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ROUGH IDEAL CONVERGENCE OF DOUBLE SEQUENCES IN INTUITIONISTIC FUZZY NORMED SPACES

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Abstract. The idea of rough statistical convergence for double sequences was studied by Özcan and Or [34] in a intuitionistic fuzzy normed space. Recently the same has been generalized in the ideal context by Hossain and Banerjee [17] for sequences. Here in this paper, we have discussed the idea of rough ideal convergence of double sequences in intuitionistic fuzzy normed spaces generalizing the idea of rough statistical convergence of double sequences. Also we have defined rough $\mathcal{I}_2$-cluster points for a double sequence and also investigated some of the basic properties associated with rough $\mathcal{I}_2$-limit set of a double sequence in a intuitionistic fuzzy normed space.
Keywords: Ideal, filter, Intuitionistic fuzzy normed space, double sequence, rough $\mathcal{I}_2$-convergence, rough $\mathcal{I}_2$-cluster point.

1. Introduction

The concept of fuzzy sets was firstly introduced by Zadeh [45] in 1965 as a generalization of the concept of crisp set. A wide range of extensive applications in various branches of modern science and engineering can be found in [4, 11, 12, 16, 26]. The idea of intuitionistic fuzzy sets was firstly given by Atanassov[1] in 1986 and later in 2004, Park [38] introduced the concept of intuitionistic fuzzy metric spaces using this idea of Atanassov. In 2006, Saadati and Park [42] extended this

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concept to the theory of intuitionistic fuzzy normed spaces. Many authors have worked on intuitionistic fuzzy normed spaces [2, 34] using the idea of Saadati and Park [42].

After the introduction of the notion statistical convergence of sequences by Fast [10] and Steinhaus [40] independently generalizing the notion of ordinary convergence of sequences, Kostyrko et al. [18] introduced two interesting extensions of this idea as $\mathcal{I}$ and $\mathcal{I}^*$-convergence of sequences using the structure of an ideal, formed by subsets of natural numbers. Several works by many authors in different directions can be found in [20, 21, 22, 23, 24, 25, 32] using the idea of [10, 40, 18].

In 2001, Phu [35] introduced the idea of rough convergence of sequences in finite dimensional normed linear spaces as a generalization of ordinary convergence of sequences. There he investigated some topological and geometrical properties of rough limit set of a sequence and also introduced the idea of rough cauchy sequences. In 2003 this concept was extended to the infinite dimensional normed linear spaces by Phu [37]. Later the idea of Phu [35, 37] was extended to rough statistical convergence using the concept of natural density by Ayter [3]. The concept of rough statistical convergence of sequences was extended to rough ideal convergence of sequences in 2013 by Pal et al. [39]. Later, numerous authors used Phu's idea in a variety of works [5, 8, 9, 13, 14].

In intuitionistic fuzzy normed spaces the idea of rough statistical convergence of sequences was defined by Antal et al. [2]. After that, Özcan and Or [34] studied the same notion in the setting of double sequences. Recently the idea of Özcan and Or [34] has been generalized in the ideal context by Hossain and Banerjee [17] for sequences. In this paper we have generalized this concept in an ideal context and investigated some of the important results using the idea of [34].

2. Preliminaries

All the time $\mathbb{N}$ and $\mathbb{R}$ denote the set of natural numbers and set of reals, respectively. First we recall some basic definitions and notations. It has been discussed in [18] that if $X \neq \emptyset$. Then a class $\mathcal{I}$ of subsets of X is said to be an ideal in X if the conditions $\emptyset \in X$; $A, B \in \mathcal{I} \implies A \cup B \in \mathcal{I}$ and $A \in \mathcal{I}$, $B \subset A \implies B \in \mathcal{I}$ hold.

Also it has been discussed that a non empty family $\mathcal{F}$ of subsets of a non empty set X is said to be filter of X if the conditions $\emptyset \notin \mathcal{F}$; $A, B \in \mathcal{F} \implies A \cap B \in \mathcal{F}$ and $A \in \mathcal{F}, A \subset B \implies B \in \mathcal{F}$ hold.

An ideal $\mathcal{I}$ is called non trivial if $X \notin \mathcal{I}$ and proper if $\mathcal{I} \neq \{\emptyset\}$. A non trivial ideal is called admissible if $\{x\} \in \mathcal{I}$ for each $x \in X$. If $\mathcal{I}$ is a non trivial ideal of X then the family $\mathcal{F}(\mathcal{I}) = \{X \setminus A : A \in \mathcal{I}\}$ is a filter on X, called filter associated with the ideal $\mathcal{I}$. Throughout the paper $\mathcal{I}$ will stand for a non trivial admissible ideal in $\mathbb{N}$.

