FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 31, No 2 (2016), 363–372

(η, γ)-FOURIER-BESSEL LIPSCHITZ FUNCTIONS IN THE SPACE $L^p_{\alpha,n}$

Radouan Daher, Salah El Ouadih and Mohamed El Hamma

Abstract. In this paper, we obtain an analog of Theorem 5.2 in Younis [5] for the generalized Fourier-Bessel transform on the real line for functions satisfying the (η, γ)-Fourier-Bessel Lipschitz condition in the space $L^p_{\alpha,n}$,$1 < p \leq 2$.

Keywords: Singular differential operator, Generalized Fourier-Bessel transform, Generalized translation operator

1. Introduction and Preliminaries

Various investigators such as Mittal and Mishra [7], Mishra et al. [8]-[12] and Mishra and Mishra [3] have determined the degree of approximation of 2π-periodic signals (functions) belonging to various classes $Lip\alpha$, $Lip(\alpha, r)$, $Lip(\xi(t), r)$ and $W(L_r, \xi(t))$, $(r \geq 1)$, of functions through trigonometric Fourier approximation using different summability matrices with monotone rows. In this direction, Theorem 5.2 of Younis [5] characterized the set of functions in $L^2(\mathbb{R})$ satisfying the Dini Lipschitz condition by means of an asymptotic estimate growth of the norm of their Fourier transforms, namely, we have

Theorem 1.1. *([5]) Let $f \in L^2(\mathbb{R})$. Then, the following statements are equivalent*

(a) $\|f(x+h) - f(x)\| = O\left(\frac{h^\eta}{(\log \frac{1}{h})^\gamma}\right)$, *as* $h \to 0, 0 < \eta < 1, \gamma \geq 0,$

(b) $\displaystyle\int_{|\lambda| \geq r} |\widehat{f}(\lambda)|^2 d\lambda = O\left(\frac{r^{-2\eta}}{(\log r)^{2\gamma}}\right)$, *as* $r \to \infty,$

where $\widehat{f}$ stands for the Fourier transform of f.

In this paper, we consider a second-order singular differential operator $\mathcal{B}$ on the half-line which generalizes the Bessel operator $\mathcal{B}_\alpha$, we obtain an analog of Theorem 1.1 for the generalized Fourier-Bessel transform associated to $\mathcal{B}$ in $L^p_{\alpha,n}$,$1 < p \leq 2$. For this purpose, we use a generalized translation operator. Some interesting applications

Received July 15, 2015; accepted November 20, 2015
2010 *Mathematics Subject Classification.* Primary 42B37; Secondary 47B48

of manuscript can be seen in ([3],[9]-[14]).
We briefly overview the theory of generalized Fourier-Bessel transform and related harmonic analysis (see [1],[6]):
Consider the second-order singular differential operator on the half line defined by

$$\mathcal{B}f(x) = \frac{d^2 f(x)}{dx^2} + \frac{(2\alpha+1)}{x}\frac{df(x)}{dx} - \frac{4n(\alpha+n)}{x^2}f(x),$$

where $\alpha > \frac{-1}{2}$ and $n = 0, 1, 2, ...$ For $n = 0$, we obtain the classical Bessel operator

$$\mathcal{B}_\alpha f(x) = \frac{d^2 f(x)}{dx^2} + \frac{(2\alpha+1)}{x}\frac{df(x)}{dx}.$$

Let M be the map defined by

$$Mf(x) = x^{2n}f(x), \quad n = 0, 1, ..$$

Let $L^p_{\alpha,n}$, $1 < p \leq 2$, be the class of measurable functions f on $[0,\infty[$ for which

$$\|f\|_{p,\alpha,n} = \|M^{-1}f\|_{p,\alpha+2n} < \infty,$$

where

$$\|f\|_{p,\alpha} = \left(\int_0^\infty |f(x)|^p x^{2\alpha+1}dx\right)^{1/p}.$$

