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VARIATIONS OF SEPARABILITY AND SUPERTIGHTNESS OF HYPERSPACES

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Abstract. For a Hausdorff non-compact space X, relationships between closure-type properties of the hyperspace (Λ, τ_Δ^+) and covering properties of that of X have been studied. We then investigate selective separability and some variations of this property. Finally supertightness of (Λ, τ_Δ^+) has been studied.
Keywords: Hausdorff space, compactness, separability, supertightness.

1. Introduction

In this paper we consider some stronger versions of separability in hyperspaces. In [27], Marion Scheepers introduced a general notation for selection principles as follows:

Let $\mathcal{A}$ and $\mathcal{B}$ be families of sets of an infinite set X. Then,
• $S_1(\mathcal{A}, \mathcal{B})$ is the selection hypothesis: for each sequence $\{A_n : n \in \mathbb{N}\}$ of elements of $\mathcal{A}$ there is a sequence $\{b_n : n \in \mathbb{N}\}$ such that for each $n, b_n \in A_n$, and $\{b_n : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$.
• $S_{fin}(\mathcal{A}, \mathcal{B})$ is the selection hypothesis: for each sequence $\{A_n : n \in \mathbb{N}\}$ of elements of $\mathcal{A}$ there is a sequence $\{B_n : n \in \mathbb{N}\}$ of finite sets such that for each $n, B_n \subseteq A_n$, and $\bigcup_{n\in\mathbb{N}} B_n \in \mathcal{B}$.

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If $\mathcal{A}$ and $\mathcal{B}$ stand for the family of all dense subsets of X (where we denote the set of all dense subsets of X by $\mathcal{D}$), then $S_{fin}(\mathcal{D},\mathcal{D})$ is called the selective separability of X. I. Juhász and S. Shelah in their paper [13] proved that a compact space X has countable π-weight whenever every dense subspace of X is separable. Selective separability of X follows from countable π-weight of X and implies that all dense subspaces of X are separable. Therefore, the above-mentioned theorem of Juhász and Shelah implies that, in compact spaces, selective separability coincides with countable π-weight.

In [3], spaces X satisfying $S_{fin}(\mathcal{D},\mathcal{D})$ or $S_1(\mathcal{D},\mathcal{D})$ are called M-separable and R-separable, respectively. Also, X is said to be H-separable if for every sequence $\{D_n : n \in \mathbb{N}\}$ of elements of $\mathcal{D}$, one can pick finite $F_n \subset D_n$ so that for every nonempty open subset O of X, the intersection $O \cap F_n$ is nonempty for all but finitely many n. Naturally, M-, R-, and H-, are motivated by analogy with well-known Menger, Rothberger, and Hurewicz properties. Recall that X is Menger if for every sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of open covers of X, there exist finite $\mathcal{V}_n \subset \mathcal{U}_n$, $n \in \mathbb{N}$, so that $\bigcup\{\mathcal{V}_n : n \in \mathbb{N}\}$ covers X; X is Rothberger if for every sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of open covers of X, there exist $U_n \in \mathcal{U}_n$, $n \in \mathbb{N}$, so that $\{U_n : n \in \mathbb{N}\}$ covers X; X is Hurewicz if for every sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of open covers of X, there exist finite $\mathcal{V}_n \subset \mathcal{U}_n$, $n \in \mathbb{N}$, so that for every $x \in X$, $x \in \bigcup \mathcal{V}_n$, for all but finitely many n. Also a family $\mathcal{P}$ of open sets in X is called a π-base for X if every nonempty open set in X contains a nonempty element of $\mathcal{P}$; where $\pi w(X) = \min\{|\mathcal{P}| : \mathcal{P}$ is a π-base for $X\}$ is the π-weight of X. The following implications are obvious:

$$\begin{array}{ccccc} & & \text{R-separable} & & \\ & \swarrow & & \nwarrow & \\ \text{Separable} \leftarrow \text{M-separable} & & & & \text{Countable } \pi\text{-weight} \\ & \nwarrow & & \swarrow & \\ & & \text{H-separable} & & \end{array}$$

Let us now recall some backgrounds of hyperspace topology. Given a Hausdorff non-compact space X, we denote the family of nonempty closed subsets (resp., closed subsets, compact subsets) of a topological space X by $CL(X)$ (resp., 2^X, $\mathbb{K}(X)$). For a subset $U \subset X$ and a family $\mathcal{U}$ of subsets of X, we write:

$$U^- = \{A \in CL(X) : A \cap U \neq \phi\},$$

$$U^+ = \{A \in CL(X) : A \subset U\},$$

$$U^c = X \setminus U,$$

$$\mathcal{U}^c = \{U^c : U \in \mathcal{U}\}.$$

The most known and popular among the topologies on 2^X are Fell topology and Vietoris topology. J. M. G. Fell [11] introduced a topology τ_F on 2^X having a subbase consisting of all sets of the form V^-, where V is an open subset of X plus all sets of the form $(K^c)^+$, where K is a compact subset of X. The Fell topology τ_F has a basic open subset of the form $(\bigcap_{i=1}^{n} V_i^-) \cap (K^c)^+$, where $V_1, V_2, ..., V_n$ are open subsets of X and K is a compact subset of X.

If compact subsets in the definition above are replaced by closed sets, we obtain the stronger Vietoris topology τ_V [21]. A basic open subset of the Vietoris topology τ_V is of the form: $< U_1, U_2, ..., U_n >= \{A \in 2^X : A \subset \bigcup_{i=1}^{n} U_i, A \cap U_i \neq \phi$, for $1 \leq i \leq n\}$, where $U_1, U_2,, U_n$ are open subsets of X, for $n \in \mathbb{N}$.

Let Δ be a subset of 2^X closed for finite unions and containing all singletons. The upper Δ-topology, denoted by Δ^+, is the topology whose subbase is the collection $\{(D^c)^+ : D \in \Delta\} \cup \{2^X\}$. If Δ is the family of all finite subsets of X (resp., the collection of compact subsets of X), the corresponding Δ^+-topology known as co-finite topology (resp., co-compact topology) will be denoted by $\mathbf{Z}^+$ (resp., $\mathbf{F}^+$).

We have the inclusions: $\mathbf{Z}^+ \subseteq \mathbf{F}^+ \subseteq \tau_F \subseteq \tau_V$.

Let $\Delta \subseteq CL(X)$ be a subfamily of $CL(X)$ closed under finite unions and containing all singletons. Then, the hit-and-miss topology on $CL(X)$ with respect to Δ (first studied in the abstract in [23] and then in [7]), denoted by τ_Δ^+, has as a base, the family

$$\{(\bigcap_{i=1}^{m} V_i^-) \cap (B^c)^+ : B \in \Delta \text{ and } V_i \in \tau \text{ for } i \in \{1, 2, ..., m\},\ m \in \mathbb{N}\}.$$

Following [32], the basic element $(\bigcap_{i=1}^{m} V_i^-) \cap (B^c)^+$ will be denoted by $(V_1, ..., V_m)_B^+$.

Two important cases of the hit-and-miss topology are the Vietoris topology, τ_V, when $\Delta = CL(X)$ ([31], [21]) and the Fell topology, τ_F, when $\Delta = \mathbb{K}(X)$ ([11]).

By a cover, we mean a nontrivial one, that is, $\mathcal{U}$ is a cover of X if $X = \cup\, \mathcal{U}$ and $X \notin \mathcal{U}$. k-covers and ω-covers play important roles in selection principles [2], [14], [15]. Different Δ-covers exposed many dualities in hyperspace topologies such as Fell topology, Vietoris topology, $\mathbf{Z}^+$, $\mathbf{F}^+$ ([5], [15], [16], [19], [10], [9], [8], [22], [26]).