In this study we were very much influenced by some ideas that can be found in [6, 27, 28, 29, 15], which really helped us improve the quality of the paper.

Definition 2.1. [30] The double natural density of the set $A \subset \mathbb{N} \times \mathbb{N}$ is defined by

$$\delta_2(A) = \lim_{m,n\to\infty} \frac{|\{(i,j) \in A : i \leq m, j \leq n\}|}{mn}$$

where $|\{(i,j) \in A : i \leq m, j \leq n\}|$ denotes the number of elements of A not exceeding m and n, respectively. It is clear that if A is finite then $\delta_2(A) = 0$.

Definition 2.2. [7] A non trivial ideal $\mathcal{I}_2$ of $\mathbb{N}\times\mathbb{N}$ is said to be strongly admissible if $\{i\} \times \mathbb{N}$ and $\mathbb{N} \times \{i\}$ belong to $\mathcal{I}_2$ for each $i \in \mathbb{N}$.

It is clear that a strongly admissible ideal is also admissible. Throughout the discussion $\mathcal{I}_2$ stands for an admissible ideal of $\mathbb{N} \times \mathbb{N}$.

Remark 2.1. (see [7]) (a) If we take $\mathcal{I}_2 = \mathcal{I}_2^0$, where $\mathcal{I}_2^0 = \{A \subset \mathbb{N} \times \mathbb{N} : \exists\ m(A) \in \mathbb{N} : i, j \geq m(A) \implies (i,j) \notin A\}$, then $\mathcal{I}_2^0$ will be a non trivial strongly admissible ideal. In this case $\mathcal{I}_2$-convergence coincides with ordinary convergence of double sequences of real numbers.
(b) If we take $\mathcal{I}_2 = \mathcal{I}_2^\delta$, where $\mathcal{I}_2^\delta = \{A \subset \mathbb{N}\times\mathbb{N} : \delta_2(A) = 0\}$, then $\mathcal{I}_2^\delta$-convergence becomes statistical convergence of double sequences of real numbers.

Now, we recall some basic definitions and notations which will be useful in the sequal.

Definition 2.3. [41] A binary operation $\star : [0,1] \times [0,1] \to [0,1]$ is said to be a continuous t-norm if the following conditions hold:

1. $\star$ is associative and commutative;
2. $\star$ is continuous;
3. $x \star 1 = x$ for all $x \in [0,1]$;
4. $x \star y \leq z \star w$ whenever $x \leq z$ and $y \leq w$ for each $x, y, z, w \in [0,1]$.

Definition 2.4. [41] A binary operation $\circ : [0,1] \times [0,1] \to [0,1]$ is said to be a continuous t-conorm if the following conditions are satisfied:

1. $\circ$ is associative and commutative;
2. $\circ$ is continuous;
3. $x \circ 0 = x$ for all $x \in [0,1]$;
4. $x \circ y \leq z \circ w$ whenever $x \leq z$ and $y \leq w$ for each $x, y, z, w \in [0,1]$.

Example 2.1. [19] The following are the examples of t-norms:

1. $x \star y = min\{x, y\}$;
2. $x \star y = x.y$;

3. $x \star y = max\{x + y - 1, 0\}$. This t-norm is known as Lukasiewicz t-norm.

Example 2.2. [19] The following are the examples of t-conorms:

1. $x \circ y = max\{x, y\}$;
2. $x \circ y = x + y - x.y$;
3. $x \circ y = min\{x + y, 1\}$. This is known as Lukasiewicz t-conorm.

Definition 2.5. [42] The 5-tuple $(X, \mu, \nu, \star, \circ)$ is said to be an intuitionistic fuzzy normed space (in short, IFNS) if X is a normed linear space, $\star$ is a continuous t-norm, $\circ$ is a continuous t-conorm and μ and ν are the fuzzy sets on $X \times (0, \infty)$ satisfying the following conditions for every $x, y \in X$ and $s, t > 0$:

1. $\mu(x, t) + \nu(x, t) \leq 1$;
2. $\mu(x, t) > 0$;
3. $\mu(x, t) = 1$ if and only if $x = 0$;
4. $\mu(\alpha x, t) = \mu(x, \frac{t}{|\alpha|})$ for each $\alpha \neq 0$;
5. $\mu(x, t) \star \mu(y, s) \leq \mu(x + y, t + s)$;
6. $\mu(x, t) : (0, \infty) \to [0, 1]$ is continuous in t;
7. $\lim_{t\to\infty} \mu(x, t) = 1$ and $\lim_{t\to 0} \mu(x, t) = 0$;
8. $\nu(x, t) < 1$;
9. $\nu(x, t) = 0$ if and only if $x = 0$;
10. $\nu(\alpha x, t) = \nu(x, \frac{t}{|\alpha|})$ for each $\alpha \neq 0$;
11. $\nu(x, t) \circ \nu(y, s) \geq \nu(x + y, s + t)$;
12. $\nu(x, t) : (0, \infty) \to [0, 1]$ is continuous in t;
13. $\lim_{t\to\infty} \nu(x, t) = 0$ and $\lim_{t\to 0} \nu(x, t) = 1$.

