FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 39, No 4 (2024), 667–688
https://doi.org/10.22190/FUMI230331044A
Original Scientific Paper

AN APPROACH TO SEMIHYPERMODULES OVER SEMIHYPERRINGS

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Abstract. In this paper, we introduce semihypermodules over semihyperrings as a generalization of semimodules over semirings. Besides studying their properties, we introduce an equivalence relation on them and use it to define factor semihypermodules. Moreover, we discuss the (semi-)isomorphism theorems for semihypermodules and present some of their interesting applications. Finally, we project our results on semihyperrings and deduce the (semi-)isomorphism theorems for semihyperrings.
Keywords: semihypermodules, semihyperrings.

1. Introduction

Naturally generalizing the concept of a group, by considering the result of the "interaction" between two elements of a non-empty set to be a non-empty set of elements (and not only one element, as for groups), Frederic Marty [14] defined the concept of a hypergroup. He presented it during the 8th congress of Scandinavian Mathematicians, held in Stockholm in 1934. The law characterizing such a structure is called *hyperoperation* and the theory of the algebraic structures endowed with at least one multi-valued operation is known as the *Hyperstructure Theory* or *Hypercompositional Algebra*. Marty's motivation to introduce hypergroups is that

Received March 31, 2023, accepted: August 11, 2024
Communicated by Abdullah Alazemi
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2020 *Mathematics Subject Classification.* Primary 16Y99; Secondary 20N20

the quotient of a group modulo any subgroup (and not necessarily a normal subgroup) is a hypergroup. Nowadays, this theory is characterized by huge diversity of character and content, and can be used to present the results in mathematics and other sciences such as physics, chemistry, biology, computer science, information technologies, social sciences, etc. Several books were written on this theory and its applications. In this regard, we refer to [2, 3, 9, 6, 7, 8, 18]

Semirings, the most natural common generalization of the theories of rings and bounded distributive lattices, abound in the mathematical world around us. The set of natural numbers under standard addition and multiplication is the easiest example of a semiring that is not a ring. Other semirings arise naturally in such diverse areas of mathematics such as functional analysis, combinatorics, graph theory, topology, commutative and non commutative ring theory, etc. Historically, semirings first appeared implicitly in Dedekind work in 1894 [11] in connection with the study of ideals of a ring. They also appeared later in connection with the axiomatization of the natural numbers and non-negative rational numbers. Semirings were first considered explicitly in Vandiver work [17] in connection with the axiomatization of arithmetic of natural numbers. Vandiver's approach was later developed in a series of articles by him and by other researchers. Over the years, semirings have been studied by various researchers either from theoretical point of view, in an attempt to broaden techniques coming from semigroup theory or ring theory, or in connection with applications. As a generalization of semirings, Ameri and Hedayati in 2007 [1] gave the notions of semihyperrings and studied the k-hyperideals of them. Later, Davvaz [5] gave the concepts of ternary semihyperrings and investigated their fuzzy hyperideals.

The semimodules over a semiring are an important tool in characterizing properties of the semiring. Moreover, many important constructions in pure and applied mathematics can be understood as semimodules over appropriate semirings. As a genralization of semimodules over semirings, our paper is concerned about semihypermodules over semihyperring and it is constructed as follows: After an Introduction, in Section 2, we present some results and examples about semirings and semihyperrings. In Section 3, we define semihypermodules over semihyperrings, present some examples, and study some of their properties. In Section 4, we define congruence relations on semihypermodules and use them to define factor semihypermodules. In Section 5, we derive (semi)-isomorphism theorems for semihypermodules and present some applications on them. Finally, in Section 6, we use our results on semihypermodules to derive (semi)-isomorphism theorems for semihyperrings.

2. Semirings and Semihyperrings

In this section, we present some results and examples about semirings and semihyperrings. For more details, we refer to [4, 9, 7, 10, 12].

Definition 2.1. [12] Let R be a non-empty set with two operations "+" and "$\cdot$".

Then $(R, +, \cdot)$ is called a *semiring* if the following conditions hold:

1. $(R, +)$ is a commutative semigroup with identity "0";
2. $(R, \cdot)$ is a semigroup;
3. $x \cdot (y + z) = x \cdot y + x \cdot z$ and $(x + y) \cdot z = x \cdot z + y \cdot z$ for all $x, y, z \in R$;
4. $x \cdot 0 = 0 \cdot x = 0$ for all $x \in R$.

Example 2.1. [12] Let $R = \{(a, b) \in \mathbb{R}^2 : a > 0, b > 0\} \cup \{(0, 0)\}$ and define "+ and "·" on R as follows:

$$(a, b) + (a', b') = \begin{cases} (a, b) & \text{if } b > b'; \\ (a', b') & \text{if } b < b'; \\ (a + a', b) & \text{if } b = b'. \end{cases}$$

and $(a, b) \cdot (a', b') = (aa', bb')$. Then $(R, +, \cdot)$ is a semiring.

Example 2.2. Let $\mathbb{N}$ be the set of non-negative integers and $\mathbb{R}$ be the set of real numbers. Then $(\mathbb{N}, +, \cdot)$, $(\mathbb{R} \cup \{-\infty\}, \vee, +)$, and $(\mathbb{R} \cup \{\pm\infty\}, \vee, \wedge)$ are infinite semirings. Here "$\vee$" and "$\wedge$" denote the maximum and minimum respectively.

Finite semirings can be presented by means of Cayley's tables.

Example 2.3. [15] Let $R = \{0, a, b, c\}$ and define $(R, +, \cdot)$ by the following tables.

$+$	0	a	b	c
0	0	a	b	c
a	a	a	a	a
b	b	a	b	c
c	c	a	c	c

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	a	a	a
b	0	b	b	b
c	0	a	a	a

Then $(R, +, \cdot)$ is a semiring.

Let H be a non-empty set. Then, a mapping $\circ : H \times H \to \mathcal{P}^*(H)$ is called a *binary hyperoperation* on H, where $\mathcal{P}^*(H)$ is the family of all non-empty subsets of H. The couple $(H, \circ)$ is called a *hypergroupoid*. In this definition, if A and B are two non-empty subsets of H and $x \in H$, then we define:

$$A \circ B = \bigcup_{\substack{a \in A \\ b \in B}} a \circ b, \;\; x \circ A = \{x\} \circ A \text{ and } A \circ x = A \circ \{x\}.$$

A hypergroupoid $(H, \circ)$ is called a *semihypergroup* if for every $x, y, z \in H$, $x \circ (y \circ z) = (x \circ y) \circ z$, that is

$$\bigcup_{u \in y \circ z} x \circ u = \bigcup_{v \in x \circ y} v \circ z.$$

The more general structure that satisfies the ring-like axioms is the hyperring in the general sense. There are different notions of hyperrings. A special case of this type is the hyperring introduced by Krasner [13] in 1983, known as Krasner hyperring, and multiplicative hyperring introduced by Rota [16] in 1982, where in the latter, the multiplication is a hyperoperation, while the addition is an operation.

Definition 2.2. [16] Let R be a non-empty set. Then $(R, +, \cdot)$ is called a *multiplicative hyperring* if the following conditions hold:

1. $(R, +)$ is an abelian group with identity "0";
2. $(R, \cdot)$ is a semihypergroup;
3. $x \cdot (y + z) = x \cdot y + x \cdot z$ and $(x + y) \cdot z = x \cdot z + y \cdot z$ for all $x, y, z \in R$;
4. $0 \in (x \cdot 0) \cap (0 \cdot x)$ for all $x \in R$.

In [1], Ameri and Hedayati discussed additive semihyperrings, i.e., "+" is a hyperoperation and "$\cdot$" is an operation. In this paper, we consider multiplicative semihyperrings.