If $p = 2$, then we have $L^2_{\alpha,n} = L^2([0,\infty[, x^{2\alpha+1}dx)$.
For $\alpha \geq \frac{-1}{2}$, we introduce the Bessel normalized function of the first kind j_α defined by

$$j_\alpha(z) = \Gamma(\alpha+1)\sum_{n=0}^{\infty}\frac{(-1)^n(\frac{z}{2})^{2n}}{n!\Gamma(n+\alpha+1)}, \quad z \in \mathbb{C}, \tag{1.1}$$

where $\Gamma(x)$ is the gamma-function (see [4]). The function $y = j_\alpha(z)$ satisfies the differential equation

$$\mathcal{B}_\alpha y + y = 0,$$

with the initial conditions $y(0) = 0$ and $y'(0) = 0$. The function $j_\alpha(z)$ is infinitely differentiable, even, and, moreover entire analytic.
From (1.1) we see that

$$\lim_{z\to 0}\frac{j_\alpha(z)-1}{z^2} \neq 0, \tag{1.2}$$

hence, there exists $c > 0$ and $\nu > 0$ satisfying

$$|z| \leq \nu \Rightarrow |j_\alpha(z) - 1| \geq c|z|^2. \tag{1.3}$$

For $\lambda \in \mathbb{C}$, and $x \in \mathbb{R}$, put

$$\varphi_\lambda(x) = x^{2n}j_{\alpha+2n}(\lambda x). \tag{1.4}$$

From [1], [6] recall the following properties.

Proposition 1.1. *(c)* φ_λ *satisfies the differential equation*

$$\mathcal{B}\varphi_\lambda = -\lambda^2 \varphi_\lambda.$$

(d) For all $\lambda \in \mathbb{C}$, *and* $x \in \mathbb{R}$

$$|\varphi_\lambda(x)| \leq x^{2n} e^{|Im\lambda||x|}.$$

The generalized Fourier-Bessel transform that we call it the integral transform defined by

$$\mathcal{F}_{\mathcal{B}} f(\lambda) = \int_0^\infty f(x)\varphi_\lambda(x) x^{2\alpha+1} dx, \lambda \geq 0, f \in L^1_{\alpha,n}.$$

From [1], we have

$$\mathcal{F}_{\mathcal{B}} = \mathcal{F}_{\alpha+2n} o M^{-1}, \tag{1.5}$$

where $\mathcal{F}_\alpha$ is the Bessel transform defined by formula (see [4], [2])

$$\mathcal{F}_\alpha(f)(\lambda) = \int_0^\infty f(x) j_\alpha(\lambda x) x^{2\alpha+1} dx.$$

Let $f \in L^1_{\alpha,n}$ such that $\mathcal{F}_{\mathcal{B}}(f) \in L^1_{\alpha+2n} = L^1([0,\infty[, x^{2\alpha+4n+1}dx)$. Then the inverse generalized Fourier-Bessel transform is given by the formula (see [1])

$$f(x) = \int_0^\infty \mathcal{F}_{\mathcal{B}} f(\lambda)\varphi_\lambda(x) d\mu_{\alpha+2n}(\lambda),$$

where

$$d\mu_{\alpha+2n}(\lambda) = a_{\alpha+2n}\lambda^{2\alpha+4n+1}d\lambda, \quad a_\alpha = \frac{1}{4^\alpha(\Gamma(\alpha+1))^2}.$$

From [1], [6] we have

Proposition 1.2. *(e) For every* $f \in L^1_{\alpha,n} \cap L^2_{\alpha,n}$ *we have the Plancherel formula*

$$\int_0^{+\infty} |f(x)|^2 x^{2\alpha+1} dx = \int_0^{+\infty} |\mathcal{F}_{\mathcal{B}} f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda).$$

(f) The generalized Fourier-Bessel transform $\mathcal{F}_{\mathcal{B}}$ *extends uniquely to an isometric isomorphism from* $L^2_{\alpha,n}$ *onto* $L^2([0,+\infty[, \mu_{\alpha+2n})$.