Throughout the paper all spaces are assumed to be Hausdorff, non-compact. Along this paper, unless we say the opposite, we will take a family $\Lambda \subseteq CL(X)$ that is closed under finite unions. Also we shall use $[X]^{<\omega}$ to denote all finite subsets of X.

2. Definitions and Results

Let us recall that an open cover $\mathcal{U}$ of a space X is called an ω-cover [12] (respectively, a k-cover [20]) if every finite (respectively, compact) subset of X is contained in a member of $\mathcal{U}$ and X is not a member of $\mathcal{U}$. An open cover $\mathcal{U}$ of X is called a γ-cover [12] if it is infinite and each $x \in X$ belongs to all but finitely many elements of $\mathcal{U}$. Notice that it is equivalent to the assertion: Each finite subset of X belongs to all but finitely many members of $\mathcal{U}$. Also Lj. D. R. Kočinac in his paper [16] introduced a stronger version of γ-cover as: an open cover $\mathcal{U}$ of a space X is called a γ_k-cover of X if each compact subset of X is contained in all but finitely many elements of $\mathcal{U}$ and X is not a member of the cover.

For a space (X, τ) and a point $x \in X$ we use

- $\mathcal{O}$: the collection of open covers of X;
- Ω : the collection of ω-covers of X;
- $\mathcal{K}$: the collection of k-covers of X;
- Γ : the collection of all γ-covers of X;
- Γ_k : the collection of all γ_k-covers of X;
- $\Omega_x = \{A \subset X : x \in ClA\}$;
- $\mathcal{D}_\tau$: the collection of all dense subsets of the space (X, τ).

As $\mathbf{F}^+$ and $\mathbf{Z}^+$ are miss type hyperspace topologies, they are dual to k-covers and ω-covers in selection principles. The Fell topology and the Vietoris topology are hit-and-miss topologies of types of subbasic open sets: those that hit a variable open subset plus those that miss a compact subset (in case of Fell topology) or a closed subset (in case of Vietoris topology). Z. Li in his paper [19] introduced the definitions of hit-and-miss type covers to study the selection principles in $CL(X)$ under τ_F and τ_V. The following definition of hit-and-miss type covers has been introduced in [6].

Definition 2.1. [6] Let (X, τ) be a topological space. A family $\mathcal{U} \subseteq \Lambda^c$ is called a $c_\Delta(\Lambda)$-cover of X, if for any $D \in \Delta$ and open subsets $V_1, ..., V_m$ of X, with $D^c \cap V_i \neq \phi$, for any $i \in \{1, ..., m\}$, there exist $U \in \mathcal{U}$ and $F \in [X]^{<\omega}$ such that $D \subseteq U$, $F \cap U = \phi$ and for each $i \in \{1, ..., m\}$, $F \cap V_i \neq \phi$. The family of all $c_\Delta(\Lambda)$-covers of X will be denoted by $\mathbb{C}_\Delta(\Lambda)$.

Next we recall the relative version of the above type of covers as follows.

Definition 2.2. [29] Let (X, τ) be a topological space and $Y \subseteq X$ with $Y \neq X$. A family $\mathcal{U} \subseteq \Lambda^c$ is called a $c_\Delta(\Lambda)$-cover of Y, if for any $D \in \Delta$ with $D \subseteq Y$ and open subsets $V_1, ..., V_m$ of X, with $Y^c \cap V_i \neq \phi$, for any $i \in \{1, ..., m\}$, there exist $U \in \mathcal{U}$ and $F \in [X]^{<\omega}$ such that $D \subseteq U$, $F \cap U = \phi$ and for each $i \in \{1, ..., m\}$, $F \cap V_i \neq \phi$. We denote by $\mathbb{C}^*_\Delta(\Lambda)$ the family of all $c_\Delta(\Lambda)$-covers of $Y \subseteq X$, with $Y \neq X$.

Lemma 2.1. *[29] Let Y be an open subset of a space X with $Y \neq X$ and $\mathcal{U} \subseteq \Lambda^c$ be a cover of Y. Then the following statements are equivalent:*

(i) $\mathcal{U}$ *is a* $c_\Delta(\Lambda)$*-cover of* Y.
(ii) $Y^c \in Cl_{\tau^+_\Delta}(\mathcal{U}^c)$.

Lemma 2.2. *For a space* X, $E \in \Lambda$ *and a collection* $\mathcal{A} \subset \Lambda$, $\mathcal{A} \in \Omega_E^{\tau^+_\Delta}$ *implies* $\{(A \cup E)^c : A \in \mathcal{A}\}$ *is a* $c_\Delta(\Lambda)$*-cover of* E^c, *where* $\Omega_E^{\tau^+_\Delta} = \{\mathcal{A} \subset CL(X) : E \in Cl_{\tau^+_\Delta}(\mathcal{A})\}$.

Proof. Let $D \in \Delta$ be such that $D \subset E^c$ and let $V_1, ..., V_m$ be open sets in X with $E \cap V_i \neq \phi$, for all $i = 1, ..., m$. Then $(V_1, ..., V_m)^+_D$ is a τ^+_Δ-neighbourhood of E. As $\mathcal{A} \in \Omega_E^{\tau^+_\Delta}$, there exists $A \in \mathcal{A}$ such that $A \in (V_1, ..., V_m)^+_D$. Now choose $x_i \in A \cap V_i$, for $1 \leq i \leq m$ and consider the set $F = \{x_i : 1 \leq i \leq m\}$. Then $F \in [X]^{<\omega}$ with $F \cap V_i \neq \phi$, for all $1 \leq i \leq m$. Also $D \subset (A \cup E)^c$ and $(A \cup E)^c \cap F = \phi$. Hence $\{(A \cup E)^c : A \in \mathcal{A}\}$ is a $c_\Delta(\Lambda)$-cover of E^c. □

We next recall the definition of $\Delta\gamma$-covers of a space as follows.

Definition 2.3. [29] Let (X, τ) be a topological space. A family $\mathcal{U} \subseteq \Lambda^c$ is called a $\Delta\gamma$-cover of X, if each $B \in \Delta$ belongs to all but finitely many elements of $\mathcal{U}$ and for any $B \in \Delta$ and open subsets $V_1, ..., V_m$ of X, with $B^c \cap V_i \neq \phi$ for any $i \in \{1, ..., m\}$, there exist $U \in \mathcal{U}$ and $F \in [X]^{<\omega}$ such that $B \subseteq U$, $F \cap U = \phi$ and for each $i \in \{1, ..., m\}$, $F \cap V_i \neq \phi$. The set of all $\Delta\gamma$-covers of X is denoted by $\Delta\Gamma$.

Next recall the relative version of the above type of covers as follows.

Definition 2.4. [28] Let (X, τ) be a topological space and $Y \subseteq X$ with $Y \neq X$. A family $\mathcal{U} \subseteq \Lambda^c$ is called a $\Delta\gamma$-cover of Y, if each $B \subseteq Y$ with $B \in \Delta$ belongs to all but finitely many elements of $\mathcal{U}$ and for any $B \subseteq Y$ with $B \in \Delta$ and open subsets $V_1, ..., V_m$ of X, with $Y^c \cap V_i \neq \phi$ for any $i \in \{1, ..., m\}$, there exist $U \in \mathcal{U}$ and $F \in [X]^{<\omega}$ such that $B \subseteq U$, $F \cap U = \phi$ and for each $i \in \{1, ..., m\}$, $F \cap V_i \neq \phi$. The set of all $\Delta\gamma$-covers of $Y \subseteq X$ is denoted by $\Delta\Gamma^*$.

Remark 2.1. If we consider $\Delta = \mathbb{K}(X)$ and $\Lambda = CL(X)$ (resp., $\Delta = \Lambda = CL(X)$) in Definitions 2.3 and 2.4 above, we get the definitions of γ_{k_F}-covers (resp., γ_{c_V}-covers) of X and also the definitions of γ_{k_F}-covers (resp., γ_{c_V}-covers) of a subset Y of X, with $Y \neq X$.