Definition 2.3. [10] Let R be a non-empty set. Then $(R, +, \cdot)$ is called a *semihyperring* if the following conditions hold:

1. $(R, +)$ is a semirgroup with identity "0";
2. $(R, \cdot)$ is a semihypergroup;
3. $x \cdot (y + z) \subseteq x \cdot y + x \cdot z$ and $(x + y) \cdot z \subseteq x \cdot z + y \cdot z$ for all $x, y, z \in R$;
4. $0 \in (x \cdot 0) \cap (0 \cdot x)$ for all $x \in R$.

Remark 2.1. Every semiring is a semihyperring and every multiplicative hyperring is a semihyperring.

A semihyperring $(R, +, \cdot)$ is called *commutative* if $(R, +)$ and $(R, \cdot)$ are commutative and we say that it has a *unity* if there exist $1 \in R$ such that $r \in (r \cdot 1) \cap (1 \cdot r)$ for all $r \in R$. It is called a *zero-sum free* if whenever $r + s = 0$ then either $r = 0$ or $s = 0$ and called *additively idempotent* if $r + r = r$ for all $r \in R$. A commutative semihyperring with unity is called *semihyperfield* if for every $r \in R - \{0\}$ there exist $s \in R - \{0\}$ such that $1 \in (r \cdot s) \cap (s \cdot r)$.

If $(R, +, \cdot)$ is a finite semihyperring, we can present it by means of Cayley's tables.

Example 2.4. Let $R = \{a, b\}$ and $(R, +, \cdot)$ be defined by the following tables.

$+$	a	b
a	a	a
b	a	b

$\cdot$	a	b
a	a	b
b	b	R

Then $(R, +, \cdot)$ is an idempotent semihyperfield that is also zero-sum free where a is the zero of R and b is it's unity.

Example 2.5. [10] Let $R = \{a, b, c\}$ and $(R, +, \cdot)$ be defined by the following tables.

$+$	a	b	c
a	a	a	a
b	a	b	c
c	a	c	c

$\cdot$	a	b	c
a	a	b	c
b	b	R	b
c	c	b	$\{a, c\}$

Then $(R, +, \cdot)$ is an idempotent semihyperfield that is also zero-sum free.

Example 2.6. [10] Let $R = \{-1, 0, 1\}$ and $(R, +, \cdot)$ be defined by the following tables.

$+$	-1	0	1
-1	-1	0	1
0	0	0	1
1	1	1	1

$\cdot$	-1	0	1
-1	-1	-1	R
0	-1	0	1
1	R	1	1

Then $(R, +, \cdot)$ is an idempotent semihyperring that it is also zero-sum free.

Example 2.7. [10] The idempotent semihyperring $(\mathbb{R} \cup \{-\infty\}, \wedge, \otimes)$ is defined as follows: For all $x, y \in \mathbb{R}$,

$$x \otimes y = \begin{cases} x \vee y & \text{if } x \neq y \\ \{t \in R \ : \ t \leq x\} & \text{if } x = y. \end{cases}$$

Definition 2.4. [1] Let $(R, +, \cdot)$ be a semihyperring. A subset I of R is called a

(1) *subsemihyperring* of R if $x + y \in I$ and $x \cdot y \subseteq I$ for all $x, y \in I$;

(2) *hyperideal* of R if I is subsemihyperring of R and $x \cdot y \subseteq I$ and $y \cdot x \subseteq I$ for all $x \in I$ and $y \in R$.

Remark 2.2. If $0 \in R$ such that $0 \cdot r = r \cdot 0 = 0$ for all $r \in R$ then $\{0\}$ is a hyperideal of R.

Example 2.8. Let $(R, +, \cdot)$ be the semihyperring in Example 2.6. Then $\{-1, 0\}$ is a subsemihyperring of R that is not a hyperideal of R. This is clear as $1 \cdot (-1) = R \nsubseteq \{-1, 0\}$.

3. Semihypermodules over Semihyperrrings

Inspired by the definition of semimodules over semirings, we define semihypermodules over semihyperrings. Moreover, we pressent some of its properties and provide different examples.

Definition 3.1. [12] Let $(M, +)$ be a commutative semigroup with 0_M, $(R, +_R, \cdot)$ be a semiring with 0_R, and define $\star : R \times M \to M$ as $(r, m) \to r \star m$. Then M is called *(left) R-semimodule* if the following conditions hold: For all $r, s \in R$, $m, n \in M$,

1. $r \star (s \star m) = (r \cdot s) \star m$;

2. $r \star (m + n) = r \star m + r \star n$;

3. $(r +_R s) \star m = r \star m + s \star m$;

4. $0_R \star m = r \star 0_M = 0_M$.

Remark 3.1. Let $(R, +, \cdot)$ be a semiring. Then R is an R-semimodule and every ideal of R is an R-semimodule.

Example 3.1. Let $(R, +, \cdot)$ be the semiring in Example 2.3. Then $\{0, a\}$ and $\{0, a, b\}$ are R-semimodules.

Definition 3.2. Let $(M, +)$ be a group, $(R, +_R, \cdot)$ be a multiplicative hyperring, and define $\star : R \times M \to P^*(M)$ as $(r, m) \to r \star m$. Then M is called a *(left) R-hypermodule* if the following conditions hold: For all $r, s \in R$, $m, n \in M$,

1. $r \star (s \star m) = (r \cdot s) \star m$;

2. $r \star (m + n) = r \star m + r \star n$;

3. $(r +_R s) \star m = r \star m + s \star m$.

Definition 3.3. Let $(M, +)$ be a semigroup, $(R, +_R, \cdot)$ be a semihyperring, and define $\star : R \times M \to P^*(M)$ as $(r, m) \to r \star m$. Then M is called *(left) R-semihypermodule* if the following conditions hold: For all $r, s \in R$, $m, n \in M$,

1. $r \star (s \star m) = (r \cdot s) \star m$;

2. $r \star (m + n) \subseteq r \star m + r \star n$;

3. $(r +_R s) \star m \subseteq r \star m + s \star m$.

Remark 3.2. Every R-semimodule and every R-hypermodule is an R-semihypermodule.

Proposition 3.1. *Let $(R, +, \cdot)$ be a semihyperring. Then every hyperideal of R is an R-semihypermodule.*

Proof. The proof is straightforward. $\square$

In what follows, we write rs instead of $r \cdot s$ and rx instead of $r \star x$ for all $r, s \in R$ and $x \in M$.

Proposition 3.2. *Let $(M, +)$ be any semigroup with identity 0_M, $R = \mathbb{N}$, and $(R, +, \cdot)$ be the semiring under standard addition and multiplication of non-negative integers. Then M is an R-semihypermodule where "$\star : R \times M \to P^*(M)$" is defined as follows:*

$$r \star m = \begin{cases} 0_M & \text{if } r = 0; \\ \{0_M, m\} & \text{if } r > 0. \end{cases}$$

Proof. Let $r, s \in R$ and $x, y \in M$. (1) We have

$$r(sx) = \begin{cases} 0_M & \text{if } r = 0; \\ \{0_M, sx\} & \text{if } r > 0. \end{cases} = \begin{cases} 0_M & \text{if } r = 0 \text{ or } s = 0; \\ \{0_M, x\} & \text{if } r > 0 \text{ and } s > 0. \end{cases} = (rs)x.$$

(2) We have

$$r(x+y) = \begin{cases} 0_M & \text{if } r = 0; \\ \{0, x+y\} & \text{if } r > 0. \end{cases} \subseteq \begin{cases} 0_M & \text{if } r = 0; \\ \{0_M, x, y, x+y\} & \text{if } r > 0. \end{cases} = rx+ry.$$

(3) We have

$$(r+s)x = \begin{cases} 0_M & \text{if } r = s = 0; \\ \{0_M, x\} & \text{otherwise.} \end{cases}$$

and

$$rx + sx = \begin{cases} 0_M & \text{if } r = s = 0; \\ \{0_M, x, x+x\} & \text{if } r > 0 \text{ and } s > 0; \\ \{0_M, x\} & \text{otherwise.} \end{cases} .$$

It is clear that $(r+s)x \subseteq rx + sx$. Therefore, M is an R-semihypermodule. □