We have the Young inequality

$$\|\mathcal{F}_\alpha(f)\|_{q,\alpha} \leq K\|f\|_{p,\alpha} \tag{1.6}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and K is positive constant.
From (1.5) and (1.6) we have

$$\|\mathcal{F}_{\mathcal{B}}(f)\|_{q,\alpha+2n} \leq K\|f\|_{p,\alpha,n} \tag{1.7}$$

Define the generalized translation operator T^h, $h \geq 0$ by the relation

$$T^h f(x) = (xh)^{2n} \tau^h_{\alpha+2n}(M^{-1}f)(x), x \geq 0,$$

where $\tau^h_{\alpha+2n}$ is the Bessel translation operators of order $\alpha + 2n$ defined by

$$\tau^h_\alpha f(x) = c_\alpha \int_0^\pi f(\sqrt{x^2 + h^2 - 2xh\cos t}) \sin^{2\alpha} t dt,$$

where

$$c_\alpha = \left(\int_0^\pi \sin^{2\alpha} t dt\right)^{-1} = \frac{\Gamma(\alpha+1)}{\Gamma(\pi)\Gamma(\alpha+\frac{1}{2})}.$$

For $f \in L^p_{\alpha,n}$, we have

$$\mathcal{F}_\mathcal{B}(T^h f)(\lambda) = \varphi_\lambda(h)\mathcal{F}_\mathcal{B}(f)(\lambda), \tag{1.8}$$

$$\mathcal{F}_\mathcal{B}(\mathcal{B} f)(\lambda) = -\lambda^2 \mathcal{F}_\mathcal{B}(f)(\lambda), \tag{1.9}$$

(see [1],[6] for details).
Let $f \in L^p_{\alpha,n}$. We define the differences of the orders $k(k = 1, 2, ..)$ with a step $h > 0$ by

$$\Delta^k_h f(x) = (T^h - h^{2n} I)^k f(x), \tag{1.10}$$

where I is the unit operator in $L^p_{\alpha,n}$.
Let $W^k_{p,\alpha,n}$, $1 < p \leq 2$, be the Sobolev space constructed by the singular differential operator $\mathcal{B}$, i.e.,

$$W^k_{p,\alpha,n} = \left\{ f \in L^p_{\alpha,n}, \mathcal{B}^m f \in L^p_{\alpha,n}, m = 1, 2, ..., k \right\}.$$

2. Fourier-Bessel Dini Lipschitz Condition

Definition 2.1. Let $f \in W^k_{p,\alpha,n}$, and define

$$\|\Delta^k_h \mathcal{B}^m f(x)\|_{p,\alpha,n} \leq C \frac{h^{\eta+2nk}}{(\log \frac{1}{h})^\gamma}, \quad \gamma \geq 0$$

i.e.,

$$\|\Delta^k_h \mathcal{B}^m f(x)\|_{p,\alpha,n} = O\left(\frac{h^{\eta+2nk}}{(\log \frac{1}{h})^\gamma}\right),$$

for all x in $\mathbb{R}^+$ and for all sufficiently small h, C being a positive constant. Then we say that f satisfies a Fourier-Bessel Dini Lipschitz of order η, or f belongs to $Lip(\eta, \gamma, p)$.

Definition 2.2. If however

$$\frac{\|\Delta_h^k \mathcal{B}^m f(x)\|_{p,\alpha,n}}{\frac{h^{\eta+2nk}}{(\log\frac{1}{h})^{\gamma}}} \to 0, \quad as \quad h\to 0, \gamma \geq 0$$

i.e.,

$$\|\Delta_h^k \mathcal{B}^m f(x)\|_{p,\alpha,n} = O\left(\frac{h^{\eta+2nk}}{(\log\frac{1}{h})^{\gamma}}\right),$$

then f is said to be belong to the little Fourier-Bessel Dini Lipschitz class $lip(\eta,\gamma,p)$.