It is easy to observe that $\Delta\Gamma \subset \mathbb{C}_\Delta(\Lambda)$.

Lemma 2.3. *[28] Let* X *be a topological space,* Y *be an open subset of* X *and* $\mathcal{U} = \{U_n : n \in \mathbb{N}\} \subseteq \Lambda^c$ *be a cover of* Y. *Then the following statements are equivalent:*

(i) $\mathcal{U}$ *is a* $\Delta\gamma$*-cover of* Y.
(ii) $\{U_n^c : n \in \mathbb{N}\}$ *converges to* Y^c *in* (Λ, τ^+_Δ).

Recall now that an open cover $\mathcal{U}$ of a space X is called
(i) ω-groupable [15], [17] (k-groupable [9]) if it can be expressed as a countable union of finite, pairwise disjoint subfamilies $\mathcal{U}_n$, $n \in \mathbb{N}$, such that for each finite (compact) set $C \subset X$, for all but finitely many n there is an $U \in \mathcal{U}_n$ such that $C \subset U$,
(ii) weakly groupable [2] (k-weakly groupable [9]) if there is a partition of $\mathcal{U}$ into countably many finite, pairwise disjoint sets $\mathcal{U}_n$, for $n \in \mathbb{N}$, such that each finite (compact) subset of X is contained in $\bigcup \mathcal{U}_n$, for some n.

Also recall that a countable element D from $\mathcal{D}$ is said to be groupable [17], [18] if there is a partition $D = \bigcup_{n\in\mathbb{N}} D_n$ into finite pairwise disjoint sets such that each nonempty open set of the space intersects D_n, for all but finitely many n. Let $\mathcal{D}^{gp}$ denote the family of groupable elements of $\mathcal{D}$.

For a space X, we denote:

- Ω^{gp} - the family of ω-groupable covers of X;
- $\mathcal{K}^{gp}$ - the family of k-groupable covers of X;
- $\mathcal{O}^{wgp}$ the family of weakly groupable covers of X;
- $\mathcal{O}^{k-wgp}$ the family of k-weakly groupable covers of X;
- $(\Omega_E^{\tau_\Delta^+})^{gp}$ - the family of groupable elements of $\Omega_E^{\tau_\Delta^+}$.

Following Definitions 5.1 and 5.5 of [19], where the classes $\mathcal{K}_F^{gp}$ of k_F-groupable covers and $\mathcal{C}_V^{gp}$ of c_V-groupable covers are introduced, we define the general notion of a Δ-groupable $c_\Delta(\Lambda)$-cover as follows.

Definition 2.5. A $c_\Delta(\Lambda)$-cover $\mathcal{U}$ of a space X is said to be Δ-groupable if it can be expressed as a union of infinitely many finite, pairwise disjoint subfamilies $\mathcal{U}_n \subset \mathcal{U}$ such that for any subset B of X with $B \in \Delta$, open sets $V_1, V_2, ..., V_m$ of X with $V_i \cap B^c \neq \phi$ $(1 \leq i \leq m)$, there exists $n_0 \in \mathbb{N}$ so that for each $n \geq n_0$, there exist $U_n \in \mathcal{U}_n$ and a finite set F_n with $F_n \cap V_i \neq \phi$ $(1 \leq i \leq m)$ such that $B \subset U_n$ and $F_n \cap U_n = \phi$. We denote the family of all Δ-groupable covers of X by $\mathbb{C}_\Delta(\Lambda)^{gp}$.

Definition 2.6. Let (X, τ) be a topological space and $Y \subseteq X$ with $Y \neq X$. A $c_\Delta(\Lambda)$-cover $\mathcal{U}$ of Y is said to be Δ-groupable if it can be expressed as a union of infinitely many finite, pairwise disjoint subfamilies $\mathcal{U}_n \subset \mathcal{U}$ such that for any subset $B \subseteq Y$ with $B \in \Delta$, open sets $V_1, V_2, ..., V_m$ of X with $V_i \cap Y^c \neq \phi$ $(1 \leq i \leq m)$, there exists $n_0 \in \mathbb{N}$ so that for each $n \geq n_0$, there exist $U_n \in \mathcal{U}_n$ and a finite set F_n with $F_n \cap V_i \neq \phi$ $(1 \leq i \leq m)$ such that $B \subset U_n$ and $F_n \cap U_n = \phi$. We denote the family of all Δ-groupable covers of $Y \subseteq X$ with $Y \neq X$ by $\mathbb{C}^*_\Delta(\Lambda)^{gp}$.

Lemma 2.4. *For a space X, $E \in \Lambda$ and a collection $\mathcal{A} \subset \Lambda$, $\mathcal{A} \in (\Omega_E^{\tau_\Delta^+})^{gp}$ implies $\{(A \cup E)^c : A \in \mathcal{A}\}$ is a Δ-groupable cover of E^c.*

Proof. Let $\mathcal{A} = \bigcup_{n\in\mathbb{N}} \mathcal{B}_n$ be a partition of $\mathcal{A}$ into finite, pairwise disjoint sets such that each τ_{Δ}^{+}-neighbourhood of E meets $\mathcal{B}_n$ for all but finitely many n. Then by Lemma 2.2, $\mathcal{U} = \{(A \cup E)^{c} : A \in \mathcal{A}\}$ is a $c_\Delta(\Lambda)$-cover of E^{c}. Write $\mathcal{U} = \bigcup_{n\in\mathbb{N}} \mathcal{V}_n$, where for each $n \in \mathbb{N}$, $\mathcal{V}_n = \{(B \cup E)^{c} : B \in \mathcal{B}_n\}$. Let $D \in \Delta$ be such that $D \subset E^{c}$ and let $V_1, ..., V_m$ be open sets in X with $E \cap V_i \neq \phi$, for all $i = 1, ..., m$. Then $(V_1, ..., V_m)_D^{+}$ is a τ_{Δ}^{+}-neighbourhood of E. Hence there exists $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$, there exists $B_n \in \mathcal{B}_n$ such that $B_n \in (V_1, ..., V_m)_D^{+}$. Now choose $x_i \in B_n \cap V_i$, for $1 \leq i \leq m$ and consider the set $F = \{x_i : 1 \leq i \leq m\}$. Then $F \in [X]^{<\omega}$ with $F \cap V_i \neq \phi$, for all $1 \leq i \leq m$. Also $D \subset (B_n \cup E)^{c}$ and $(B \cup E)^c \cap F = \phi$. Hence $\{(A \cup E)^{c} : A \in \mathcal{A}\}$ is a Δ-groupable cover of E^{c}. □

Definition 2.7. A cover $\mathcal{U}$ of a space X is weakly Δ-groupable if it can be expressed as a union of infinitely many finite, pairwise disjoint subfamilies $\mathcal{U}_n \subset \mathcal{U}$ such that for any subset B of X with $B \in \Delta$, open sets $V_1, V_2, ..., V_m$ of X with $V_i \cap B^{c} \neq \phi$ $(1 \leq i \leq m)$, there exist $\mathcal{U}_n$ and a finite set F with $F \cap V_i \neq \phi$ $(1 \leq i \leq m)$ such that $B \subset \cup\, \mathcal{U}_n$ and $F \cap (\cup\, \mathcal{U}_n) = \phi$. We denote the family of all weakly Δ-groupable covers of X by $\mathbb{C}_{\Delta}^{wgp}$.