Example 3.2. Let $R = \mathbb{N}$ and $(R, +, \cdot)$ be the semiring under standard addition and multiplication of non-negative integers. Using Proposition 3.2, we get that $(\mathbb{N}, +)$ and $(\mathbb{N}, \vee)$ are both R-semihypermodules where

$$r \star m = r \star m = \begin{cases} 0 & \text{if } r = 0; \\ \{0, m\} & \text{if } r > 0. \end{cases}$$

Proposition 3.3. *Let $(R, +_R, \cdot)$ be a semihyperring, E be any non-empty set, and $R^E = \{f : E \to R\}$. Then R^E is an R-semihypermodule. Here, for all $f, g \in R^E, r \in R$, $x \in E$, we have $(f+g)(x) = f(x) +_R g(x)$ and $r \star f(x) = r \cdot (f(x))$.*

Proof. The proof is straightforward. □

Example 3.3. Let $(R, +, \cdot)$ be the semihyperring in Example 2.4 and $E = \{1, 2\}$. By setting $f(1) = f(2) = a$, $g(1) = g(2) = b$, $h(1) = a, h(2) = b$, and $i(1) = b, i(2) = a$, we get $R^E = \{f, g, h, i\}$. We present the R-semihypermodule R^E by the following tables.

$+$	f	g	h	i
f	f	f	f	f
g	f	g	h	i
h	f	h	h	f
i	f	i	f	i

$\star$	f	g	h	i
a	f	g	h	i
b	g	R^E	$\{g, i\}$	$\{g, h\}$

Proposition 3.4. *Let $(R, +, \cdot)$ be semihyperring and M_α be an R-semihypermodule for every $\alpha \in \Gamma$. Then $\prod_{\alpha \in \Gamma} M_\alpha$ is an R-semihypermodule. Here $(x_\alpha) \oplus (y_\alpha) = (x_\alpha +_\alpha y_\alpha)$ and $r \star (x_\alpha) = (r \star_\alpha x_\alpha)$.*

Proof. The proof is straightforward. □

Corollary 3.1. *Let $(R, +, \cdot)$ be a semihyperring, n a positive integer, and $V_n(R) = \{(a_1, \ldots, a_n) : a_i \in R, 1 \leq i \leq n\}$. Then $V_n(R)$ is an R-semihypermodule.*

Proof. The proof follows from Proposition 3.4. □

Example 3.4. Let $(R, +, \cdot)$ be the semihyperring in Example 2.4. Then $V_2(R)$ is an R-semihypermodule and it is presented by the following tables.

$+$	(a,a)	(a,b)	(b,a)	(b,b)
(a,a)	(a,a)	(a,a)	(a,a)	(a,a)
(a,b)	(a,a)	(a,b)	(a,a)	(a,b)
(b,a)	(a,a)	(a,a)	(b,a)	(b,a)
(b,b)	(a,a)	(a,b)	(b,a)	(b,b)

$\star$	(a,a)	(a,b)	(b,a)	(b,b)
a	(a,a)	(a,b)	(b,a)	(b,b)
b	(b,b)	$\{(b,a),(b,b)\}$	$\{(a,b),(b,b)\}$	$V_2(R)$

Proposition 3.5. *Let $(R, +, \cdot)$ be a semihyperring, $M_{n,k}(R)$ be the set of all $n \times k$ matrices with entries from R. Then $M_{n,k}(R)$ is an R-semihypermodule. Here, $(a_{ij})+(b_{ij}) = (a_{ij}+b_{ij})$ and $r\star(a_{ij}) = (r\star a_{ij})$ for all matrices $(a_{ij}), (b_{ij}) \in M_{n,k}(R)$ and $r \in R$.*

Proof. The proof is straightforward. □

Definition 3.4. Let $(R, +, \cdot)$ be a semihyperring and M be an R-semihypermodule. A non-empty subset N of M is a subsemihypermodule of M if and only if it is an R-semihypermodule.

Proposition 3.6. *Let $(R, +, \cdot)$ be a semihyperring and M be an R-semihypermodule. A non-empty subset N of M is a subsemihypermodule of M if and only if $N+N \subseteq N$ and $r \star N \subseteq N$ for all $r \in R$.*

Proof. The proof is straightforward. □

Remark 3.3. Let $(R, +, \cdot)$ be a semihyperring and M be an R-semihypermodule. If $0 \in M$ and $r \star 0 = 0$ for all $r \in R$ then $\{0\}$ is a subsemihypermodule of M (beside M).

A subsemihypermodule of M is called *subtractive* if $x + y \in N$ $(y + x \in N)$ and $y \in N$ then $x \in N$.

Example 3.5. Let $M = (\mathbb{N}, +)$ be the R-semihypermodule defined in Example 3.2. Then $N = 2M = \{0, 2, 4, 6, 8, 10, \ldots\}$ is a subtractive subsemihypermodule of M. In general, $nM = \{0, n, 2n, 3n, \ldots\}$ is a subtractive subsemihypermodule of M for all $n \in \mathbb{N}$.

Proposition 3.7. *Let $M = (\mathbb{N}, +)$ be the R-semihypermodule defined in Example 3.2 and N be a non-trivial subsemihypermodule of M. Then N is subtractive in M if and only if N is a positive multiple of M.*

Proof. If N is a positive multiple of M then by Example 3.5, we get that N is subtractive in M.

Conversely, let $N \neq \{0\}$ be a subtractive subsemihypermodule of M. Since M consists of non-negative integers, it follows that there exist a least positive integer, say n in N. Let $x \in M$. Using Division Algorithm, we get that there exist $q \geq 0, 0 \leq r < x$ such that $x = qn + r$. Since $x, qn \in N$, it follows that $r \in N$ (as N is subtractive.). Thus, $r = 0$ and hence, x is a multiple of n. □

Proposition 3.8. *Let R be a multiplicative hyperring and M be an R-hypermodule. Then every subhypermodule of M is subtractive.*

Proof. The proof is straightforward as $(M, +)$ is a group. □

Remark 3.4. Not every subsemihypermodule is subtractive.

Example 3.6. Let $M = (\mathbb{N}, +)$ be the R-semihypermodule defined in Example 3.2. Then $N = 2M - \{2\} = \{0, 4, 6, 8, 10, \ldots\}$ is a subsemihypermodule of M. But it is not a subtractive subsemihypermodule of M because it can not be written as a positive multiple of M (using Proposition 3.7). Or one can easily see that $2 + 4 = 6 \in N$, $4, 6 \in N$ and $2 \notin N$.

Example 3.7. Let $R = \mathbb{N}$ and $(R, +, \cdot)$ be the semiring under standard addition and multiplication of non-negative integers. And $(\mathbb{N}, \vee)$ be the R-semihypermodule described in Example 3.2 where

$$r \star m = \begin{cases} 0 & \text{if } r = 0; \\ \{0, m\} & \text{if } r > 0. \end{cases}$$

Let N be any non-empty subset of $\mathbb{N}$ containing 0 and $x \leq y \in N$. Since $x \vee y = y \in N$ and $r \star x \subseteq \{0, x\} \subseteq N$ then N is a subsemihypermodule of $\mathbb{N}$. Moreover, the only proper subtractive subsemihypermodules of $\mathbb{N}$ are of the form $A_n = \{0, 1, \ldots, n\}$ for every non-negative integer n. This is clear as if $N \neq A_n$ for all $n \in \mathbb{N}$ then there exist $x, y \in N$, $z \in \mathbb{N}$ with $x < z < y$ and $z \notin N$. Then having $z \vee y = y \in N$ and $z \notin N$ contradicts our assumption that N is subtractive.