Remark. It follows immediately from these definitions that

$$lip(\eta,\gamma,p) \subset Lip(\eta,\gamma,p).$$

Theorem 2.1. *Let $\eta > 1$. If $f \in Lip(\eta,\gamma,p)$, then $f \in lip(1,\gamma,p)$.*

Proof. For $x \in \mathbb{R}^+$ and h small, $f \in Lip(\eta,\gamma,p)$ we have

$$\|\Delta_h^k \mathcal{B}^m f(x)\|_{p,\alpha,n} \leq C\frac{h^{\eta+2nk}}{(\log\frac{1}{h})^{\gamma}}.$$

Then

$$(\log\frac{1}{h})^{\gamma}\|\Delta_h^k \mathcal{B}^m f(x)\|_{p,\alpha,n} \leq Ch^{\eta+2nk}.$$

Therefore

$$\frac{(\log\frac{1}{h})^{\gamma}}{h^{1+2nk}}\|\Delta_h^k \mathcal{B}^m f(x)\|_{p,\alpha,n} \leq Ch^{\eta-1},$$

which tends to zero with $h\to 0$. Thus

$$\frac{(\log\frac{1}{h})^{\gamma}}{h^{1+2nk}}\|\Delta_h^k \mathcal{B}^m f(x)\|_{p,\alpha,n} \to 0, \quad h\to 0.$$

Then $f \in lip(1,\gamma,p)$. □

Theorem 2.2. *If $\eta < \nu$, then $Lip(\eta,0,p) \supset Lip(\nu,0,p)$ and $lip(\eta,0,p) \supset lip(\nu,0,p)$.*

Proof. We have $0 \leq h \leq 1$ and $\eta < \nu$, then $h^{\nu} \leq h^{\eta}$.
Then the proof of the theorem is immediate. □

3. New Results on Fourier-Bessel Dini Lipschitz Class

Lemma 3.1. *For $f \in W^k_{p,\alpha,n}$, we have*

$$\left(h^{2qnk}\int_0^{\infty}\lambda^{2qm}|j_{\alpha+2n}(\lambda h)-1|^{qk}|\mathcal{F}_{\mathcal{B}}f(\lambda)|^q d\mu_{\alpha+2n}(\lambda)\right)^{\frac{1}{q}} \leq K\|\Delta_h^k \mathcal{B}^m f(x)\|_{p,\alpha,n},$$

where $\frac{1}{p}+\frac{1}{q}=1$ and $m=0,1,2...,k$.

Proof. From formula (1.8), we obtain

$$\mathcal{F}_{\mathcal{B}}(\mathcal{B}^m f)(\lambda) = (-1)^m \lambda^{2m} \mathcal{F}_{\mathcal{B}} f(\lambda); m = 0, 1, ... \tag{3.1}$$

By using the formulas (1.4), (1.8) and (3.1), we conclude that

$$\mathcal{F}_{\mathcal{B}}(T^h \mathcal{B}^m f)(\lambda) = (-1)^m h^{2n} j_{\alpha+2n}(\lambda h) \lambda^{2m} \mathcal{F}_{\mathcal{B}} f(\lambda). \tag{3.2}$$

From the definition of finite difference (1.10) and formula (3.2), the image $\Delta_h^k \mathcal{B}^r f(x)$ under the generalized Fourier-Bessel transform has the form

$$\mathcal{F}_{\mathcal{B}}(\Delta_h^k \mathcal{B}^m f)(\lambda) = (-1)^m h^{2nk} (j_{\alpha+2n}(\lambda h) - 1)^k \lambda^{2m} \mathcal{F}_{\mathcal{B}} f(\lambda).$$

Now by formula (1.7), we have the result. □

Theorem 3.1. *Let $\eta > 2k$. If f belongs to the Fourier-Bessel Dini Lipschitz class, i.e.,*

$$f \in Lip(\eta, \gamma, p), \quad \eta > 2k, \gamma \geq 0.$$

Then f is equal to the null function in $\mathbb{R}^+$.

Proof. Assume that $f \in Lip(\eta, \gamma, p)$. Then we have

$$\|\Delta_h^k \mathcal{B}^m f(x)\|_{p,\alpha,n} \leq C \frac{h^{\eta+2nk}}{(\log \frac{1}{h})^{\gamma}}, \quad \gamma \geq 0.$$

From Lemma 3.1, we get

$$h^{2qnk} \int_0^\infty \lambda^{2qm} |j_{\alpha+2n}(\lambda h) - 1|^{qk} |\mathcal{F}_{\mathcal{B}} f(\lambda)|^q d\mu_{\alpha+2n}(\lambda) \leq K^q C^q \frac{h^{q\eta+2qnk}}{(\log \frac{1}{h})^{q\gamma}}.$$