Lemma 2.5. *[6] A family $\mathcal{U} \subseteq \Lambda^{c}$ is a $c_\Delta(\Lambda)$-cover of X if and only if the family $\mathcal{U}^{c}$ is a dense subset of $(\Lambda, \tau_{\Delta}^{+})$.*

Lemma 2.6. *For a space X and a countable subset $\mathcal{A} \subset CL(X)$, the following statements are equivalent:*

(i) $\mathcal{A}$ is a groupable dense subset of $(CL(X), \tau_{\Delta}^{+})$.
(ii) $\mathcal{A}^{c}$ is a Δ-groupable cover of X.

Proof. (i) ⇒ (ii): Let $\mathcal{A} = \bigcup_{n\in\mathbb{N}} \mathcal{B}_n$ be a partition into finite pairwise disjoint sets such that each open set of $(CL(X), \tau_{\Delta}^{+})$ intersects $\mathcal{B}_n$ for all but finitely many n. We claim that $\mathcal{A}^{c} = \bigcup_{n\in\mathbb{N}} \mathcal{B}_n^{c}$ is a Δ-groupable cover of X. Indeed, let $K \in \Delta$ be a subset of X and $V_1, ..., V_m$ be open in X with $(X \setminus K) \cap V_i \neq \phi$, for $1 \leq i \leq m$. Then $(V_1, ..., V_m)_K^{+}$ is a τ_{Δ}^{+}-open set in $CL(X)$. Hence there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, there exists $B_n \in \mathcal{B}_n$ such that $B_n \in (V_1, ..., V_m)_K^{+}$. Let $U_n = B_n^{c}$, for $n \geq n_0$. Then $U_n \in \mathcal{B}_n^{c}$. Choose $x_i^{(n)} \in V_i \cap B_n$, for $1 \leq i \leq m$ and consieder $F = \{x_i^{(n)} : 1 \leq i \leq m\}$. Then F is a finite subset of X with $F \cap V_i \neq \phi$, for all $1 \leq i \leq m$. Also $K \subset U_n$ and $F \cap U_n = \phi$. Hence $\mathcal{B}_n^{c}$ is a $c_\Delta(CL(X))$-cover of X.

(ii) ⇒ (i): Let $\mathcal{A}^{c} = \bigcup_{n\in\mathbb{N}} \mathcal{U}_n$ be a partition of $\mathcal{A}^{c}$ that witnesses (ii). We claim that $\mathcal{A}$ is a groupable dense subset of $(CL(X), \tau_{\Delta}^{+})$. Let $(V_1, ..., V_m)_D^{+}$ be a τ_{Δ}^{+}-open set in $(CL(X), \tau^+_\Delta)$. Then there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, there exist $U_n \in \mathcal{U}_n$ and $F_n \in [X]^{<\omega}$ with $F_n \cap V_i \neq \phi$, for all $i = 1, ..., m$ such that $D \subseteq U_n$ and $U_n \cap F_n = \phi$. Hence $U_n^c \in (V_1, ..., V_m)^+_D$, for all $n \geq n_0$, so that $\mathcal{A}$ is a groupable dense subset of $(CL(X), \tau^+_\Delta)$. □

3. Selective separability of the hyperspace (Λ, τ^+_Δ)

In this section we first start with the relationships between closure-type properties of the hyperspace (Λ, τ^+_Δ) and covering properties of that of X. We then discuss about the selective separability and variations of separability in (Λ, τ^+_Δ).

Theorem 3.1. *Let $\star \in \{1, fin\}$. Then for a space X the following statements are equivalent:*

(i) X satisfies $S_\star(\mathbb{C}_\Delta(\Lambda), \mathbb{C}_\Delta(\Lambda))$.
(ii) (Λ, τ^+_Δ) satisfies $S_\star(\mathcal{D}_{\mathbb{C}_\Delta(\Lambda)}, \mathcal{D}_{\mathbb{C}_\Delta(\Lambda)})$.
(where $\mathcal{D}_{\mathbb{C}_\Delta(\Lambda)}$ denotes the family of dense subsets of (Λ, τ^+_Δ)).

Proof. We prove the theorem for $\star = fin$, the other part being similar.

(i) ⇒ (ii): Let $\{D_i : i \in \mathbb{N}\}$ be a family of dense subsets of (Λ, τ^+_Δ) such that $D_i \in \mathcal{D}_{\mathbb{C}_\Delta(\Lambda)}$, for each $i \in \mathbb{N}$. Then by Lemma 2.5, $\{D_i^c : i \in \mathbb{N}\}$ is a family of open covers of X such that $D_i^c \in \mathbb{C}_\Delta(\Lambda)$, for all $i \in \mathbb{N}$. As X satisfies $S_{fin}(\mathbb{C}_\Delta(\Lambda), \mathbb{C}_\Delta(\Lambda))$, there exists a sequence $\{A_i : i \in \mathbb{N}\}$ of finite sets such that $A_i \subseteq D_i^c$ and $\bigcup_{i\in\mathbb{N}} A_i \in \mathbb{C}_\Delta(\Lambda)$, for each $i \in \mathbb{N}$. Then $\bigcup_{i\in\mathbb{N}} A_i^c \in \mathcal{D}_{\mathbb{C}_\Delta(\Lambda)}$.

(ii) ⇒(i): Assume that $\{\mathcal{U}_n : n \in \mathbb{N}\}$ is a family of open covers of X such that $\mathcal{U}_n \in \mathbb{C}_\Delta(\Lambda)$. Consider $\mathcal{A}_n = \mathcal{U}_n^c$, for each $n \in \mathbb{N}$. Then by Lemma 2.5, $\mathcal{A}_n$ is a dense subset of (Λ, τ^+_Δ) for each $n \in \mathbb{N}$ such that $\mathcal{A}_n \in \mathcal{D}_{\mathbb{C}_\Delta(\Lambda)}$. As (Λ, τ^+_Δ) satisfies $S_{fin}(\mathcal{D}_{\mathbb{C}_\Delta(\Lambda)}, \mathcal{D}_{\mathbb{C}_\Delta(\Lambda)})$, there exists a sequence $\{A_n : n \in \mathbb{N}\}$ of finite subsets such that $A_n \subseteq \mathcal{A}_n$, for each $n \in \mathbb{N}$ and $\bigcup_{i\in\mathbb{N}} A_i \in \mathcal{D}_{\mathbb{C}_\Delta(\Lambda)}$. Then $U_n = A_n^c$, for $n \in \mathbb{N}$ is such that $\bigcup_{n\in\mathbb{N}} U_n$ is an open cover of X and $\bigcup_{n\in\mathbb{N}} U_n \in \mathbb{C}_\Delta(\Lambda)$. □

Corollary 3.1. *(Theorem 3.6 in [19]) For a space X, the following are equivalent:*

(i) $(CL(X), \tau_V)$ satisfies $S_1(\mathcal{D}, \mathcal{D})$.
(ii) X satisfies $S_1(\mathbb{C}_V, \mathbb{C}_V)$.

Corollary 3.2. *(Theorem 3.4 in [19]) For a space X, the following are equivalent:*

(i) $(CL(X), \tau_F)$ satisfies $S_1(\mathcal{D}, \mathcal{D})$.
(ii) X satisfies $S_1(\mathbb{K}_F, \mathbb{K}_F)$.

Corollary 3.3. *(Theorem 4.4 in [19]) For a space X, the following are equivalent:*

(i) $(CL(X), \tau_V)$ *satisfies* $S_{fin}(\mathcal{D}, \mathcal{D})$.
(ii) X *satisfies* $S_{fin}(\mathbb{C}_V, \mathbb{C}_V)$.

Corollary 3.4. *(Theorem 4.2 in [19]) For a space X, the following are equivalent:*

(i) $(CL(X), \tau_F)$ *satisfies* $S_{fin}(\mathcal{D}, \mathcal{D})$.
(ii) X *satisfies* $S_{fin}(\mathbb{K}_F, \mathbb{K}_F)$.