In this case (μ, ν) is called an intuitionistic fuzzy norm on X.

Example 2.3. [42] Let $(X, \|\cdot\|)$ be a normed space. Denote $a \star b = ab$ and $a \circ b = min\{a + b, 1\}$ for all $a, b \in [0, 1]$ and let μ and ν be fuzzy sets on $X \times (0, \infty)$ defined as follows:

$$\mu(x, t) = \frac{t}{t + \|x\|}, \ \nu(x, t) = \frac{\|x\|}{t + \|x\|}.$$

Then $(X, \mu, \nu, \star, \circ)$ is an intuitionistic fuzzy normed space.

Definition 2.6. [43] Let $(X,\mu,\nu,\star,\circ)$ be an IFNS with intuitionistic fuzzy norm (μ,ν). For $r>0$, the open ball $B(x,\lambda,r)$ with center $x\in X$ and radius $0<\lambda<1$, is the set

$$B(x,\lambda,r)=\{y\in X:\mu(x-y,r)>1-\lambda,\ \nu(x-y,r)<\lambda\}.$$

Similarly, closed ball is the set $\overline{B(x,\lambda,r)}=\{y\in X:\mu(x-y,r)\geq 1-\lambda,\ \nu(x-y,r)\leq \lambda\}$.

Definition 2.7. [44] Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in an IFNS $(X,\mu,\nu,\star,\circ)$. Then a point $\gamma\in X$ is called a $\mathcal{I}$-cluster point of $\{x_n\}_{n\in\mathbb{N}}$ with respect to the intuitionistic fuzzy norm (μ,ν) if for every $\varepsilon>0$, $\lambda\in(0,1)$, the set $\{n\in\mathbb{N}:\mu(x_n-\gamma,\varepsilon)>1-\lambda \text{ and } \nu(x_n-\gamma,\varepsilon)<\lambda\}\notin\mathcal{I}$.

Definition 2.8. [31] Let $\{x_{mn}\}$ be a double sequence in an IFNS $(X,\mu,\nu,\star,\circ)$. Then $\{x_{mn}\}$ is said to be convergent to $\xi\in X$ with respect to the intuitionistic fuzzy norm (μ,ν) if for every $\varepsilon>0$ and $\lambda\in(0,1)$ there exists $\mathcal{N}_\varepsilon\in\mathbb{N}$ such that $\mu(x_{mn}-\xi,\varepsilon)>1-\lambda$ and $\nu(x_{mn}-\xi,\varepsilon)<\lambda$ for all $m,n\geq\mathcal{N}_\varepsilon$. In this case we write $(\mu,\nu)\text{-}\lim x_{mn}=\xi$ or $x_{mn}\xrightarrow{(\mu,\nu)}\xi$.

Definition 2.9. [33] Let $\mathcal{I}_2$ be a non trivial ideal of $\mathbb{N}\times\mathbb{N}$ and $(X,\mu,\nu,\star,\circ)$ be an intuitionistic fuzzy normed space. A double sequence $x=\{x_{mn}\}$ of elements of X is said to be $\mathcal{I}_2$-convergent to $L\in X$ if for each $\varepsilon>0$ and $t>0$, $\{(m,n)\in\mathbb{N}\times\mathbb{N}:\mu(x_{mn}-L,t)\leq 1-\varepsilon \text{ or } \nu(x_{mn}-L,t)\geq\varepsilon\}\in\mathcal{I}_2$. In this case we write $\mathcal{I}_2^{(\mu,\nu)}\text{-}\lim x=L$ or $x_{mn}\xrightarrow{\mathcal{I}_2^{(\mu,\nu)}}L$.

Definition 2.10. [34] Let $\{x_{mn}\}$ be a double sequence in an IFNS $(X,\mu,\nu,\star,\circ)$ and r be a non negative real number. Then $\{x_{mn}\}$ is said to be rough convergent (in short r-convergent) to $\xi\in X$ with respect to the intuitionistics fuzzy norm (μ,ν) if for every $\varepsilon>0$ and $\lambda\in(0,1)$ there exists $\mathcal{N}_\lambda\in\mathbb{N}$ such that $\mu(x_{mn}-\xi,r+\varepsilon)>1-\lambda$ and $\nu(x_{mn}-\xi,r+\varepsilon)<\lambda$ for all $m,n\geq\mathcal{N}_\lambda$. In this case we write $r_2^{(\mu,\nu)}-\lim x_{mn}=\xi$ or $x_{mn}\xrightarrow{r_2^{(\mu,\nu)}}\xi$.

