) + g(\nabla_W R(e_i, V)\xi, \xi) + g(R(e_i, V)\xi, \nabla_W \xi) = 0.$$

On simplification, the above equation is reduced to,

$$g(\nabla_W R(e_i, V)\xi, \xi) = -g(R(e_i, V)\xi, \nabla_W \xi). \tag{4.8}$$

Using (4.8) into (4.6), we find

$$g((\nabla_W R)(e_i, V)\xi, \xi) = -g(R(e_i, V)\xi, \nabla_W \xi) - g(R(e_i, V)\nabla_W \xi, \xi),$$

or,

$$g(\nabla_W R)(e_i, V)\xi, \xi) = -g(R(e_i, V)\xi, W + \eta(W)\xi) - g(R(e_i, V)(W + \eta(W)\xi, \xi).$$

On evaluation, the above equation becomes,

$$g((\nabla_W R)(e_i, V)\xi, \xi) = -g((R(e_i, V)\eta(W))\xi), \xi).$$

Since

$$(R(e_i, V)\eta(W)) = 0,$$

so,

$$g((\nabla_W R)(e_i, V)\xi, \xi) = 0.$$

With the aid of the above equation, the equation (4.3) turns into,

$$(\nabla_W S)(V, \xi) = 0,$$

or,

$$(\nabla_W S)(V, \xi) = \nabla_W(S(V, \xi)) - S(\nabla_W V, \xi) - S(V, \nabla_W \xi).$$

With the help of the equations (2.6) and (2.12), the above relation provides,

$$S(V, W) = (n-1)g(V, W), \tag{4.9}$$

which shows that a ϕ-symmetric LP-Kenmotsu manifold is an Einstein manifold. So, we state the following theorem:

Theorem 4.1. *: A ϕ-symmetric LP-Kenmotsu manifold is an Einstein manifold.*

5. Conformally flat locally ϕ-symmetric LP-Kenmotsu manifolds

Let (M^n, g) be an n-dimensional $(n > 3)$ connected pseudo-Riemannian manifold of class C^∞ and ∇ be the Levi-Civita connection, then the conformal curvature tensor C of (M, g) is defined by

(5.1)
$$C(U, V)Z = R(U, V)Z - \frac{1}{n-2}[S(V, Z)U - S(U, Z)V + g(V, Z)QU - g(U, Z)QV] + \frac{r}{(n-1)(n-2)}[g(V, Z)U - g(U, Z)V],$$

where, r is the scalar curvature, S is the Ricci tensor and Q is the Ricci operator s.t. $S(U, V) = g(QU, V)$ [14, 16]. We assume that the manifold is conformally flat, so, $C(U, V)Z = 0$. Hence the equation (5.1) turns into,

$$R(U, V)Z = \frac{1}{n-2}[S(V, Z)U - S(U, Z)V + g(V, Z)QU - g(U, Z)QV] - \frac{r}{(n-1)(n-2)}[g(V, Z)U - g(U, Z)V]. \tag{5.2}$$

Replacing $U = Z = \xi$ in the above equation and using (2.11) together with (2.12), we obtain

$$QU = (\frac{r}{n-1} - 1)U + (\frac{r}{n-1} - n)\eta(U)\xi. \tag{5.3}$$

According to the definition, $S(U,V) = g(QU,V)$, we get

$$S = (\frac{r}{n-1} - 1)g + (\frac{r}{n-1} - n)\eta \otimes \eta, \tag{5.4}$$

by virtue of the equations (5.3), (5.4), the equation (5.2) turns into

$$\begin{aligned} R(U,V)Z = {} & (\frac{1}{n-2})(\frac{r}{n-1} - 2)[g(V,Z)U - g(U,Z)V] \\ & + (\frac{1}{n-2})(\frac{r}{n-1} - n)[g(V,Z)\eta(U)\xi - g(U,Z)\eta(V)\xi] \\ & + (\frac{1}{n-2})(\frac{r}{n-1} - n)[\eta(V)\eta(Z)U - \eta(U)\eta(Z)V]. \end{aligned} \tag{5.5}$$