Recall here that a space X is M-separable [3] if for every sequence $\{D_n : n \in \mathbb{N}\}$ of dense subspaces of X one can select finite $F_n \subset D_n$ so that $\bigcup\{F_n : n \in \mathbb{N}\}$ is dense in X. Thus we have the following theorem.

Theorem 3.2. *For a space X, (Λ, τ_Δ^+) is M-separable if and only if X satisfies* $S_{fin}(\mathbb{C}_\Delta(\Lambda), \mathbb{C}_\Delta(\Lambda))$.

Again a space X is R-separable [3] if for every sequence $\{D_n : n \in \mathbb{N}\}$ of dense subspaces of X one can pick $x_n \in D_n$ so that $\{x_n : n \in \mathbb{N}\}$ is dense in X. Thus we have the following theorem.

Theorem 3.3. *For a space X, (Λ, τ_Δ^+) is R-separable if and only if X satisfies* $S_1(\mathbb{C}_\Delta(\Lambda), \mathbb{C}_\Delta(\Lambda))$.

Theorem 3.4. *Let $\Phi, \Psi \in \{\Delta\Gamma^*, \mathbb{C}^*_\Delta(\Lambda)\}, \star \in \{1, fin\}$. Then for a space X the following statements are equivalent:*

(i) Each open set $Y \subset X$ with $Y \in \Lambda^c$ has the property $S_\star(\Phi, \Psi)$.
(ii) Each $E \in (\Lambda, \tau_\Delta^+)$ satisfies $S_\star(\Phi_E, \Psi_E)$.
(where Φ_E denotes the Φ family of covers of E and Ψ_E denotes the Ψ family of covers of E).

Proof. We prove the theorem for $\star = 1$, the other parts being similar.

(i) $\Rightarrow$ (ii): Let $E \in \Lambda$ and let $\{\mathcal{A}_n : n \in \mathbb{N}\}$ be a sequence such that for each $n \in \mathbb{N}$, $\mathcal{A}_n \in \Phi_E$. Then $\{\mathcal{A}_n^c : n \in \mathbb{N}\}$ is a sequence of open covers of E^c such that for each $n \in \mathbb{N}$, $\mathcal{A}_n^c \in \Phi$. As E^c has the property $S_1(\Phi, \Psi)$, there exists a sequence $\{A_n^c : n \in \mathbb{N}\}$ such that for each $n \in \mathbb{N}$, $A_n^c \in \mathcal{A}_n^c$ and $\{A_n^c : n \in \mathbb{N}\}$ is an open cover of E^c such that $\{A_n^c : n \in \mathbb{N}\} \in \Psi$. Hence $\{A_n : n \in \mathbb{N}\} \in \Psi_E$.

(ii) $\Rightarrow$ (i): Let Y be an open subset of X with $Y \in \Lambda^c$ and $\{\mathcal{F}_n : n \in \mathbb{N}\}$ be a sequence of open covers of Y such that $\mathcal{F}_n \in \Phi_Y$, for $n \in \mathbb{N}$. Let $E = X \setminus Y$. Put $\mathcal{A}_n = \mathcal{F}_n^c, n \in \mathbb{N}$. Then $\mathcal{A}_n \subset \Lambda$ and $\mathcal{A}_n \in \Phi_E$, for $n \in \mathbb{N}$. As E satisfies $S_1(\Phi_E, \Psi_E)$, there exists a sequence $\{A_n : n \in \mathbb{N}\}$ such that $A_n \in \mathcal{A}_n$, for each $n \in \mathbb{N}$ and $\{A_n : n \in \mathbb{N}\} \in \Psi_E$. Hence $\{F_n = A_n^c : n \in \mathbb{N}\} \in \Psi$. $\square$

Recall that a space X has countable fan tightness [1] if whenever $x \in ClA_n$ for all $n \in \mathbb{N}$, one can choose finite $F_n \subset A_n$ so that $x \in Cl(\cup\{F_n : n \in \mathbb{N}\})$ and X has countable strong fan tightness [25] if whenever $x \in ClA_n$ for $n \in \mathbb{N}$, there are $x_n \in A_n$ such that $x \in Cl(\{x_n : n \in \mathbb{N}\})$. In view of these definitions we can restate the above theorem as follows.

Theorem 3.5. *For a space X, $(\Lambda, \tau_{\Delta}^{+})$ has countable strong fan tightness if and only if each open subset $Y \subsetneq X$ with $Y^c \in \Lambda$ satisfies $S_1(\mathbb{C}_{\Delta}^{*}(\Lambda), \mathbb{C}_{\Delta}^{*}(\Lambda))$.*

Proof. First let $Y \subsetneq X$ be open in X with $Y^c \in \Lambda$ and $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of $c_\Delta(\Lambda)$-covers of Y. Then by Lemma 2.1, $Y^c \in Cl_{\tau_{\Delta}^{+}}(\mathcal{U}_n^c)$. As $(\Lambda, \tau_{\Delta}^{+})$ has countable strong fan tightness, there exists $U_n^c \in \mathcal{U}_n^c$, for $n \in \mathbb{N}$ such that $Y^c \in Cl_{\tau_{\Delta+}}(\{U_n^c : n \in \mathbb{N}\})$. Hence $\{U_n : n \in \mathbb{N}\}$ is a $c_\Delta(\Lambda)$-cover of Y.

Conversely, let $E \in \Lambda$ be such that $E \in Cl(\mathcal{U}_n)$. Then by Lemma 2.1, $\{\mathcal{U}_n^c : n \in \mathbb{N}\}$ is a sequence of $c_\Delta(\Lambda)$-covers of E^c. By the given condition, there exists $U_n^c \in \mathcal{U}_n^c$, for $n \in \mathbb{N}$ such that $\{U_n^c : n \in \mathbb{N}\}$ is a $c_\Delta(\Lambda)$-cover of E^c. Hence $E \in Cl_{\tau_{\Delta}^{+}}(\{U_n : n \in \mathbb{N}\})$, so that $(\Lambda, \tau_{\Delta}^{+})$ has countable strong fan tightness. □

Theorem 3.6. *For a space X, $(\Lambda, \tau_{\Delta}^{+})$ has countable fan tightness if and only if each open subset $Y \subsetneq X$ with $Y^c \in \Lambda$ satisfies $S_{fin}(\mathbb{C}_{\Delta}^{*}(\Lambda), \mathbb{C}_{\Delta}^{*}(\Lambda))$.*

Proof. First let $Y \subsetneq X$ be open in X with $Y^c \in \Lambda$ and $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of $c_\Delta(\Lambda)$-covers of Y. Then by Lemma 2.1, $Y^c \in Cl_{\tau_{\Delta}^{+}}(\mathcal{U}_n^c)$. As $(\Lambda, \tau_{\Delta}^{+})$ has countable fan tightness, there exist finite $\mathcal{V}_n^c \subset \mathcal{U}_n^c$, for $n \in \mathbb{N}$, such that $Y^c \in Cl_{\tau_{\Delta+}}(\bigcup\{\mathcal{V}_n^c : n \in \mathbb{N}\})$. Hence $\bigcup\{\mathcal{V}_n : n \in \mathbb{N}\}$ is a $c_\Delta(\Lambda)$-cover of Y.

Conversely, let $E \in \Lambda$ be such that $E \in Cl(\mathcal{U}_n)$. Then by Lemma 2.1, $\{\mathcal{U}_n^c : n \in \mathbb{N}\}$ is a sequence of $c_\Delta(\Lambda)$-covers of E^c. By the given condition, there exist finite $\mathcal{V}_n^c \subset \mathcal{U}_n^c$, for $n \in \mathbb{N}$, such that $\bigcup\{\mathcal{V}_n^c : n \in \mathbb{N}\}$ is a $c_\Delta(\Lambda)$-cover of E^c. Hence $E \in Cl_{\tau_{\Delta}^{+}}(\bigcup\{\mathcal{V}_n : n \in \mathbb{N}\})$. □

Corollary 3.5. *(Theorem 3.2 of [19]) For a space X, the following are equivalent:*

(i) $(CL(X), \tau_V)$ has countable strong fan tightness.
(ii) Each open subset Y of X with $Y \subset X$ satisfies $S_1(\mathbb{C}_V^{*}, \mathbb{C}_V^{*})$.