Definition 2.11. [34] Let $\{x_{mn}\}$ be a double sequence in an IFNS $(X,\mu,\nu,\star,\circ)$ and r be a non negative real number. Then $\{x_{mn}\}$ is said to be rough statistically convergent to $\xi\in X$ with respect to the intuitionistic fuzzy norm (μ,ν) if for every $\varepsilon>0$ and $\lambda\in(0,1)$, $\delta_2(\{(m,n)\in\mathbb{N}\times\mathbb{N}:\mu(x_{mn}-\xi,r+\varepsilon)\leq 1-\lambda \text{ or } \nu(x_{mn}-\xi,r+\varepsilon)\geq\lambda\})=0$. In this case we write $r\text{-}st_2^{(\mu,\nu)}\text{-}\lim x_{mn}=\xi$ or $x_{mn}\xrightarrow{r-st_2^{(\mu,\nu)}}\xi$.

3. Main Results

We first introduce the notion of rough ideal convergence of double sequences in an IFNS and then investigate some important results associated with rough $\mathcal{I}_2$-cluster points in the same space.

Definition 3.1. Let $\{x_{mn}\}$ be a double sequence in an IFNS $(X, \mu, \nu, \star, \circ)$ and r be a non negative real number. Then $\{x_{mn}\}$ is said to be rough $\mathcal{I}_2$-convergent to $\xi \in X$ with respect to the intuitionistic fuzzy norm (μ, ν) if for every $\varepsilon > 0$ and $\lambda \in (0,1)$, $\{(m,n) \in \mathbb{N}\times\mathbb{N} : \mu(x_{mn}-\xi, r+\varepsilon) \leq 1-\lambda \text{ or } \nu(x_{mn}-\xi, r+\varepsilon) \geq \lambda\} \in \mathcal{I}_2$. In this case ξ is called r-$\mathcal{I}_2^{(\mu,\nu)}$-limit of $\{x_{mn}\}$ and we write r-$\mathcal{I}_2^{(\mu,\nu)}$- $\lim x_{mn} = \xi$ or $x_{mn} \xrightarrow{r-\mathcal{I}_2^{(\mu,\nu)}} \xi$.

Remark 3.1. (a) If we put $r = 0$ in Definition 3.1 then the notion of rough $\mathcal{I}_2$-convergence with respect to the intuitionistic fuzzy norm (μ, ν) coincides with the notion of $\mathcal{I}_2$-convergence with respect to the intuitionistic fuzzy norm (μ, ν). So, our main interest is on the fact $r > 0$.
(b) If we use $\mathcal{I}_2 = \mathcal{I}_2^0$ in Definition 3.1, then the notion of rough $\mathcal{I}_2$-convergence with respect to the intuitionistic fuzzy norm (μ, ν) coincides with the notion of rough convergence of double sequences with respect to the intuitionistic fuzzy norm (μ, ν).
(c) If we take $\mathcal{I}_2 = \mathcal{I}_2^\delta$ in Definition 3.1, then the notion of rough $\mathcal{I}_2$-convergence with respect to the intuitionistic fuzzy norm (μ, ν) coincides with the notion of rough statistical convergence of double sequences with respect to the intuitionistic fuzzy norm (μ, ν).

Note 3.1. *From Definition 3.1, we get* r-$\mathcal{I}_2^{(\mu,\nu)}$*-limit of* $\{x_{mn}\}$ *is not unique. So, in this regard we denote* $\mathcal{I}_2^{(\mu,\nu)}$-$LIM^r_{x_{mn}}$ *to mean the set of all* r-$\mathcal{I}_2^{(\mu,\nu)}$*-limit of* $\{x_{mn}\}$*, i.e.,* $\mathcal{I}_2^{(\mu,\nu)}$-$LIM^r_{x_{mn}} = \{\xi \in X : x_{mn} \xrightarrow{r-\mathcal{I}_2^{(\mu,\nu)}} \xi\}$. *The double sequence* $\{x_{mn}\}$ *is called rough* $\mathcal{I}_2$*-convergent if* $\mathcal{I}_2^{(\mu,\nu)}$-$LIM^r_{x_{mn}} \neq \emptyset$.