Proposition 3.9. *Let $(R, +, \cdot)$ be a semihyperring, M be an R-semihypermodule, and N_α be a subsemihypermodule of M. Then $\bigcap_{\alpha \in \Gamma} N_\alpha$ is a subsemihypermodule of M. Moreover, if N_α is subtractive then so $\bigcap_{\alpha \in \Gamma} N_\alpha$.*

Proof. The proof is straightforward. □

Proposition 3.10. *Let R be a semihyperring, M be a commutative R-semihypermodule, and N_1, N_2 be subsemihypermodules of M. Then $N_1 + N_2$ is a subsemihypermodule of M.*

Proof. Let $x, y \in N_1 + N_2$ and $r \in R$. Then there exist $n_1, n_1' \in N_1$, $n_2, n_2' \in N_2$ such that $x = n_1 + n_2$ and $y = n_1' + n_2'$. Since M is commutative, it follows that $x + y = n_1 + n_2 + n_1' + n_2' = n_1 + n_1' + n_2 + n_2' \in N_1 + N_2$. Also, we have $r \star (n_1 + n_2) \subseteq r \star n_1 + r \star n_2 \subseteq N_1 + N_2$. Thus, $N_1 + N_2$ is a subsemihypermodule of M. □

Definition 3.5. Let $(R, +, \cdot)$ be a semihyperring and M, N be R-semihypermodules. A function $f : M \to N$ is called a

1. *homorphism* if $f(x+y) = f(x)+f(y)$ and $f(rx) = rf(x)$ for all $x, y \in M$ and $r \in R$;
2. *isomorphism* if f is a bijective homomorphism. In this case, we say that M and N are isomorphic R-semihypermodules and we write $M \cong N$.

Example 3.8. Let $(R, +, \cdot)$ be a semihyperring and M be an R-semihypermodule. Then the identity map ($f : M \to M$ defined as $f(x) = x$ for all $x \in M$) defines an isomorphism.

Example 3.9. Let $(R, +, \cdot)$ be a semihyperring and M, N be R-semihypermodules with $0 \in N$ and $r \star 0 = 0$ for all $r \in R$. Then $f : M \to N$ defined as $f(x) = 0$ for all $x \in M$ defines a homomorphism. This homomorphism is known by the *trivial homomorphism.*

In what follows, all R-semihypermodules and their subsemihypermodules have an identity 0 and $r \star 0 = 0$ for all $r \in R$. Also, if f is an R-semihypermodule homomorphism then $f(0) = 0$.

Definition 3.6. Let $(R, +, \cdot)$ be a semihyperring, M, N be R-semihypermodules, and $f : M \to N$ be a homomorphism. Then the *kernel* of f, denoted by $ker(f)$ is defined as $ker(f) = \{m \in M : f(m) = 0\}$. And *image* of f, denoted by $im(f)$, is defined as $im(f) = \{f(m) : m \in M\}$.

Proposition 3.11. *Let $(R, +, \cdot)$ be a semihyperring, M, N be R-semihypermodules, and $f : M \to N$ be a homomorphism. Then $ker(f)$ is a subtractive subsemihypermodule of M.*

Proof. Let $x, y \in ker(f)$. Having $f(x + y) = f(x) + f(y) = 0 + 0 = 0$ and $f(rx) = rf(x) = r(0) = 0$ implies that $x + y \in ker(f)$ and $rx \subseteq ker(f)$. Thus, $ker(f)$ is a subsemihypermodule of M. To prove that $ker(f)$ is substractive, let $x, x + y \in ker(f)$. Then $f(x + y) = f(x) + f(y) = 0$. Having $f(x) = 0$ implies that $f(y) = 0$ and hence, $y \in ker(f)$. Therefore, $ker(f)$ is subtractive. □

Proposition 3.12. *Let $(R, +, \cdot)$ be a semihyperring, M, N be R-semihypermodules, and $f : M \to N$ be a homomorphism. Then $im(f)$ is a subsemihypermodule of N.*

Proof. The proof is straightforward. □

Definition 3.7. Let $(R, +, \cdot)$ be a semihyperring, M, N be R-semihypermodules, and $f : M \to N$ be a homomorphism. Then f is called *semi-isomorphism* if f is a surjective homomorphism and $ker(f) = \{0\}$. And we write $M \cong_s N$.

Remark 3.5. Let $(R, +, \cdot)$ be a semihyperring, M, N be R-semihypermodules, and $f : M \to N$ be a homomorphism. If f is injective then $ker(f) = \{0\}$. Thus, every isomorphism is a semi-isomorphism.

4. Factor Semihypermodules

In this section, we define congruence relations on semihypermodules and use them to discuss factor semihypermodules.

Definition 4.1. Let $(R, +, \cdot)$ be a semihyperring, M be an R-semihypermodule, and ρ an equivalence relation on M. Then ρ is called a *congruence relation* on M if (1) $x\rho y$ and $z\rho w$ implies $(x+z)\rho(y+w)$ and (2) $x\rho y$ implies $(r \star x)\rho(r \star y)$ for all $r \in R$.

If A, B are non-empty sets, by $A\rho B$, we mean that for every $a \in A$ there exist $b \in B$ such that $a\rho b$ and for every $b \in B$ there exist $a \in A$ such that $a\rho b$.

Let $M/\rho = \{[m] : m \in M\}$ be the set of all equivalence classes of M with respect to the relation ρ and define $\oplus$ and $\odot$ as follows: $[x] \oplus [y] = [x+y]$ and $r \odot [x] = [r \star x] = \{[t] : t \in rx\}$ for all $x, y \in M$ and $r \in R$.

Proposition 4.1. *"$\oplus$" and "$\odot$" are well defined.*

Proof. Let $[x] = [y]$ and $[z] = [w]$ in M/ρ. Then $x\rho y$ and $z\rho w$. Since ρ is a congruence on M, it follows that $(x+z)\rho(y+w)$. Thus, $[x+z] = [y+w]$.

Let $[x] = [y]$ in M/ρ. Then having $x \in [y]$ implies that $x\rho y$. Having ρ a congruence relation on M implies that $(rx)\rho(ry)$. Let $t \in rx$. Then there exist $t' \in ry$ such that $t\rho t'$. The latter implies that $[rx] \subseteq [ry]$. Thus, $[rx] \subseteq [ry]$. Similarly, we get $[ry] \subseteq [rx]$. $\square$

Theorem 4.1. *Let $(R, +, \cdot)$ be a semihyperring, M be an R-semihypermodule, and ρ a congruence relation on M. Then M/ρ is an R-semihypermodule.*

Proof. The proof is straightforward. $\square$

NOTATION 1. *M/ρ is called the factor semihypermodule.*

Remark 4.1. Every R-semihypermodule has at least two congruence relations: the **trivial congruence** $(\sim_t)$ and the **universal congruence** $(\sim_u)$, where $m \sim_t n$ if and only if $m = n$ and $m \sim_u n$ if and only if $m, n \in M$. It is clear that $M/\sim_t \cong M$ and $M/\sim_u \cong \{0\}$.

Lemma 4.1. *Let $(R, +, \cdot)$ be a semihyperring, M be an R-semihypermodule, and ρ a congruence relation on M. Then $f : M \to M/\rho$ is a surjective homomorphism. Here, $f(m) = [m]$ for all $m \in M$.*

Proof. It is clear that f is surjective since $m \in [m]$ for all $m \in M$. Let $x, y \in M$ and $r \in R$. Then $f(x+y) = [x+y] = [x] \oplus [y] = f(x) \oplus f(y)$ and $f(rx) = [rx] = r \odot [x] = r \odot f(x)$. Thus, f is a homomorphism. $\square$

Proposition 4.2. *Let $(R,+,\cdot)$ be a semihyperring, M,N be R-semihypermodules, and $f:M\to N$ be a homomorphism. Define $\sim_f$ on M as follows:*

$$x \sim_f y \Leftrightarrow f(x)=f(y) \text{ for all } x,y\in M.$$

Then $\sim_f$ is a congruence relation on M. Moreover, if f is injective then $M/\sim_f \cong M$.