Corollary 3.6. *(Theorem 3.1 of [19]) For a space X, the following are equivalent:*

(i) $(CL(X), \tau_F)$ has countable strong fan tightness.
(ii) Each open subset Y of X with $Y \subset X$ satisfies $S_1(\mathbb{K}_F^{*}, \mathbb{K}_F^{*})$.

Corollary 3.7. *(Theorem 4.3 of [19]) For a space X, the following are equivalent:*

(i) $(CL(X), \tau_V)$ has countable fan tightness.
(ii) Each open subset Y of X with $Y \subset X$ satisfies $S_{fin}(\mathbb{C}_V^{*}, \mathbb{C}_V^{*})$.

Corollary 3.8. *(Theorem 4.1 of [19]) For a space X, the following are equivalent:*

(i) $(CL(X), \tau_F)$ has countable fan tightness.
(ii) Each open subset Y of X with $Y \subset X$ satisfies $S_{fin}(\mathbb{K}^*_F, \mathbb{K}^*_F)$.

Theorem 3.7. *For a space X, the following statements are equivalent:*

(i) X satisfies $S_1(\mathbb{C}_\Delta(CL(X)), \mathbb{C}_\Delta(CL(X))^{gp})$.
(ii) $(CL(X), \tau^+_\Delta)$ satisfies $S_1(\mathcal{D}_{\tau^+_\Delta}, \mathcal{D}^{gp}_{\tau^+_\Delta})$.

Proof. (i) $\Rightarrow$ (ii): Let $\{\mathcal{D}_n : n \in \mathbb{N}\}$ be a sequence of dense subsets of $(CL(X), \tau^+_\Delta)$. For each $n \in \mathbb{N}$, put $\mathcal{U}_n = \mathcal{D}^c_n$. Then $\mathcal{U}_n$ is a $c_\Delta(CL(X))$-cover of X, for each $n \in \mathbb{N}$. By (i) applied to $\{\mathcal{U}_n : n \in \mathbb{N}\}$, there exists a sequence $\{D^c_n : n \in \mathbb{N}\}$ such that for each $n \in \mathbb{N}$, $D^c_n \in \mathcal{U}_n$ and $\{D^c_n : n \in \mathbb{N}\}$ is a Δ-groupable cover of X. Hence by Lemma 2.6, $\{D_n : n \in \mathbb{N}\}$ is a groupable dense subset of $(CL(X), \tau^+_\Delta)$.

(ii) $\Rightarrow$ (i): Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of $c_\Delta(CL(X))$-covers of X. Put $\mathcal{A}_n = \mathcal{U}^c_n$, $n \in \mathbb{N}$. Then by Lemma 2.5 for each $n \in \mathbb{N}$, $\mathcal{A}_n$ is a sequence of dense subsets of $(CL(X), \tau^+_\Delta)$. By (ii), there exists a sequence $\{A_n : n \in \mathbb{N}\}$ such that for each $n \in \mathbb{N}$, $A_n \in \mathcal{A}_n$ and $\mathcal{B} = \{A_n : n \in \mathbb{N}\} \in \mathcal{D}^{gp}_{\tau^+_\Delta}$. Again by Lemma 2.6, $\mathcal{B}^c$ is a Δ-groupable cover of X. Hence $\{A^c_n : n \in \mathbb{N}\}$ guarantees for $\{\mathcal{U}_n : n \in \mathbb{N}\}$ that X satisfies $S_1(\mathbb{C}_\Delta(CL(X)), \mathbb{C}_\Delta(CL(X))^{gp})$. $\square$

Next recall that a space X is H-separable [3] if for every sequence $\{D_n : n \in \mathbb{N}\}$ of dense subspaces of X, one can pick finite $F_n \subset D_n$ so that for every nonempty open set $O \subset X$, the intersection $O \cap F_n$ is nonempty for all but finitely many n. Thus we have the following theorem.

Theorem 3.8. *For a space X, $(CL(X), \tau^+_\Delta)$ is H-separable if and only if X satisfies $S_{fin}(\mathbb{C}_\Delta(CL(X)), \mathbb{C}_\Delta(CL(X))^{gp})$.*

Proof. First let, $(CL(X), \tau^+_\Delta)$ be H-separable and $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of $c_\Delta(CL(X))$-covers of X. Then by Lemma 2.5, $\{\mathcal{U}^c_n : n \in \mathbb{N}\}$ is a sequence of dense subsets of $CL(X)$. By H-separability of $(CL(X), \tau^+_\Delta)$, there exist finite $\mathcal{V}^c_n \subset \mathcal{U}^c_n$, $n \in \mathbb{N}$, such that for every non-empty open set W of $CL(X)$, $W \cap \mathcal{V}^c_n \neq \phi$, for all but finitely many $n \in \mathbb{N}$. We claim that $\bigcup \mathcal{V}_n$ is a Δ-groupable cover of X. Indeed, Let $D \in \Delta$ and $V_1, ..., V_m$ be open in X with $D^c \cap V_i \neq \phi$, for all $1 \leq i \leq m$. Then $(V_1, ..., V_m)^+_D$ is a τ^+_Δ-open set in $CL(X)$ and hence there exists $n_0 \in \mathbb{N}$ such that $(V_1, ..., V_m)^+_D \cap \mathcal{V}^c_n \neq \phi$, for all $n \geq n_0$. Choose $V^c_n \in (V_1, ..., V_m)^+_D \cap \mathcal{V}^c_n$, for all $n \geq n_0$. Next choose $x^{(n)}_i \in (V_1, ..., V_m)^+_D \cap V^c_n$, for all $1 \leq i \leq m$ and consider the set $F_n = \{x^{(n)}_i : 1 \leq i \leq m\}$. Then $F_n \in [X]^{<\omega}$ with $F_n \cap V_i \neq \phi$, for all $1 \leq i \leq m$. Also, $D \subset V_n$ and $V_n \cap F_n = \phi$, for all $n \geq n_0$. Hence $\bigcup \mathcal{V}_n$ is a Δ-groupable cover of X.

Conversely, let $\{\mathcal{D}_n : n \in \mathbb{N}\}$ be a sequence of dense subsets of $CL(X)$. By Lemma 2.5, $\{\mathcal{D}_n^c : n \in \mathcal{N}\}$ is a sequence of $c_\Delta(CL(X))$-covers of X. As X satisfies $S_{fin}(\mathbb{C}_\Delta(CL(X)), \mathbb{C}_\Delta(CL(X))^{gp})$, there exist finite $\mathcal{B}_n^c \subset \mathcal{D}_n^c$, $n \in \mathbb{N}$, such that $\bigcup \mathcal{B}_n^c$ is a Δ-groupable cover of X. Then every τ_Δ^+-open set intersects all but finitey many$\mathcal{B}_n$. Hence $(CL(X), \tau_\Delta^+)$ is H-separable. □

Corollary 3.9. *(Theorem 5.4 of [19]) For a space X, the following statements are equivalent:*

(i) $(CL(X), \tau_V))$ satisfies $S_1(\mathcal{D}, \mathcal{D}^{gp})$.
(ii) X satisfies $S_1(\mathbb{C}_V, \mathbb{C}_V^{gp})$.

Corollary 3.10. *(Theorem 5.2 of [19]) For a space X, the following statements are equivalent:*

(i) $(CL(X), \tau_F)$ satisfies $S_1(\mathcal{D}, \mathcal{D}^{gp})$.
(ii) X satisfies $S_1(\mathbb{K}_F, \mathbb{K}_F^{gp})$.