We denote $LIM^{r(\mu,\nu)}_{x_{mn}}$ to mean the set of all rough convergent limits of the double sequence $\{x_{mn}\}$ with respect to the intuitionistic fuzzy norm (μ, ν). The sequence $\{x_{mn}\}$ is called rough convergent if $LIM^{r(\mu,\nu)}_{x_{mn}} \neq \emptyset$. If the sequence is unbounded then $LIM^{r(\mu,\nu)}_{x_{mn}} = \emptyset$ [34], although in such cases $\mathcal{I}_2^{(\mu,\nu)}$-$LIM^r_{x_{mn}} \neq \emptyset$ may happen which will be shown in the following example.

Example 3.1. Let $(X, \|\cdot\|)$ be a real normed linear space and $\mu(x,t) = \frac{t}{t+\|x\|}$ and $\nu(x,t) = \frac{\|x\|}{t+\|x\|}$ for all $x \in X$ and $t > 0$. Also, let $a \star b = ab$ and $a \circ b = \min\{a+b, 1\}$. Then $(X, \mu, \nu, \star, \circ)$ is an IFNS. Now let us consider ideal $\mathcal{I}_2$ consisting of all those subsets of $\mathbb{N}\times\mathbb{N}$ whose double natural density are zero. Let us consider the double sequence $\{x_{mn}\}$ by $x_{mn} = \begin{cases} (-1)^{m+n}, & \text{if } m,n \neq i^2, i \in \mathbb{N} \\ mn, & \text{otherwise} \end{cases}$. Then $\mathcal{I}_2^{(\mu,\nu)}$-$LIM^r_{x_{mn}} = \begin{cases} \emptyset, & r < 1 \\ [1-r, r-1], & r \geq 1 \end{cases}$ and $LIM^{r(\mu,\nu)}_{x_{mn}} = \emptyset$ for any r.

Remark 3.2. From Example 3.1, we have $\mathcal{I}_2^{(\mu,\nu)}$-$LIM^r_{x_{mn}} \neq \emptyset$ does not imply that $LIM^{r(\mu,\nu)}_{x_{mn}} \neq \emptyset$. But, whenever $\mathcal{I}_2$ is an admissible ideal then $LIM^{r(\mu,\nu)}_{x_{mn}} \neq \emptyset$ implies $\mathcal{I}_2^{(\mu,\nu)}$-$LIM^r_{x_{mn}} \neq \emptyset$ as $\mathcal{I}_2^0 \subset \mathcal{I}_2$.

Now we define $\mathcal{I}_2$-bounded of double sequences in an IFNS analogue to ([34], Definition 3.6).

Definition 3.2. Let $\{x_{mn}\}$ be a double sequence in an IFNS $(X, \mu, \nu, \star, \circ)$. Then $\{x_{mn}\}$ is said to be $\mathcal{I}_2$-bounded with respect to the intuitionistic fuzzy norm (μ, ν) if for every $\lambda \in (0, 1)$ there exists a positive real number M such that $\{(m, n) \in \mathbb{N} \times \mathbb{N} : \mu(x_{mn}, M) \leq 1 - \lambda \text{ or } \nu(x_{mn}, M) \geq \lambda\} \in \mathcal{I}_2$.

Theorem 3.1. *Let $\{x_{mn}\}$ be a double sequence in an IFNS $(X, \mu, \nu, \star, \circ)$. Then $\{x_{mn}\}$ is $\mathcal{I}_2$-bounded if and only if $\mathcal{I}_2^{(\mu,\nu)}$-$LIM^r_{x_{mn}} \neq \emptyset$ for all $r > 0$.*

Proof. First suppose that $\{x_{mn}\}$ is $\mathcal{I}_2$-bounded in X with respect to the intuitionistic fuzzy norm (μ, ν). Then for every $\lambda \in (0, 1)$ there exists a positive real number M such that $\{(m, n) \in \mathbb{N} \times \mathbb{N} : \mu(x_{mn}, M) \leq 1 - \lambda \text{ or } \nu(x_{mn}, M) \geq \lambda\} \in \mathcal{I}_2$. Let $K = \{(m, n) \in \mathbb{N} \times \mathbb{N} : \mu(x_{mn}, M) \leq 1 - \lambda \text{ or } \nu(x_{mn}, M) \geq \lambda\}$. Now for $(i, j) \in K^c$, we have $\mu(x_{ij} - \theta, r + M) \geq \mu(x_{ij}; M) \star \mu(\theta, r) > (1 - \lambda) \star 1 = 1 - \lambda$ and $\nu(x_{ij} - \theta, r + M) \leq \nu(x_{ij}; M) \circ \nu(\theta, r) < \lambda \circ 0 = \lambda$, where θ is the zero element of X. Therefore $\{(i, j) \in \mathbb{N} \times \mathbb{N} : \mu(x_{ij} - \theta, r + M) \leq 1 - \lambda \text{ or } \nu(x_{mn}, M) \geq \lambda\} \subset K$. Since $K \in \mathcal{I}_2$, $\theta \in \mathcal{I}_2$. Hence $\mathcal{I}_2^{(\mu,\nu)}$-$LIM^r_{x_{mn}} \neq \emptyset$.