Differentiating covariantly the equation (5.5) with respect to W, we find

$$\begin{aligned} (\nabla_W R)(U,V)Z = {} & (\frac{1}{n-2})\frac{dr(W)}{(n-1)}[g(V,Z)U - g(U,Z)V] \\ & + (\frac{1}{n-2})\frac{dr(W)}{(n-1)}[g(V,Z)\eta(U)\xi - g(U,Z)\eta(V)\xi + \eta(V)\eta(Z)U - \eta(U)\eta(Z)V)] \\ & + (\frac{1}{n-2})(\frac{r}{n-1} - n)[g(V,Z)(\nabla_W\eta)(U)\xi + g(V,Z)\eta(U)\nabla_W\xi - g(U,Z)(\nabla_W\eta)(V)\xi \\ & - g(U,Z)\eta(V)\nabla_W\xi + (\nabla_W\eta)(V)\eta(Z)U + \eta(V)(\nabla_W\eta)(Z)U \\ & - (\nabla_W\eta)(U))\eta(Z)V - \eta(U)(\nabla_W\eta)(Z)V]. \end{aligned} \tag{5.6}$$

Now, operating ϕ^2 on both sides of the equation (5.6), we get

$$\begin{aligned} \phi^2((\nabla_W R)(U,V)Z) = {} & \phi^2((\frac{1}{n-2})\frac{dr(W)}{(n-1)}[g(V,Z)U - g(U,Z)V] \\ & + (\frac{1}{n-2})\frac{dr(W)}{(n-1)}[g(V,Z)\eta(U)\xi - g(U,Z)\eta(V)\xi + \eta(V)\eta(Z)U - \eta(U)\eta(Z)V)] \\ & + (\frac{1}{n-2})(\frac{r}{n-1} - n)[g(V,Z)(\nabla_W\eta)(U)\xi + g(V,Z)\eta(U)\nabla_W\xi - g(U,Z)(\nabla_W\eta)(V)\xi \\ & - g(U,Z)\eta(V)\nabla_W\xi + (\nabla_W\eta)(V)\eta(Z)U + \eta(V)(\nabla_W\eta)(Z)U \\ & - (\nabla_W\eta)(U)\eta(Z)V - \eta(U)(\nabla_W\eta)(Z)V]). \end{aligned} \tag{5.7}$$

On simplification, the above equation becomes,

(5.8)
$$\begin{aligned}\phi^2((\nabla_W R)(U,V)Z) = (\frac{1}{n-2})\frac{dr(W)}{(n-1)}[g(V,Z)U - g(U,Z)V + g(V,Z)\eta(U)\xi - \\ g(U,Z)\eta(V)\xi - \eta(U)\eta(Z)V + \eta(V)\eta(Z)U] \\ + (\frac{1}{n-2})(\frac{r}{n-1} - n)[(\nabla_W\eta)(V)\eta(Z)U + \eta(V)(\nabla_W\eta)(Z)U - \\ (\nabla_W\eta)(U)\eta(Z)V - \eta(U)(\nabla_W\eta(Z)V) + (\nabla_W\eta)(V)\eta(U)\eta(Z)\xi - \\ (\nabla_W\eta)(U)\eta(V)\eta(Z)\xi + g(U,Z)\eta(V)W - g(V,Z)\eta(U)W \\ + g(U,Z)\eta(V)\eta(W)\xi - g(V,Z)\eta(U)\eta(W)\xi].\end{aligned}$$

Let U, V, Z be orthogonal to ξ, therefore the equation (5.8) becomes,

$$\phi^2((\nabla_W R)(U,V)Z) = (\frac{1}{n-2})\frac{dr(W)}{(n-1)}[g(V,Z)U - g(U,Z)V]. \tag{5.9}$$

If M is locally ϕ-symmetric, then the equation (5.9) reduces to

$$(\frac{1}{n-2})\frac{dr(W)}{(n-1)}[g(V,Z)U - g(U,Z)V] = 0. \tag{5.10}$$

Hence, we state the following theorem:

Theorem 5.1. *A conformally flat LP-Kenmotsu manifold is locally ϕ-symmetric, iff the scalar curvature is constant.*