Theorem 3.9. *For a space X, the following statements are equivalent:*

(i) $(CL(X), \tau_\Delta^+)$ satisfies: for each sequence $\{\mathcal{D}_n : n \in \mathbb{N}\}$ of dense subsets of $(CL(X), \tau_\Delta^+)$ there is a finite $\mathcal{B}_n \subset \mathcal{D}_n$ such that $\bigcup_{n\in\mathbb{N}} \mathcal{B}_n$ can be partitioned into a union of finite sets $\mathcal{C}_n$, $n \in \mathbb{N}$, so that $\{\bigcap \mathcal{C}_n : n \in \mathbb{N}\}$ is dense in $(CL(X), \tau_\Delta^+)$.
(ii) X satisfies $S_{fin}(\mathbb{C}_\Delta(CL(X)), \mathbb{C}_\Delta^{wgp})$.

Proof. (i) ⇒ (ii): Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of $c_\Delta(CL(X))$-open covers of X. Then for each $n \in \mathbb{N}$, $\mathcal{A}_n = \mathcal{U}_n^c$ is a dense subset of $(CL(X), \tau_\Delta^+)$. By (i), there exist finite $\mathcal{B}_n \subset \mathcal{A}_n$, for each $n \in \mathbb{N}$, such that $\mathcal{B} = \bigcup_{n\in\mathbb{N}} \mathcal{B}_n$ is a union of finite pairwise disjoint sets $\mathcal{C}_n$ and $\{\bigcap \mathcal{C}_n : n \in \mathbb{N}\}$ is dense in $(CL(X), \tau_\Delta^+)$. Let $\mathcal{V} = \mathcal{B}^c$ and $\mathcal{W}_n = \mathcal{C}_n^c$, for each $n \in \mathbb{N}$. We now claim that $\mathcal{V} = \bigcup_{n\in\mathbb{N}} \mathcal{W}_n$ is a weakly Δ-groupable cover of X. Let $K \in \Delta$, $V_1, V_2, ..., V_m$ be open sets of X with $V_i \cap K^c \neq \phi$ $(1 \leq i \leq m)$. Then there exists a $n_0 \in \mathbb{N}$ such that $\bigcap \mathcal{C}_{n_0} \in (V_1, ..., V_m)_K^+$. Choose $x_i \in V_i \cap (\bigcap \mathcal{C}_{n_0})$, for $1 \leq i \leq m$. Now consider $F = \{x_i : 1 \leq i \leq m\}$. Hence $K \subset (\bigcap \mathcal{C}_{n_0})^c = \bigcup \mathcal{W}_{n_0}$ and $F \cap (\bigcup \mathcal{W}_{n_0}) = \phi$.

(ii) ⇒ (i): Let $\{\mathcal{D}_n : n \in \mathbb{N}\}$ be a sequence of dense subsets of $(CL(X), \tau_\Delta^+)$. For each $n \in \mathbb{N}$, let $\mathcal{U}_n = \mathcal{D}_n^c$. Then $\{\mathcal{U}_n : n \in \mathbb{N}\}$ is a sequence of $c_\Delta(CL(X))$-covers of X. By (ii), for each $n \in \mathbb{N}$, there is a finite subset $\mathcal{V}_n$ of $\mathcal{U}_n$ such that $\mathcal{V} = \bigcup_{n\in\mathbb{N}} \mathcal{V}_n$ is a weakly Δ-groupable cover of X. Thus $\mathcal{V} = \bigcup_{n\in\mathbb{N}} \mathcal{W}_n$ is a union of countably many finite pairwise disjoint sets $\mathcal{W}_n$ satisfying: for each subset $K \in \Delta$, open sets $V_1, V_2, ..., V_m$ of X with $V_i \cap K^c \neq \phi$ $(1 \leq i \leq m)$, there exist a n_0 and a finite set F with $F \cap V_i = \phi$, for $1 \leq i \leq m$ such that $K \subset \bigcup \mathcal{W}_{n_0}$ and $F \cap (\bigcup \mathcal{W}_{n_0}) = \phi$. Hence $\bigcap \mathcal{C}_{n_0} \in (V_1, ..., V_m)^+_K$. Let $\mathcal{B}_n = \mathcal{V}^c_n$ and $\mathcal{C}_n = \mathcal{W}^c_n$, for each $n \in \mathbb{N}$. Then $\mathcal{B}_n$ is finite set of $\mathcal{D}_n$ such that $\bigcup_{n\in\mathbb{N}} \mathcal{B}_n$ can be partitioned into a union $\bigcup_{n\in\mathbb{N}} \mathcal{C}_n$ of finite sets $\mathcal{C}_n$, for $n \in \mathbb{N}$, such that $\{\bigcap \mathcal{C}_n : n \in \mathbb{N}\}$ is dense in $(CL(X), \tau^+_\Delta)$. □

Recall that a space X is weakly Fréchet in the strict sense [24] if whenever $x \in ClA_n$ for all $n \in \mathbb{N}$, there are finite $F_n \subset A_n$ such that every neighbourhood of x intersects all but finitely many F_n.

Theorem 3.10. *For a space X, (Λ, τ^+_Δ) is weakly Fréchet in the strict sense if and only if each open subset $Y \subsetneq X$ with $Y^c \in \Lambda$ has $S_{fin}(\mathbb{C}^*_\Delta(\Lambda), \mathbb{C}^*_\Delta(\Lambda)^{gp})$.*

Proof. First let $Y \subsetneq X$ be such that $Y^c \in \Lambda$ and $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of $c_\Delta(\Lambda)$-covers of Y. Then by Lemma 2.1, $\{\mathcal{U}^c_n : n \in \mathbb{N}\}$ is a sequence of subsets of (Λ, τ^+_Δ) such that $Y^c \in Cl_{\tau^+_\Delta}\mathcal{U}^c_n$, for each $n \in \mathbb{N}$. Since (Λ, τ^+_Δ) is weakly Fréchet in the strict sense, there exist finite $\mathcal{V}^c_n \subset \mathcal{U}^c_n$, $n \in \mathbb{N}$, such that each neighbourhood of Y^c intersects all but finitely many $\mathcal{V}^c_n$. We now show that $\bigcup\{\mathcal{V}_n : n \in \mathbb{N}\}$ is a Δ-groupable cover of Y. Let $B \subseteq Y$ with $B \in \Delta$ and $V_1, ..., V_m$ be open subsets of X with $Y^c \cap V_i \neq \phi$, for $1 \leq i \leq m$ so that $(V_1, ..., V_m)^+_B \cap \Lambda$ is a τ^+_Δ-neighbourhood of Y^c in the space (Λ, τ^+_Δ). Thus there exists $n_0 \in \mathbb{N}$ such that $(V_1, ..., V_m)^+_B \cap \mathcal{V}^c_n \cap \Lambda \neq \phi$, for all $n \geq n_0$. Let $V^c_n \in \mathcal{V}^c_n$ be such that $V^c_n \in (V_1, ..., V_m)^+_B \cap \Lambda$ and choose $x^{(n)}_i \in V^c_n \cap V_i$, for $1 \leq i \leq m$. Now form the set $F_n = \{x^{(n)}_1, ..., x^{(n)}_m\}$. Then $F_n \in [X]^{<\omega}$ with $F_n \cap V_i \neq \phi$, for $1 \leq i \leq m$, $F_n \cap V_n = \phi$ and $B \subseteq V_n$, for all $n \geq n_0$.