Conversely, suppose that $\mathcal{I}_2^{(\mu,\nu)}$-$LIM^r_{x_{mn}} \neq \emptyset$. Then there exists $\beta \in \mathcal{I}_2^{(\mu,\nu)}$-$LIM^r_{x_{mn}}$ such that for every $\varepsilon > 0$ and $\lambda \in (0, 1)$ such that $\{(m, n) \in \mathbb{N} \times \mathbb{N} : \mu(x_{mn} - \beta, r + \varepsilon) \leq 1 - \lambda \text{ or } \nu(x_{mn} - \beta, r + \varepsilon) \geq \lambda\} \in \mathcal{I}_2$. This shows that almost all x_{mn} are contained in some ball with center β. Hence $\{x_{mn}\}$ is $\mathcal{I}_2$-bounded. This completes the proof. $\square$

Now we will discuss some algebraic characterization of rough $\mathcal{I}_2$-convergence in an IFNS.

Theorem 3.2. *Let $\{x_{mn}\}$ and $\{y_{mn}\}$ be two double sequences in an IFNS $(X, \mu, \nu, \star, \circ)$. Then for some $r > 0$, the following statements hold:*

1. *If $x_{mn} \xrightarrow{r-\mathcal{I}_2^{(\mu,\nu)}} \xi$ and $y_{mn} \xrightarrow{r-\mathcal{I}_2^{(\mu,\nu)}} \eta$, then $x_{mn} + y_{mn} \xrightarrow{r-\mathcal{I}_2^{(\mu,\nu)}} \xi + \eta$.*
2. *If $x_{mn} \xrightarrow{r-\mathcal{I}_2^{(\mu,\nu)}} \xi$ and $k(\neq 0) \in \mathbb{R}$, then $kx_{mn} \xrightarrow{r-\mathcal{I}_2^{(\mu,\nu)}} k\xi$.*

Proof. Let $\{x_{mn}\}$ and $\{y_{mn}\}$ be two double sequences in an IFNS $(X, \mu, \nu, \star, \circ)$, $r > 0$ and $\lambda \in (0, 1)$.

1. Let $x_{mn} \xrightarrow{r-\mathcal{I}_2^{(\mu,\nu)}} \xi$ and $y_{mn} \xrightarrow{r-\mathcal{I}_2^{(\mu,\nu)}} \eta$. Also, let $\varepsilon > 0$ be given. Now, for a given $\lambda \in (0, 1)$, choose $s \in (0, 1)$ such that $(1 - s) \star (1 - s) > 1 - \lambda$ and $s \circ s < \lambda$. Then $A, B \in \mathcal{I}_2$, where $A = \{(m, n) \in \mathbb{N} \times \mathbb{N} : \mu(x_{mn} - \xi, \frac{r+\varepsilon}{2}) \leq 1 - s \text{ or } \nu(x_{mn} - \xi, \frac{r+\varepsilon}{2}) \geq s\}$ and $B = \{(m, n) \in \mathbb{N} \times \mathbb{N} : \mu(y_{mn} - \eta, \frac{r+\varepsilon}{2}) \leq 1 - s \text{ or } \nu(y_{mn} - \eta, \frac{r+\varepsilon}{2}) \geq s\}$. So, $A^c \cap B^c \in \mathcal{F}(\mathcal{I}_2)$. Now for $(i, j) \in A^c \cap B^c$, we have $\mu(x_{ij} + y_{ij} - (\xi + \eta), r + \varepsilon) \geq \mu(x_{ij} - \xi, \frac{r+\varepsilon}{2}) \star \mu(y_{ij} - \eta, \frac{r+\varepsilon}{2}) > (1 - s) \star (1 - s) > 1 - \lambda$ and $\nu(x_{ij} + y_{ij} - (\xi + \eta), r + \varepsilon) \leq \nu(x_{ij} - \xi, \frac{r+\varepsilon}{2}) \circ \nu(y_{ij} - \eta, \frac{r+\varepsilon}{2}) < s \circ s < \lambda$. Therefore $\{(i, j) \in \mathbb{N} \times \mathbb{N} : \mu(x_{ij} + y_{ij} - (\xi +$