Therefore

$$\int_0^\infty \lambda^{2qm} |j_{\alpha+2n}(\lambda h) - 1|^{qk} |\mathcal{F}_{\mathcal{B}} f(\lambda)|^q d\mu_{\alpha+2n}(\lambda) \leq K^q C^q \frac{h^{q\eta}}{(\log \frac{1}{h})^{q\gamma}}.$$

Then

$$\frac{\int_0^\infty \lambda^{2qm} |j_{\alpha+2n}(\lambda h) - 1|^{qk} |\mathcal{F}_{\mathcal{B}} f(\lambda)|^q d\mu_{\alpha+2n}(\lambda)}{h^{2qk}} \leq K^q C^q \frac{h^{q\eta-2qk}}{(\log \frac{1}{h})^{q\gamma}},$$

Since $\eta > 2k$ we have

$$\lim_{h \to 0} \frac{h^{q\eta-2qk}}{(\log \frac{1}{h})^{q\gamma}} = 0.$$

Thus

$$\lim_{h \to 0} \int_0^\infty \left(\frac{|1 - j_\alpha(\lambda h)|}{\lambda^2 h^2} \right)^{qk} \lambda^{2qk+2qm} |\mathcal{F}_{\mathcal{B}} f(\lambda)|^q d\mu_{\alpha+2n}(\lambda) = 0.$$

and also from the formula (1.2) and Fatou theorem, we obtain

$$\int_0^\infty \lambda^{2qk+2qm} |\mathcal{F}_{\mathcal{B}} f(\lambda)|^q d\mu_{\alpha+2n}(\lambda) = 0.$$

Hence $\lambda^{2k+2m} \mathcal{F}_{\mathcal{B}} f(\lambda) = 0$ for all $\lambda \in \mathbb{R}^+$, and so $f(x)$ is the null function. □

Analog of the Theorem 3.1, we obtain this theorem.

Theorem 3.2. *Let $f \in W^k_{p,\alpha,n}$. If f belong to $lip(2,0,p)$, i.e.,*

$$\|\Delta^k_h \mathcal{B}^m f(x)\|_{p,\alpha,n} = O(h^2), \quad as \quad h \to 0,$$

Then f is equal to null function in $\mathbb{R}^+$.

Now, we give another main result of this paper analog of Theorem 1.1.

Theorem 3.3. *Let f belong to $Lip(\eta,\gamma,p)$. Then*

$$\int_r^\infty \lambda^{2qm}|\mathcal{F}_\mathcal{B} f(\lambda)|^q d\mu_{\alpha+2n}(\lambda) = O\left(\frac{r^{-q\eta}}{(\log r)^{q\gamma}}\right), \quad as \quad r \to \infty$$

where $\frac{1}{p}+\frac{1}{q}=1$ and $m = 0,1,2,...,k$.

Proof. Let $f \in Lip(\eta,\gamma,p)$. Then

$$\|\Delta^k_h \mathcal{B}^m f(x)\|_{p,\alpha,n} = O\left(\frac{h^{\eta+2nk}}{(\log \frac{1}{h})^\gamma}\right), \quad as \quad h \to 0$$

From Lemma 3.1, we have

$$\int_0^\infty \lambda^{2qm}|j_{\alpha+2n}(\lambda h)-1|^{qk}|\mathcal{F}_\mathcal{B} f(\lambda)|^q d\mu_{\alpha+2n}(\lambda) \leq \frac{K^q}{h^{2qnk}}\|\Delta^k_h \mathcal{B}^m f(x)\|^q_{p,\alpha,n}.$$

By formula (1.3), we get

$$\int_{\frac{\nu}{2h}}^{\frac{\nu}{h}} \lambda^{2qm}|j_{\alpha+2n}(\lambda h)-1|^{qk}|\mathcal{F}_\mathcal{B} f(\lambda)|^q d\mu_{\alpha+2n}(\lambda) \geq \frac{c^{qk}\nu^{2qk}}{2^{2qk}}\int_{\frac{\nu}{2h}}^{\frac{\nu}{h}} \lambda^{2qm}|\mathcal{F}_\mathcal{B} f(\lambda)|^q d\mu_{\alpha+2n}(\lambda).$$