Proof. It is clear that $\sim_f$ is an equivalence relation on M. To prove that $\sim_f$ is a congruence, let $r\in R$, $x\sim_f y$, and $z\sim_f w$. Having $f(x)=f(y)$, $f(z)=f(w)$, and f is homomorphism implies that $f(x+z)=f(y+w)$. Thus, $(x+z)\sim_f (y+w)$. Also, we get that $f(rx)=rf(x)=rf(y)=f(ry)$. Thus, $rx\sim_f ry$.

Let f be injective and $x\sim_f y$. Then $f(x)=f(y)$ implies that $x=y$. The latter implies that $x\sim_t y$. Thus, $M/\sim_f\cong M/\sim_t\cong M$. □

Proposition 4.3. *Let R be a semihyperring, M be a commutative R-semihypermodule, N subsemihypermodule of M, and define $\sim_N$ on M as follows:*

$$x\sim_N y \Leftrightarrow \text{there exist } n_1,n_2\in N \text{ with} x+n_1=y+n_2.$$

Then $\sim_N$ is an equivalence relation on M. Moreover, if $x\sim_N y$ and $z\sim_N w$ then $(x+z)\sim_N (y+w)$.

Proof. It is clear that $\sim_N$ is reflexive and symmetric. To prove that $\sim_N$ is transitive, let $x\sim_N y$ and $y\sim_N z$. Then there exist $n_1,n_2,n_3,n_4\in N$ such that $x+n_1=y+n_2$ and $y+n_3=z+n_4$. Having M commutative implies that $x+n_1+n_3=y+n_2+n_3=y+n_3+n_2=z+n_4+n_2$. Having $n_1+n_3,n_4+n_2\in N$ implies that $x\sim_N z$.

Let $x\sim_N y$ and $z\sim_N w$. Then there exist $n_1,n_2,n_3,n_4\in N$ such that $x+n_1=y+n_2$, $z+n_3=w+n_4$. The latter and having M commutative implies that $x+z+n_1+n_3=y+w+n_2+n_4$. Thus, $(x+z)\sim_N (y+w)$. □

Let R be a semihyperring, M be a commutative R-semihypermodule, and N subsemihypermodule of M. If $x\sim_N y$ and $r\in R$ then $(rx)\sim_N (ry)$ may not be satisfied.

Definition 4.2. Let R be a semihyperring, M a commutative R-semihypermodule, and N a subsemihypermodule of M. If $\sim_N$ defines a congruence on M then N is called a *congruence subsemihypermodule*.

Proposition 4.4. *Let R be a multiplicative hyperring, M be a commutative R-hypermodule, and N subhypermodule of M. If $x\sim_N y$ and $r\in R$ then $(rx)\sim_N (ry)$.*

Proof. Let $x\sim_N y$. Then there exist $n_1,n_2\in N$ such that $x+n_1=y+n_2$ and hence $rx+rn_1=r(x+n_1)=r(x+n_2)=rx+rn_2$. Having $rx+rn_1=rx+rn_2$ implies that for every $z_1\in rx$, there exist $n,n'\in N$ and $z_2\in ry$ such that $z_1+n=z_2+n'$. Thus, $z_1\sim_n z_2$. Similarly, we can take $w_1\in ry$ and show that there exist $w_2\in rx$ such that $w_1\sim_N w_2$. □

Corollary 4.1. *Every subhypermodule of an R-hypermodule is a congruence subhypermodule.*

Proof. The proof follows from Propositions 4.3 and 4.4. □

In the next proposition, Proposition 4.5, we show that there exist subsemihypermodules of semihypermodules (and are not hypermodules) that are congruence subsemihypermodules.

Proposition 4.5. *Let $M = (\mathbb{N}, +)$ be the R-semihypermodule defined in Example 3.2. Then every subsemihypermodule of M is a congruence subsemihypermodule.*

Proof. By Proposition 4.3, it suffices to show that if $x \sim_N y$ then $(rx) \sim_N (ry)$ for all $r \in R$. Let N be a subsemihypermodule of M and $x \sim_N y$. Then there exist $n_1, n_2 \in N$ such that $x + n_1 = y + n_2$. Let $r \in R$. If $r = 0$ then $0 \sim_N 0$ and we are done. If $r > 0$ then $rx = \{0, x\}$ and $ry = \{0, y\}$. For $0 \in rx$, we have $0 \in ry$ such that $0 \sim_N 0$ and for $x \in rx$, we have $y \in ry$ such that $x \sim_N y$. Thus, for every $z \in rx$ there is $w \in ry$ such that $z \sim_N w$ (and similarly, for every $w \in ry$ there is $z \in rx$ such that $z \sim_N w$). Therefore, $(rx) \sim_N (ry)$. □

Remark 4.2. Let R be the semiring of non-negative integers under standard addition and multiplication of integers and M be the R-semihypermodule defined in Proposition 3.2. Then using the same proof of Proposition 4.5, we get that every subsemihypermodule of M is a congruence subsemihypermodule.

Notation 2. *Let $(R, +, \cdot)$ be a semihyperring, M a commutative R-semihypermodule, and N a congruence subsemihypermodule of M. Then $M/\sim_N$ is written as*

$$M/N = \{m + N : m \in M\}.$$

Corollary 4.2. *Let R be a semihyperring, M, N commutative R-semihypermodules, and $f : M \to N$ be a homomorphism. If $ker(f)$ is a congruence subsemihypermodule of M then $M/ker(f)$ is an R-semihypermodule.*

Proof. The proof follows from Proposition 4.3. □

Remark 4.3. Let K, N be congruence subsemihypermodules of M and $K \subseteq N$. If K is a congruence subsemihypermodule of M then K is a congruence subsemihypermodule of N.

Theorem 4.2. *Let R be a semihyperring, M be a commutative R-semihypermodule, and N a congruence subsemihypermodule of M. Then a subset S of M/N is subsemihypermodule of M/N if and only if there exist a subsemihypermodule K of M containing N such that $S = K/N$.*

Proof. Let S be a subsemihypermodule of M/N and $K = \{x \in M : x + N \in S\}$. It is clear that K is a subsemihypermodule of M. We prove now that $N \subseteq K$. Let $n \in N$. Then $0 \sim_N n$ (as $0+n = n+0$) and $n+N \in S$ as $0+N \in S$ ($n+N = 0+N$ as equivalence classes.). The latter implies that $n \in K$. Moreover, it is clear that $S = K/N$.

Conversely, let K be a subsemihypermodule of M containing N. Remark 4.3 asserts that $S = K/N$ is a semihypermodule. One can easily see that $S = K/N \subseteq M/N$. Therefore, $S = K/N$ is subsemihypermodule of M/N □

Proposition 4.6. *Let $(R, +, \cdot)$ be a semihyperring, M an R-semihypermodule, and $f : M \to N$ a homomorphism. If $x \sim_{ker(f)} y$ then $x \sim_f y$.*

Proof. Let $x \sim_{ker(f)} y$. Then there exist $k_1, k_2 \in ker(f)$ such that $x + k_1 = y + k_2$. The latter implies that $f(x) + f(k_1) = f(x + k_1) = f(y + k_2) = f(y) + f(k_2)$. But $f(k_1) = f(k_2) = 0$. Thus, $f(x) = f(y)$. Therefore, $x \sim_f y$. □

Example 4.1. Let $M = (\mathbb{N}, +)$ be the R-semihypemodule in Example 3.2. Proposition 4.5 asserts that $M/2M$ is an R-semihypermodule. Let $x, y \in M$ and $n = 2k, n' = 2k' \in 2M$. Then $x + 2k = y + 2k'$ if and only if x and y are both even integers or are both odd integers. The latter implies that we have only two equivalence classes: $0 + 2M$ and $1 + 2M$. Hence, $M/2M = \{0 + 2M, 1 + 2M\}$ and it is given by the following tables. For $r \in R$,

$+$	$0+2M$	$1+2M$
$0+2M$	$0+2M$	$1+2M$
$1+2M$	$1+2M$	$0+2M$

$\star$	$0+2M$	$1+2M$
0	$0+2M$	$0+2M$
$r(r>0)$	$0+2M$	$M/2M$

Definition 4.3. Let $(R, +, \cdot)$ be a semihyperring, M, N be R-semihypermodules, and $f : M \to N$ be a homomorphism. Then f is *steady* if $\sim_f$ and $\sim_{ker(f)}$ coincide.