$\eta), r+\varepsilon) \leq 1-\lambda$ or $\nu(x_{ij}-\xi, \frac{r+\varepsilon}{2}) \geq \lambda\} \subset A \cup B$. Since $A \cup B \in \mathcal{I}_2$, $\{(i,j) \in \mathbb{N}\times\mathbb{N} : \mu(x_{ij}+y_{ij}-(\xi+\eta), r+\varepsilon) \leq 1-\lambda$ or $\nu(x_{ij}-\xi, \frac{r+\varepsilon}{2}) \geq \lambda\} \in \mathcal{I}_2$. Therefore $x_{mn}+y_{mn} \xrightarrow{r-\mathcal{I}_2^{(\mu,\nu)}} \xi+\eta$.

2. Let $x_{mn} \xrightarrow{r-\mathcal{I}_2^{(\mu,\nu)}} \xi$ and $k(\neq 0) \in \mathbb{R}$. Then, $\{(m,n) \in \mathbb{N}\times\mathbb{N} : \mu(x_{mn}-\xi, \frac{r+\varepsilon}{|k|}) \leq 1-\lambda$ or $\nu(x_{mn}-\xi, \frac{r+\varepsilon}{|k|}) \geq \lambda\} \in \mathcal{I}_2$. Therefore, $\{(m,n) \in \mathbb{N}\times\mathbb{N} : \mu(kx_{mn}-k\xi, r+\varepsilon) \leq 1-\lambda$ or $\nu(kx_{mn}-k\xi, r+\varepsilon) \geq \lambda\} \in \mathcal{I}_2$. Hence $kx_{mn} \xrightarrow{r-\mathcal{I}_2^{(\mu,\nu)}} k\xi$. This completes the proof.

□

Now we prove some topological and geometrical properties of the set $\mathcal{I}_2^{(\mu,\nu)}$-$LIM^r_{x_{mn}}$.

Theorem 3.3. *Let $\{x_{mn}\}$ be a double sequence in an IFNS $(X,\mu,\nu,\star,\circ)$. Then for all $r>0$, the set $\mathcal{I}_2^{(\mu,\nu)}$-$LIM^r_{x_{mn}}$ is closed.*

Proof. If $\mathcal{I}_2^{(\mu,\nu)}$-$LIM^r_{x_{mn}} = \emptyset$ then there is nothing to prove. So, let $\mathcal{I}_2^{(\mu,\nu)}$-$LIM^r_{x_{mn}} \neq \emptyset$. Suppose thet $\{z_{mn}\}$ is a double sequence in $\mathcal{I}_2^{(\mu,\nu)}$-$LIM^r_{x_{mn}}$ such that $z_{mn} \xrightarrow{(\mu,\nu)} y_0$. Now, for a given $\lambda \in (0,1)$, choose $s \in (0,1)$ such that $(1-s)\star(1-s) > 1-\lambda$ and $s \circ s < \lambda$. Let $\varepsilon > 0$ be given. Then there exists $m_0 \in \mathbb{N}$ such that $\mu(z_{mn}-y_0, \frac{\varepsilon}{2}) > 1-s$ and $\nu(z_{mn}-y_0, \frac{\varepsilon}{2}) < s$ for all $m,n \geq m_0$. Suppose $i,j > m_0$. Then $\mu(z_{ij}-y_0, \frac{\varepsilon}{2}) > 1-s$ and $\nu(z_{ij}-y_0, \frac{\varepsilon}{2}) < s$. Also, $P = \{(m,n) \in \mathbb{N}\times\mathbb{N} : \mu(x_{mn}-z_{ij}, r+\frac{\varepsilon}{2}) \leq 1-s$ or $\nu(x_{mn}-z_{ij}, r+\frac{\varepsilon}{2}) \geq s\} \in \mathcal{I}_2$. Now, for $(p,q) \in P^c$, we have $\mu(x_{pq}-y_0, r+\varepsilon) \geq \mu(x_{pq}-z_{ij}, r+\frac{\varepsilon}{2})\star\mu(z_{ij}-y_0, \varepsilon) > (1-s)\star(1-s) > 1-\lambda$ and $\nu(x_{pq}-y_0, r+\varepsilon) \leq \nu(x_{pq}-z_{ij}, r+\frac{\varepsilon}{2}) \circ \nu(z_{ij}-y_0, \frac{\varepsilon}{2}) < s \circ s < \lambda$. Therefore $\{(m,n) \in \mathbb{N}\times\mathbb{N} : \mu(x_{mn}-y_0, r+\frac{\varepsilon}{2}) \leq 1-s$ or $\nu(x_{mn}-y_0, r+\frac{\varepsilon}{2}) \geq s\} \subset P$. Since $P \in \mathcal{I}_2$, $y_0 \in \mathcal{I}_2^{(\mu,\nu)}$-$LIM^r_{x_{mn}}$. Therefore $\mathcal{I}_2^{(\mu,\nu)}$-$LIM^r_{x_{mn}}$ is closed. This completes the proof. □