Note that there exists a positive constant C such that

$$\begin{aligned}
\int_{\frac{\nu}{2h}}^{\frac{\nu}{h}} \lambda^{2qm}|\mathcal{F}_\mathcal{B} f(\lambda)|^q d\mu_{\alpha+2n}(\lambda) &\leq C\int_{\frac{\nu}{2h}}^{\frac{\nu}{h}} \lambda^{2qm}|j_{\alpha+2n}(\lambda h)-1|^{qk}|\mathcal{F}_\mathcal{B} f(\lambda)|^q d\mu_{\alpha+2n}(\lambda) \\
&\leq \frac{CK^q}{h^{2qnk}}\|\Delta^k_h \mathcal{B}^m f\|^q_{p,\alpha,n} \\
&= O\left(\frac{h^{q\eta}}{(\log\frac{1}{h})^{q\gamma}}\right).
\end{aligned}$$

So we obtain

$$\int_r^{2r} \lambda^{2qm}|\mathcal{F}_\mathcal{B} f(\lambda)|^q d\mu_{\alpha+2n}(\lambda) \leq C'\frac{r^{-q\eta}}{(\log r)^{q\gamma}}$$

where C' is a positive constant. Now, we have

$$\begin{aligned}
\int_r^{\infty} \lambda^{2qm}|\mathcal{F}_{\mathcal{B}}f(\lambda)|^q d\mu_{\alpha+2n}(\lambda) &= \sum_{i=0}^{\infty}\int_{2^i r}^{2^{i+1}r} \lambda^{2qm}|\mathcal{F}_{\mathcal{B}}f(\lambda)|^q d\mu_{\alpha+2n}(\lambda) \\
&\leq C'\left(\frac{r^{-q\eta}}{(\log r)^{q\gamma}} + \frac{(2r)^{-q\eta}}{(\log 2r)^{q\gamma}} + \frac{(4r)^{-q\eta}}{(\log 4r)^{q\gamma}} + \cdots\right) \\
&\leq C'\left(\frac{r^{-q\eta}}{(\log r)^{q\gamma}} + \frac{(2r)^{-q\eta}}{(\log r)^{q\gamma}} + \frac{(4r)^{-q\eta}}{(\log r)^{q\gamma}} + \cdots\right) \\
&\leq C'\frac{r^{-q\eta}}{(\log r)^{q\gamma}}\left(1 + 2^{-q\eta} + (2^{-q\eta})^2 + (2^{-q\eta})^3 + \cdots\right) \\
&\leq K_\eta \frac{r^{-q\eta}}{(\log r)^{q\gamma}},
\end{aligned}$$

where $K_\eta = C'(1-2^{-q\eta})^{-1}$ since $2^{-q\eta} < 1$.
Consequently

$$\int_r^{\infty} \lambda^{2qm}|\mathcal{F}_{\mathcal{B}}f(\lambda)|^q d\mu_{\alpha+2n}(\lambda) = O\left(\frac{r^{-q\eta}}{(\log r)^{q\gamma}}\right), \quad as \quad r\to\infty.$$

□

Corollary 3.1. *Let $f \in W^k_{p,\alpha,n}$. If*

$$\|\Delta_h^k \mathcal{B}^m f(x)\|_{p,\alpha,n} = O\left(\frac{h^{\eta+2nk}}{(\log \frac{1}{h})^{\gamma}}\right), \quad as \quad h\to 0,$$

then

$$\int_r^{\infty} |\mathcal{F}_{\mathcal{B}}f(\lambda)|^q d\mu_{\alpha+2n}(\lambda) = O\left(\frac{r^{-2qm-q\eta}}{(\log r)^{q\gamma}}\right), \quad as \quad r\to\infty,$$

where $\frac{1}{p}+\frac{1}{q}=1$ and $m=0,1,2...,k$.

Acknowledgements
The authors would like to thank the referee for his valuable comments and suggestions.