Let (M^n, g) be an n-dimensional $(n > 3)$ connected pseudo-Riemannian manifold of class C^∞ and ∇ be the Levi-Civita connection, then the conformal curvature tensor C of (M, g) is defined by,

(5.11)
$$\begin{aligned}C(U,V)Z = R(U,V)Z - \frac{1}{n-2}[S(V,Z)U - S(U,Z)V + g(V,Z)QU - g(U,Z)QV] \\ + \frac{r}{(n-1)(n-2)}[g(V,Z)U - g(U,Z)V],\end{aligned}$$

where, r, S and Q are scalar curvature, Ricci tensor and Ricci operator, respectively, such that $S(U,V) = g(QU,V)$.
If M is ϕ-symmetric, then from the theorem 4.1 together with the equation (4.9), S is found as,

$$S(U,V) = (n-1)g(U,V). \tag{5.12}$$

Using $S(U,V) = g(QU,V)$ in the equation(5.12) we yield,

$$QU = (n-1)U. \tag{5.13}$$

Contracting the equation (5.12),

$$r = n(n-1). \tag{5.14}$$

Using equations (5.12), (5.13) and (5.14) in the equation (5.11), we get

$$C(U,V)Z = R(U,V)Z - \frac{(n^2-3n+2)}{(n-1)(n-2)}[g(V,Z)U - g(U,Z)V],$$

or,

$$C(U,V)Z = R(U,V)Z - \{g(V,Z)U - g(U,Z)V\}. \tag{5.15}$$

We assume that M is conformally flat, i.e. $C \equiv 0$. Hence, from this result, the equation (5.15) reduces to

$$R(U,V)Z = \{g(V,Z)U - g(U,Z)V\}. \tag{5.16}$$

Thus, we state the following theorem:

Theorem 5.2. *ϕ-symmetric conformally flat LP-Kenmotsu manifold M of dimension greater than 3 is a space of constant curvature 1.*

6. example

Example 6.1. Conformally flat LP-Kenmotsu manifold M of dimension n ($n > 3$), together with scalar curvature $r = n(n-1)$, is ϕ-symmetric.

Example 6.2. We take a 3-dimensional smooth manifold $M^3 = \{(u,v,w) \in R^3 : (u,v,w) \neq (0,0,0)\}$, where (u,v,w) is the standard coordinates in 3-dimensional real space R^3. Consider the set $\{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$ of vector fields at every point of M^3, which are linearly independent, are defined as,

$$\bar{e}_1 = e^{u+w}\frac{\partial}{\partial u}, \qquad \bar{e}_2 = e^{v+w}\frac{\partial}{\partial v}, \qquad \bar{e}_3 = \frac{\partial}{\partial w}.$$

We define the Lorentz metric g on M^3 as:

$$g_{ij} = g(\bar{e}_i, \bar{e}_j) = \begin{cases} 0 & \text{if } i \neq j \\ -1 & \text{if } i = j = 3 \\ 1 & \text{i = j = 1 or 2,} \end{cases}$$

Assume η to be the 1-form corresponding to the Lorentz metric g by

$$\eta(U) = g(U, \bar{e}_3),$$

for any $U \in \Gamma(M^3)$, where $\Gamma(M^3)$ is the set of all smooth vector fields on M^3. We define the $(1,1)$-tensor field ϕ as follows:

$$\phi(\bar{e}_1) = \bar{e}_1, \qquad \phi(\bar{e}_2) = \bar{e}_2, \qquad \phi(\bar{e}_3) = 0.$$

From linearity property of ϕ and g, we simply prove the results given below:

$$\eta(\bar{e}_3) = -1, \qquad \phi^2(U) = U + \eta(U)\bar{e}_3, \qquad g(\phi U, \phi V) = g(U, V) + \eta(U)\eta(V),$$