Definition 4.4. Let $(R, +, \cdot)$ be a semihyperring, M be an R-semihypermodule. Then M is called *simple* if it has only two congruence relations.

Theorem 4.3. *Let R be a semihyperring and M a simple commutative R-semihypermodule. If N is a subtractive congruence subsemihypermodule of M then $N = \{0\}$ or $N = M$.*

Proof. Let N be a subtractive congruence subsemihypermodule of M. Then Proposition 4.3 asserts that $\sim_N$ is a congruence on M. We get that $\sim_N$ is either $\sim_t$ or $\sim_u$. If $\sim_N$ is $\sim_t$ then $M/\sim_N \cong M/\sim_t \cong M$. Thus, $N = \{0\}$. If $\sim_N$ is $\sim_u$ then $m \sim_N 0$ for all $m \in M$. The latter implies that there exist $n, n' \in N$ such that $m + n = 0 + n' = n'$. Having $n' \in N$ implies that $m + n \in N$. Since $n \in N$ and N is subtractive, it follows that $m \in N$. Thus, $N = M$. □

Corollary 4.3. *Let $(R, +, \cdot)$ be a semihyperring, M, N be R-semihypermodules, and $f : M \to N$ be a non-trivial homomorphism. If M is simple and $ker(f)$ is a congruence subsemihypermodule of M then the following hold.*

1. *$ker(f) = \{0\}$;*

2. *if f is steady then it is injective;*

3. *if f is steady and surjective then it is an isomorphism.*

Proof. (1) The proof follows from Proposition 3.11 and Theorem 4.3.

(2) Let $f(x) = f(y)$. Having f a steady function implies that $x \sim_{ker(f)} y$. The latter implies that there exist $k_1, k_2 \in ker(f)$ such that $x + k_1 = y + k_2$. By (1), we know that $ker(f) = \{0\}$. Thus, $x = y$.

(3) The proof is immediate consequence of (2). □

Proposition 4.7. *Let R be a multiplicative hyperring and M be a commutative R-hypermodule. If M has no proper non-trivial subhypermodules then M is simple.*

Proof. Let ρ be a congruence on M and $N = \{m \in M : m\rho 0\}$. Having N a subhypermodule of M implies that $N = \{0\}$ or $N = M$. If $N = \{0\}$ then ρ coincides with $\sim_t$ and if $N = M$ then ρ coincides with $\sim_u$. Therefore, M is simple. □

Corollary 4.4. *Let R be a multiplicative hyperring and M be a commutative R-hypermodule. Then M is simple if and only if it has no proper non-trivial subhypermodules.*

Proof. The proof follows from Theorem 4.3 Proposition 4.7. □

Proposition 4.8. *Let R be a semihyperring, M be a commutative R-semihypermodule, and N a congruence subsemihypermodule of M. Then the following hold.*

1. *if $a \in N$ then $N \subseteq a + N = 0 + N$;*

2. *if N is subtractive and $a \in N$ then $a + N = b + N$ if and only if $b \in N$;*

3. *if N is subtractive then $a + N = 0 + N = N$ if and only if $a \in N$.*

Proof. (1) Having $a + 0 = 0 + a$ implies that $a \sim_N 0$. The latter implies that $a + N = 0 + N$. Moreover, having $n \sim_N 0$ for all $n \in N$ implies that $n \in 0 + N$ for all $n \in N$. Thus, $N \subseteq 0 + N$.

(2) Let $a + N = b + N$. Then there exist $n_1, n_2 \in N$ such that $a + n_1 = b + n_2$. Having $a \in N$ implies that $b + n_2 \in N$. Since N is subtractive, it follows that $b \in N$. Conversely, let $b \in N$. Then by (1), we get that $a + N = 0 + N$ and $b + N = 0 + N$. Thus, $a + N = b + N$.

(3) The proof of $a + N = 0 + N$ follows from (1) and (2). We need to show that $0 + N \subseteq N$. Let $x \in 0 + N$. Then there exist $n_1, n_2 \in N$ such that $x + n_1 = 0 + n_2 = n_2$. We get now that $x + n_1 \in N$ and having N subtractive implies that $x \in N$. □

Theorem 4.4. *Let R be a semihyperring, M be a commutative R-semihypermodule, and N be a congruence subsemihypermodule of M. Then N is subtractive if and only if it is the kernel of a surjective homomorphism.*

Proof. Let N be a subsemihypermodule of M. If N is the kernel of a surjective homomorphism then it is subtractive by Proposition 3.11. Conversely, let N be a congruence subtractive subsemihypermodule of M. Then $\sim_N$ defines a congruence on M. Lemma 4.1 asserts that $f : M \to M/\sim_N$, defined by $f(m) = m + N$ for all $m \in M$, is a surjective homomorphism. It is clear that N is the kernel of $f : M \to M/\sim_N$. $\square$

5. Semi-isomorphism Theorems for Semihypermodules and Their Applications

In this section, we prove (semi)-isomorphism theorems for semihypermodules over semihyperring and present some of their interesting applications. The importance of (semi)-isomorphism theorems is to describe the relationship between factor semihypermodules, homomorphism, and subsemihypermodules and how they interact with the intersection and addition of semihypermodules.

Theorem 5.1. (First (semi)-isomorphism theorem for semihypermodules.) *Let $(R, +, \cdot)$ be a semihyperring, M, N be a R-semihypermodules, $f : M \to N$ be a surjective homomorphism, and $ker(f)$ a congruence subsemihypermodule of M. Then $M/ker(f) \cong_s N$. Moreover, if f is steady then $M/ker(f) \cong N$.*

Proof. Let $\phi : M/ker(f) \to N$ be defined as $\phi(x + ker(f)) = f(x)$. It is clear that ϕ is a surjective homomorphism. Having $ker(\phi) = \{x + ker(f) \in M/ker(f) : f(x) = 0\} = \{x + ker(f) \in M/ker(f) : x \in ker(f)\}$. Proposition 4.8 asserts that $ker(\phi) = \{0 + ker(f)\} = \{0_{M/ker(f)}\}$. Therefore, ϕ is a semi-isomorphism.

Let f be steady and $\phi(x + ker(f)) = \phi(y + ker(f))$. Then $f(x) = f(y)$. Since $x \sim_f y$ and f is steady, it follows that $x \sim_{ker(f)} y$. The latter implies that there exist $k_1, k_2 \in ker(f)$ such that $x + k_1 = y + k_2$. Thus, $x + ker(f) = y + ker(f)$. We get now that ϕ is injective and hence, ϕ is an isomorphism by Corollary 4.3. $\square$

Proposition 5.1. *Let $(R, +, \cdot)$ be a semihyperring, M_i be an R-semihypermodules, and N_i be a congruence subsemihypermodule of M_i for all $i = 1, \ldots, n$. Then $\prod_{i=1}^n N_i$ is a congruence subsemihypermodule of $\prod_{i=1}^n M_i$.*

Proof. It is easy to see that $\prod_{i=1}^n N_i$ is a subsemihypermodule of $\prod_{i=1}^n M_i$. We prove that $\prod_{i=1}^n N_i$ is a congruence subsemihypermodule of $\prod_{i=1}^n M_i$. It suffices to show that if $(x_1, \ldots, x_n) \sim_{\prod_{i=1}^n N_i} (y_1, \ldots, y_n)$ and $r \in R$ then $r(x_1, \ldots, x_n) \sim_{\prod_{i=1}^n N_i} r(y_1, \ldots, y_n)$. Let $(z_1, \ldots, z_n) \in r(x_1, \ldots, x_n) = (rx_1, \ldots, rx_n)$. Then $z_i \in rx_i$ for all $i = 1, \ldots, n$. Having $(x_1, \ldots, x_n) \sim_{\prod_{i=1}^n N_i} (y_1, \ldots, y_n)$ implies that there exist $(k_1, \ldots, k_n), (k'_1, \ldots, k'_n) \in \prod_{i=1}^n N_i$ such that

$$(x_1, \ldots, x_n) + (k_1, \ldots, k_n) = (y_1, \ldots, y_n) + (k'_1, \ldots, k'_n).$$