REFERENCES

1. R. F. Al Subaie and M. A. Mourou: *The continuous wavelet transform for a Bessel type operator on the half line*, Mathematics and Statistics 1(4): 196-203, 2013.
2. S. M. Nikol'skii: *Aproximation of functions of several variables and embedding theorems*, (Nauka,Moscow.1977)[in Russian].
3. V. N. Mishra and L. N. Mishra: *Trigonometric approximation of signals (functions) in L_p, $(p \geq 1)$-norm*, International Journal of Contemporary Mathematical Sciences, Vol. 7, no. 19, 2012, pp. 909-918.

4. B. M. Levitan: *Expansion in Fourier series and integrals over Bessel functions*, Uspekhi Math.Nauk, 6,No.2,102-143,1951.
5. M. S. Younis: *Fourier transforms of Dini-Lipschitz functions*, Int. J. Math. Math. Sci. 1986, 9 (2), 301312. doi:10.1155/S0161171286000376
6. R. F. Al Subaie and M. A. Mourou: *Transmutation operators associated with a Bessel type operator on the half line and certain of their applications*, Tamsii. oxf. J. Inf. Math. Scien 29 (3) (2013), pp. 329-349.
7. M. L. Mittal and V. N. Mishra: *Approximation of signals (functions) belonging to the Weighted $W(L_p, \xi(t))$, $(p \geq 1)$-Class by almost matrix summability method of its Fourier series*, Int. J. of Math. Sci. and Engg. Appls. Vol. 2 (2008), No. IV, 1- 9.
8. V. N. Mishra, K. Khatri and L. N. Mishra: *Product $(N, p_n)(E, q)$ summability of a sequence of Fourier coefficients*, Mathematical Sciences (Springer open access) 6:38 (2012), DOI: 10.1186/2251 7456-6-38.
9. V. N. Mishra, K. Khatri and L. N. Mishra: *Using linear operators to approximate signals of $Lip(\alpha, p)$, $(p \geq 1)$-class*, Filomat 27:2 (2013), 353-363.
10. V. N. Mishra, K. Khatri and L. N. Mishra: *Product summability transform of conjugate series of Fourier series*, International Journal of Mathematics and Mathematical Sciences, Vol. 2012(2012), Article ID 298923, 13 pages, DOI: 10.1155/2012/298923.
11. V. N. Mishra, K. Khatri and L. N. Mishra: *Product $(N, p_n)(C, 1)$ summability of a sequence of Fourier coefficients*, Mathematical Sciences, 2012, 6:38.
12. L. N. Mishra, V. N. Mishra, K. Khatri and Deepmala: *On the trigonometric approximation of signals belonging to generalized weighted Lipschitz $W(L^r, \xi(t))$, $(r \geq 1)-$ class by matrix (C^1, N_p) operator of conjugate series of its Fourier series*, Applied Mathematics and Computation, Vol. 237 (2014) 252-263.
13. V. N. Mishra, K. Khatri, L. N. Mishra and Deepmala: *Trigonometric approximation of periodic signals belonging to generalized weighted Lipschitz $W'(L^r, \xi(t))$, $(r \geq 1)-$ class by Nörlund-Euler $(N, p_n)(E, q)$ operator of conjugate series of its Fourier series*, Journal of Classical Analysis, Volume 5, Number 2(2014), 91-105. doi:10.7153/jca-05-08.
14. Deepmala, L. N. Mishra and V. N. Mishra: *Trigonometric approximation of signals (functions) belonging to the $W(L^r, \xi(t))$, $(r \geq 1)-$ class by $(E, q)(q > 0)$- means of the conjugate series of its Fourier series*, GJMS Special Issue for Recent Advances in Mathematical Sciences and Applications-13, Global Journal of Mathematical Sciences, Vol 2. No. 2, pp. 61-69, (2014).

Radouan Daher

Department of Mathematics and Informatics

Faculty of Science Aïn Chock

University Hassan II, Casablanca, Morocco

rjdaher024@gmail.com

Salah El Ouadih
Department of Mathematics and Informatics
Faculty of Science Aïn Chock
University Hassan II, Casablanca, Morocco
salahwadih@gmail.com

Mohamed EL Hamma
Department of Mathematics and Informatics
Faculty of Science Aïn Chock
University Hassan II, Casablanca, Morocco
m-elhamma@yahoo.fr