$\forall\ U, V \in \Gamma(M^3)$. This implies that $\bar{e}_3 = \xi$, the structure (ϕ, ξ, η, g) goes to a Lorentzian paracontact structure and the manifold M^3 equipped with the Lorentzian paracontact structure is called the Lorentzian paracontact manifold of dimension 3.
We represent $[U, V]$ as the Lie-derivative of vector fields U and V, defined by $[U, V] = UV - VU$. The non-zero constituents of the Lie-bracket are calculated as:

$$[\bar{e}_1, \bar{e}_3] = -\bar{e}_1, \qquad [\bar{e}_2, \bar{e}_3] = -\bar{e}_2.$$

Let Levi-Civita connection with respect to the Lorentzian metric tensor g be denoted by ∇. Then for $\bar{e}_3 = \xi$, the Koszul's formula

$$\begin{aligned} 2g(\nabla_U V, Z) = Ug(V, Z) + Vg(Z, U) - Zg(U, V) \\ - g([V, Z], U) + g([Z, U], V) + g([U, V], Z) \end{aligned}$$

gives,

$$\begin{aligned} &\nabla_{\bar{e}_1}\bar{e}_1 = -\bar{e}_3, && \nabla_{\bar{e}_1}\bar{e}_2 = 0, && \nabla_{\bar{e}_1}\bar{e}_3 = -\bar{e}_1, \\ &\nabla_{\bar{e}_2}\bar{e}_1 = 0, && \nabla_{\bar{e}_2}\bar{e}_2 = -\bar{e}_3, && \nabla_{\bar{e}_2}\bar{e}_3 = -\bar{e}_2, \\ &\nabla_{\bar{e}_3}\bar{e}_1 = 0, && \nabla_{\bar{e}_3}\bar{e}_2 = 0, && \nabla_{\bar{e}_3}\bar{e}_3 = 0. \end{aligned}$$

Let $U \in \Gamma(M^3)$. So, $U = \sum_{i=1}^{3} U^i \bar{e}_i = U^1\bar{e}_1 + U^2\bar{e}_2 + U^3\bar{e}_3$. From the above equations, it can be verified that $\nabla_U \bar{e}_3 = -\{U + \eta(U)\bar{e}_3\}$ holds for each $U \in \Gamma(M^3)$. Hence, the Lorentzian paracontact manifold is a LP-Kenmotsu manifold of dimension 3. From the above equations, the non-zero constituents of R are evaluated underneath:

$$\begin{aligned} &R(\bar{e}_1, \bar{e}_2)\bar{e}_2 = \bar{e}_1, && R(\bar{e}_2, \bar{e}_3)\bar{e}_2 = -\bar{e}_3, \\ &R(\bar{e}_1, \bar{e}_3)\bar{e}_3 = -\bar{e}_1, && R(\bar{e}_2, \bar{e}_3)\bar{e}_3 = -\bar{e}_2, \\ &R(\bar{e}_2, \bar{e}_1)\bar{e}_1 = \bar{e}_2, && R(\bar{e}_1, \bar{e}_3)\bar{e}_1 = -\bar{e}_3. \end{aligned}$$

The above relations indicates that the M^3 under consideration is locally ϕ-symmetric. We have

$$R(U, V)Z = g(V, Z)U - g(U, Z)V,$$

so, it is the space of constant curvature 1.
The definition of the Ricci tensor S of M^3 gives,

$$S(U, V) = \varepsilon_1 g(R(\bar{e}_1, U)V, \bar{e}_1) + \varepsilon_2 g(R(\bar{e}_2, U)V, \bar{e}_2) + \varepsilon_3 g(R(\bar{e}_3, U)V, \bar{e}_3)$$

where, $\varepsilon_i = g(\bar{e}_i, \bar{e}_i)$, $i \in \{1, 2, 3\}$.
The matrix representation of S is given by

$$S = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

and the scalar curvature $r = \varepsilon_1 S(\bar{e}_1, \bar{e}_1) + \varepsilon_2 S(\bar{e}_2, \bar{e}_2) + \varepsilon_3 S(\bar{e}_3, \bar{e}_3) = 6$, where, $\varepsilon_i = g(\bar{e}_i, \bar{e}_i)$, $i \in \{1, 2, 3\}$. This shows that the manifold under consideration possesses the constant scalar curvature 6.

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