The latter implies that $x_i + k_i = y_i + k'_i$ for all $i = 1, \ldots n$. Thus, $x_i \sim_{N_i} y_i$ for all $i = 1, \ldots n$. Having $z_i \in rx_i$ implies that there exist $z'_i \in ry_i$ such that $z_i \sim_{N_i} z'_i$. We can find $n_i, n'_i \in N_i$ such that $z_i + n_i = z'_i + n'_i$ for $i = 1, \ldots, n$. As a result, we have

$$(z_1, \ldots, z_n) + (n_1, \ldots, n_n) = (y_1, \ldots, y_n) + (n'_1, \ldots, n'_n).$$

Therefore, $r(x_1, \ldots, x_n) \sim_{\prod_{i=1}^n N_i} r(y_1, \ldots, y_n)$. □

Theorem 5.2. *Let R be a semihyperring, M_i be a commutative R-semihypermodule, and N_i be a congruence subtractive subsemihypermodule of M_i for all $i = 1, \ldots, n$. Then*

$$(\prod_{i=1}^n M_i)/(\prod_{i=1}^n N_i) \cong \prod_{i=1}^n (M_i/N_i).$$

Proof. Let $f : \prod_{i=1}^n M_i \to \prod_{i=1}^n (M_i/N_i)$ be defined as

$$f((x_1, \ldots, x_n)) = (x_1 + N_1, \ldots, x_n + N_n).$$

It is clear that f is surjective. We show that f is a homomorphism. Let $(x_1, \ldots, x_n)$, $(y_1, \ldots, y_n) \in \prod_{i=1}^n M_i$ and $r \in R$. Then $f((x_1, \ldots, x_n) + (y_1, \ldots, y_n)) = f((x_1 + y_1, \ldots, x_n + y_n)) = (x_1 + y_1 + N_1, \ldots, x_n + y_n + N_n) = f((x_1, \ldots, x_n)) + f((y_1, \ldots, y_n))$. Moreover,

$$f(r(x_1, \ldots, x_n)) = f((rx_1, \ldots, rx_n)) = (rx_1 + N_1, \ldots, rx_n + N_n) = r(f((x_1, \ldots, x_n))).$$

We have $ker(f) = \{(x_1, \ldots, x_n) \in \prod_{i=1}^n M_i : (x_1 + N_1, \ldots, x_n + N_n) = (0 + N_1, \ldots, 0 + N_n)\}$. The latter implies that $x_i + N_i = 0 + N_i$ for all $i = 1, \ldots, n$. Since N_i is subtractive in M, it follows by Proposition 4.8 that $x_i \in N_i$. Thus, $ker(f) = \prod_{i=1}^n N_i$. To prove that f is steady, it suffices (by Proposition 4.6) to show that if $(x_1, \ldots, x_n) \sim_f (y_1, \ldots, y_n)$ then $(x_1, \ldots, x_n)_{ker(f)}(y_1, \ldots, y_n)$. Let $f((x_1, \ldots, x_n)) = f((y_1, \ldots, y_n))$. Then $(x_1 + N_1, \ldots, x_n + N_n) = (y_1 + N_1, \ldots, y_n + N_n)$. We get now that $x_i + N_i = y_i + N_i$ for all $i = 1, \ldots, n$. The latter implies that there exist $k_i, k'_i \in N_i$ such that $x_i + k_i = y_i + k'_i$. Having $(k_1, \ldots, k_n), (k'_1, \ldots, k'_n) \in ker(f)$ and $(x_1, \ldots, x_n) + (k_1, \ldots, k_n) = (y_1, \ldots, y_n) + (k'_1, \ldots, k'_n)$ implies that $\sim_f$ and $\sim_{ker(f)}$ coincide. Thus, f is steady. Theorem 5.1 completes the proof. □

Corollary 5.1. *Let R be a semihyperring, M_i be a commutative R-semihypermodule, and N_i be a congruence subtractive subsemihypermodule of M_i for all $i = 1, \ldots, n$. Then $\prod_{i=1}^n N_i$ is a congruence subtractive subsemihypermodule of $\prod_{i=1}^n M_i$.*

Proof. Since $\prod_{i=1}^n N_i$ is the kernel of the homomorphism presented in the proof of Theorem 5.2, it follows by using Proposition 3.11 that $\prod_{i=1}^n N_i$ is a subtractive subsemihypermodule of $\prod_{i=1}^n M_i$. □

Theorem 5.3. (Second semi-isomorphism theorem for semihypermodules.) *Let $(R,+,\cdot)$ be a semihyperring, M a commutative R-semihypermodule, N,K congruence subsemihypermodules of M, $N\cap K$ is congruence in N, and K is subtractive. Then*

$$N/(N\cap K)\cong_s (N+K)/K.$$

Proof. Remark 4.3 asserts that K is a congruence subsemihypermodule of $N+K$. So, $(N+K)/K$ is an R-semihypermodule. Let $f:N\to(N+K)/K$ defined as $f(x)=x+K$ for all $x\in N$. It is easy to see that f is a well-defined surjective homomorphism. We have $ker(f)=\{x\in K : x+K=0+K\}$. Since K is subtractive, it follows by Proposition 4.8 that $ker(f)=\{x\in N : x\in K\}=N\cap K$. Theorem 5.1 completes the proof. □

Corollary 5.2. *Let R be a semihyperring, M a commutative R-semihypermodule, N,K congruence subsemihypermodules of M, $N\cap K$ is a congruence subsemihypermodule of N, and K is subtractive. Then $N\cap K$ is a subtractive subsemihypermodule of K.*

Proof. Since $N\cap K$ is the kernel of the homomorphism presented in the proof of Theorem 5.3, it follows by using Proposition 3.11 that $N\cap K$ is a subtractive subsemihypermodule of K. □

Proposition 5.2. *Let $(R,+,\cdot)$ be a semihyperring, M a commutative R-semihypermodule, N,K congruence subsemihypermodules of M such that $K\subseteq N$, and N is subtractive. Then N/K is congruence subsemihypermodule of M/K.*

Proof. Using Proposition 4.3, it suffices to show that if $m+K\sim_{N/K} m'+K$ then $r(m+K)\sim_{N/K} r(m'+K)$. Let $m+K\sim_{N/K} m'+K$ and $z+K\in r(m+K)=rm+K$. Then there exist $n_1+K,n_2+K\in N/K$ such that $m+n_1+K=m+K+n_1+K=m'+K+n_2+K=m'+n_2+K$. The latter implies that $(m+n_1)\sim_K(m'+n_2)$ and consequently, there exist $k_1,k_2\in K$ such that $m+n_1+k_1=m'+n_2+k_2$. Since $K\subseteq N$, it follows that $m\sim_N m'$. Having $z+K\in r(m+K)=rm+K$ implies that $z\in rm$. And since N is a congruence subsemihypermodule of M, it follows that $(rm)\sim_N(rm')$. The latter implies that if $z\in rm$ then there exist $z'\in rm'$ such that $z\sim_N z'$. Thus, there exist $n,n'\in N$ such that $z+n=z'+n'$. Since $z+n+0=z'+n'+0$ and $0\in K$, it follows that $(z+K)+(n+K)=z+n+K=z'+n'+K=(z'+K)+(n'+K)$ with $n+K,n'+K\in N/K$. Thus, $(z+K)\sim_{N/K}(z'+K)$. □