Conversely, let $\{\mathcal{A}_n : n \in \mathbb{N}\}$ be a sequence of subsets of Λ and $E \in \Lambda$ be such that $E \in Cl_{\tau^+_\Delta}(\mathcal{A}_n)$, for $n \in \mathbb{N}$. Then $\{\mathcal{A}^c_n : n \in \mathbb{N}\}$ is a sequence of $c_\Delta(\Lambda)$-covers of E^c, for each $n \in \mathbb{N}$. Hence by the given condition there exist finite $\mathcal{B}^c_n \subset \mathcal{A}^c_n$, $n \in \mathbb{N}$, such that $\bigcup \mathcal{B}^c_n$ is a Δ-groupable cover of E^c. Hence (Λ, τ^+_Δ) is weakly Fréchet in the strict sense. □

4. Supertightness of (Λ, τ^+_Δ)

In [29], the authors have posed an open problem as: "Is it possible to characterize the supertightness of the hyperspace Λ by means of $c_\Delta(\Lambda)$-covers of Y, for some open subset $Y \subseteq X$?" In this section we give an affirmative answer to the question. Let us first recall that a family $\mathcal{P}$ of nonempty subsets of a space X is said to be a π-network at p [30] if every neighbourhood of p contains some member of $\mathcal{P}$.

Definition 4.1. [30, 24] A space X is said to have countable supertightness if $p \in X$ and $\mathcal{P}$ is a π-network at p consisting of finite subsets of X, then there is a countable subfamily $\mathcal{F} \subset \mathcal{P}$ such that $\mathcal{F}$ is a π-network at p.

We now define the following.

Definition 4.2. Let Y be a subspace of X. A partitioned $c_{\Delta}(\Lambda)$-cover $\mathcal{U} = \bigcup_{\alpha \in A} \mathcal{U}_{\alpha}$ (where $\mathcal{U} \subseteq \Lambda^c$) is called a finite p-$c_{\Delta}(\Lambda)$-cover of Y if each $\mathcal{U}_{\alpha}$ is finite and for any subset $B \subseteq Y$ with $B \in \Delta$, open sets $V_1, V_2, ..., V_m$ of X with $V_i \cap B^c \neq \phi$ $(1 \leq i \leq m)$, there exists $\alpha \in A$ and $F \in [X]^{<\omega}$ with $F \cap V_i \neq \phi$, for all $i = 1, 2, ..., m$ such that $B \subset U$ and $F \cap U = \phi$, for each $U \in \mathcal{U}_{\alpha}$.

Theorem 4.1. *For a space X, the following are equivalent:*

(i) $(\Lambda, \tau_{\Delta}^{+})$ has countable supertightness.
(ii) For each open subset $Y \subsetneq X$ with $Y^c \in \Lambda$ and each finite p-$c_{\Delta}(\Lambda)$-cover $\mathcal{U} = \bigcup_{\alpha \in A} \mathcal{U}_{\alpha}$ of Y, there exists a countable subset $A' \subset A$ such that $\bigcup_{\alpha \in A'} \mathcal{U}_{\alpha}$ is a finite p-$c_{\Delta}(\Lambda)$-cover of Y.

Proof. (i) $\Rightarrow$ (ii): Let $Y \subsetneq X$ be an open subset of X with $Y^c \in \Lambda$ and $\mathcal{U} = \bigcup_{\alpha \in A} \mathcal{U}_{\alpha}$ be a finite p-$c_{\Delta}(\Lambda)$-cover of Y. Then $\{\mathcal{U}_{\alpha}^{c} : \alpha \in A\}$ is a π-network at Y^c. Indeed let $Y^c \in (V_1, ..., V_m)_D^{+} \cap \Lambda$. Then there exists $\alpha \in A$ and $F \in [X]^{<\omega}$ with $F \cap V_i \neq \phi$, for all $i = 1, ..., m$ such that $D \subset U$ and $F \cap U = \phi$, for all $U \in \mathcal{U}_{\alpha}$. Then $U^c \in (V_1, ..., V_m)_D^{+} \cap \Lambda$, for each $U \in \mathcal{U}_{\alpha}$. Hence $\{\mathcal{U}_{\alpha}^{c} : \alpha \in A\}$ is a π-network at Y^c consisting of finite subsets of Λ. As $(\Lambda, \tau_{\Delta}^{+})$ has countable supertightness, there exists a countable subset $A' \subset A$ such that $\{\mathcal{U}_{\alpha}^{c} : \alpha \in A'\}$ is a π-network at Y^c. Hence $\bigcup_{\alpha \in A'} \mathcal{U}_{\alpha}$ is a finite p-$c_{\Delta}(\Lambda)$-cover of Y.

(ii) $\Rightarrow$ (i): Let $E \in \Lambda$ and $\{\mathcal{A}_{\alpha} : \alpha \in A\}$ be a π-network at E, where each $\mathcal{A}_{\alpha}$ is a finite subset of A. Then for any neighbourhood $(V_1, ..., V_m)_D^{+} \cap \Lambda$ of E, there exists $\alpha \in A$ such that $\mathcal{A}_{\alpha} \subset (V_1, ..., V_m)_D^{+} \cap \Lambda$. Let

$$A' = \{\alpha \in A : E^c \cap F^c \neq \phi, \text{ for each } F \in \mathcal{A}_{\alpha}\}.$$

Then $A' \neq \phi$ and $\{\mathcal{A}_{\alpha} : \alpha \in A'\}$ is a π-network at A. Hence $\bigcup_{\alpha \in A'} \mathcal{A}_{\alpha}^{c}$ is a finite p-$c_{\Delta}(\Lambda)$-cover of E^c. By (ii), there exists a countable family $\{\mathcal{A}_{\alpha_n} : n \in \mathbb{N}\} \subset \{\mathcal{A}_{\alpha} : \alpha \in A'\}$ such that $\bigcup_{n \in \mathbb{N}} \mathcal{A}_{\alpha_n}^{c}$ is a finite p-$c_{\Delta}(\Lambda)$-cover of E^c. Hence $\{\mathcal{A}_{\alpha_n} : n \in \mathbb{N}\}$ is a π-network at E, so that $(\Lambda, \tau_{\Delta}^{+})$ has countable supertightness. $\square$

Definition 4.3. [4] A space X is supertight at $p \in X$ if whenever $\mathcal{P}$ is a π-network at p consisting of countable subsets of X, there is a countable subfamily $\mathcal{F} \subset \mathcal{P}$ such that $\mathcal{F}$ is a π-network at p. A space is supertight if all its points are supertight.

Definition 4.4. Let Y be a subspace of X. A partitioned $c_\triangle(\Lambda)$-cover $\mathcal{U} = \bigcup_{\alpha \in A} \mathcal{U}_\alpha$ (where $\mathcal{U} \subseteq \Lambda^c$) is called a countable p-$c_\triangle(\Lambda)$-cover of Y if each $\mathcal{U}_\alpha$ is countable and for any subset $B \subseteq Y$ with $B \in \Delta$, open sets $V_1, V_2, ..., V_m$ of X with $V_i \cap B^c \neq \phi (1 \leq i \leq m)$, there exists $\alpha \in A$ and $F \in [X]^{<\omega}$ with $F \cap V_i \neq \phi$, for all $i = 1, 2, ..., m$ such that $B \subset U$ and $F \cap U = \phi$, for each $U \in \mathcal{U}_\alpha$.

Theorem 4.2. *For a space X, the following are equivalent:*

(i) $(\Lambda, \tau^+_\triangle)$ *is supertight.*
(ii) For each open subset $Y \subseteq X$ with $Y \neq X$ and each countable p-$c_\triangle(\Lambda)$-groupable cover $\mathcal{U} = \bigcup_{\alpha \in A} \mathcal{U}_\alpha$ of Y, there exists a countable subset $A' \subset A$ such that $\bigcup_{\alpha \in A'} \mathcal{U}_\alpha$ is a countable p-$c_\triangle(\Lambda)$-cover of Y.

Proof. Same as Theorem 4.1. □

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