Theorem 5.4. (Third isomorphism theorem for semihypermodules.) *Let $(R,+,\cdot)$ be a semihyperring, M a commutative R-semihypermodule, N,K congruence subsemihypermodules of M such that $K\subseteq N$, and N is subtractive. Then M/K then*

$$(M/K)/(N/K)\cong M/N.$$

Proof. Let $f : M/K \to M/N$ be defined as $f(x+K) = x+N$. It is easy to see that f is a well-defined surjective homomorphism. Moreover, $ker(f) = \{x+K : x+N = 0+N\}$. Since N is subtractive, it follows that $ker(f) = \{x+K : x \in N\} = N/K$ and it is a congruence subsemihypermodule of M/K (by Proposition 5.2.). Thus, by using Theorem 5.1, we get that f is a semi-isomorphism. We prove now that f is steady. Let $f((x+K)) = f((y+K))$. Then $x+N = y+N$. The latter implies that there exist $n_1, n_2 \in N$ such that $x+n_1 = y+n_2$ and hence, $x+n_1+K = y+n_2+K$. We get now that $x+K+n_1+K = y+K+n_2+K$. Thus, $x+K \sim_{ker(f)} y+K$. □

Corollary 5.3. *$(R,+,\cdot)$ be a semihyperring, M a commutative R-semihypermodule, N, K congruence subsemihypermodules of M such that $K \subseteq N$, and N is subtractive subsemihypermodule of M. Then N/K is a subtractive subsemihypermodule in M/K.*

Proof. Since N/K is the kernel of the homomorphism presented in the proof of Theorem 5.4, it follows by using Proposition 3.11 that N/K is a subtractive subsemihypermodule of M/K. □

We present some interesting applications on the (semi)-isomorphism theorems for semihypermodules.

Let $M = \{0, 1, 2, \ldots\}$ and $(M,+)$ be the R-semihypermodule in Example 3.2. Moreover, nM is a congruence subtractive subsemihyperodules of M for all $n = 1, 2, \ldots$. Proposition 3.2 asserts that $(\mathbb{Z}_n, +)$ is an R-semihypermodule where $\mathbb{Z}_n$ is the set of integers modulo n and "+" is taken as standard addition of integers modulo n and is given as $\mathbb{Z}_n = \{0 \pmod n, 1 \pmod n, \ldots, (n-1) \pmod n\}$.

Using Remark 4.2, we deduce that all the subsemihypermodules that we are dealing with in the following applications are congruence subsemihypermodules.

Application 1. Let n be any positive integer. Then $M/nM \cong \mathbb{Z}_n$.

Solution. Let $f : M \to \mathbb{Z}_n$ be defined as $f(x) = x \pmod n$. It is clear that f is surjective. Let $x, y \in M$ and $r \in R$. Then $f(x+y) = (x+y) \pmod n = x \pmod n + y \pmod n = f(x) + f(y)$. And

$$f(rx) = \begin{cases} 0 \pmod n & \text{if } r = 0 \\ f(\{0, x\}) & \text{if } r > 0. \end{cases} = \begin{cases} 0 \pmod n & \text{if } r = 0 \\ \{0 \pmod n, x \pmod n\} & \text{if } r > 0. \end{cases}$$

On the other hand, we have,

$$rf(x) = \begin{cases} 0 \pmod n & \text{if } r = 0 \\ \{0 \pmod n, f(x)\} & \text{if } r > 0. \end{cases} = \begin{cases} 0 \pmod n & \text{if } r = 0 \\ \{0 \pmod n, x \pmod n\} & \text{if } r > 0. \end{cases}$$

Thus, f is a homomorphism. We have $ker(f) = \{x \in M : x \pmod n = 0 \pmod n\} = nM$ is a congruence subsemihypermodule of M. Theorem 5.1 asserts that f defines a semi-isomorphism. To prove that f is an isomorphism, we prove that f is steady. Let $f(x) = f(y)$. Then by getting $x \pmod n = y \pmod n$, we deduce that there exist $k_1, k_2 \in M$ such that $x + nk_1 = y + nk_2$. Since $nk_1, nk_2 \in ker(f)$, it follows that $x \sim_{ker(f)} y$. Thus, f is steady.

Application 2. $\mathbb{Z}_3 \cong (M - \{1\})/3M$.

Solution. Let $N = 2M, K = 3M$. Then by using Second semi-isomorphism theorem, we get that $2M/(2M \cap 3M) \cong_s (2M + 3M)/3M$. One can easily see that $2M \cap 3M = 6M$ and $N + K = M - \{1\}$. Thus, $2M/6M \cong_s (M - \{1\})/3M$. Using same procedure as in Application 1, one can prove that $2M/6M \cong \mathbb{Z}_3$. Therefore, $\mathbb{Z}_3 \cong_s (M - \{1\})/3M$. Since $\mathbb{Z}_3$ and $(M - \{1\})/3M$ have each only three elements and $\mathbb{Z}_3 \cong_s (M - \{1\})/3M$, it follows that $\mathbb{Z}_3 \cong (M - \{1\})/3M$.

Application 3. $\mathbb{Z}_4/\mathbb{Z}_2 \cong \mathbb{Z}_2$.

Solution. Let $K = 4M \subset N = 2M$. Then by Third isomorphism theorem, we get that $(M/4M)/(2M/4M) \cong M/2M$. Using the results in Application 1, we get that $M/4M \cong \mathbb{Z}_4$ and $M/2M \cong \mathbb{Z}_2$. And in a similar manner, we can prove that $2M/4M \cong \mathbb{Z}_2$.

6. Applications of Our Results to Semihyperrings

In this section, we use our results on semihypermodules to deduce some results for semihyperrings. In particular, we derive (semi-)isomorphism theorems for semihyperrings.

Since every semihyperring R can be viewed as an R-semihypermodule and every hyperideal of it can be viewed as a subsemihypermodule, then the results of the previous sections can be applied to semihyperrings.

In what follows, all semihyperrings and their hyperideals have an identity 0 under addition "+" and the operation "+" is commutative. And if f is a homomorphism between semihyperrings then $f(0) = 0$.

Notation 3. *Let R be a semihyperring and I a hyperideal of R. Then I is a congruence hyperideal if I is a congruence subsemihypemodule of R when viewed as an R-semihypermodule.*

Theorem 6.1. *Let $(R, +, \cdot)$ be a semihyperring and I be a congruence hyperideal of R. Then I is subtractive if and only if it is the kernel of a surjective homomorphism.*

Theorem 6.2. (First semi-isomorphism theorem for semihyperrings.) *Let R, S be semihyperrings and $f : R \to S$ be a surjective homomorphism. If $ker(f)$ is a congruence hyperideal of R then $R/ker(f) \cong_s S$. Moreover, if f is steady then $R/ker(f) \cong S$.*

Proposition 6.1. *Let R_i be semihyperrings and K_i be congruence hyperideals of R_i for all $i = 1, \ldots, n$. Then $(\prod_{i=1}^n R_i)/(\prod_{i=1}^n K_i) \cong \prod_{i=1}^n (R_i/K_i)$.*

Theorem 6.3. (Second semi-isomorphism theorem for semihyperrings.) *Let $(R, +, \cdot)$ be a semihyperring with $(R, +)$ commutative, I, K congruence hyperideals of R, $I \cap K$ is congruence in I, and K is subtractive. Then*

$$I/(I \cap K) \cong_s (I + K)/K.$$

Theorem 6.4. (Third semi-isomorphism theorem for semihyperrings.) *Let $(R, +, \cdot)$ be a semihyperring with $(R, +)$ commutative, I, K congruence hyperideals of R such that $K \subseteq I$, and I is subtractive. Then*

$$(R/K)/(I/K) \cong R/I.$$

7. Conclusion

This paper dealt with semihypermodules over semihyperrings. Some properties of semihypermodules were discussed and different examples were presented. By means of a certain equivalence relation on semihypermodules, (Semi-)Isomorphism theorems for semihypermodules were derived and several applications were pointed. The results of this paper can be considered as a generalization for semimodules and for hypermodules.

For future work, we can make the following question: "Is it possible to derive the (semi-)isomorphism theorems for semihypermodules with less conditions?"

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