Theorem 3.4. *Let $\{x_{mn}\}$ be a double sequence in an IFNS $(X,\mu,\nu,\star,\circ)$. Then for all $r>0$, the set $\mathcal{I}_2^{(\mu,\nu)}$-$LIM^r_{x_{mn}}$ is convex.*

Proof. Let $x_1, x_2 \in \mathcal{I}_2^{(\mu,\nu)}$-$LIM^r_{x_{mn}}$ and $\kappa \in (0,1)$. Let $\lambda \in (0,1)$. Choose $s \in (0,1)$ such that $(1-s)\star(1-s) > 1-\lambda$ and $s\circ s < \lambda$. Then for every $\varepsilon > 0$, the sets $H, T \in \mathcal{I}_2$ where $H = \{(m,n) \in \mathbb{N}\times\mathbb{N} : \mu(x_{mn}-x_1, \frac{r+\varepsilon}{2(1-\kappa)}) \leq 1-s$ or $\nu(x_{mn}-x_1, \frac{r+\varepsilon}{2(1-\kappa)}) \geq s\}$ and $T = \{(m,n) \in \mathbb{N}\times\mathbb{N} : \mu(x_{mn}-x_2, \frac{r+\varepsilon}{2\kappa}) \leq 1-s$ or $\nu(x_{mn}-x_2, \frac{r+\varepsilon}{2\kappa}) \geq s\}$. Now for $(m,n) \in H^c \cap T^c$, we have $\mu(x_{mn}-[(1-\kappa)x_1+\kappa x_2], r+\varepsilon) \geq \mu((1-\kappa)(x_{mn}-x_1), \frac{r+\varepsilon}{2})\star\mu(\kappa(x_{mn}-x_2), \frac{r+\varepsilon}{2}) = \mu(x_{mn}-x_1, \frac{r+\varepsilon}{2(1-\kappa)})\star\mu(x_{mn}-x_2, \frac{r+\varepsilon}{2\kappa}) > (1-s)\star(1-s) > 1-\lambda$ and $\nu(x_{mn}-[(1-\kappa)x_1+\kappa x_2], r+\varepsilon) \leq \nu((1-\kappa)(x_{mn}-x_1), \frac{r+\varepsilon}{2})\circ \nu(\kappa(x_{mn}-x_2), \frac{r+\varepsilon}{2}) = \nu(x_{mn}-x_1, \frac{r+\varepsilon}{2(1-\kappa)})\circ\nu(x_{mn}-x_2, \frac{r+\varepsilon}{2\kappa}) < s\circ s < \lambda$, which gives that $\{(m,n) \in \mathbb{N}\times\mathbb{N} : \mu(x_{mn}-[(1-\kappa)x_1+\kappa x_2], r+\varepsilon) \leq 1-\lambda$ or $\nu(x_{mn}-[(1-\kappa)x_1+$

$\kappa x_2], r+\varepsilon) \geq \lambda\} \subset H \cup T$. Since $H \cup T \in \mathcal{I}_2$, $(1-\kappa)x_1 + \kappa x_2 \in \mathcal{I}_2^{(\mu,\nu)}\text{-}LIM^r_{x_{mn}}$. Therefore $\mathcal{I}_2^{(\mu,\nu)}\text{-}LIM^r_{x_{mn}}$ is convex. This completes the proof. □

Theorem 3.5. *A double sequence $\{x_{mn}\}$ in an IFNS $(X, \mu, \nu, \star, \circ)$ is rough $\mathcal{I}_2$-convergent to $\beta \in X$ with respect to the intuitionistic fuzzy normed spaces (μ, ν) for some $r > 0$ if there exists a double sequence $\{y_{mn}\}$ in X such that $y_{mn} \xrightarrow{\mathcal{I}_2^{(\mu,\nu)}} \beta$ and for every $\lambda \in (0,1)$, $\mu(x_{mn} - y_{mn}, r) > 1 - \lambda$ and $\nu(x_{mn} - y_{mn}, r) < \lambda$ for all $m, n \in \mathbb{N}$.*

Proof. Let $\varepsilon > 0$ be given. Now, for a given $\lambda \in (0,1)$ choose $s \in (0,1)$ such that $(1-s) \star (1-s) > 1-\lambda$ and $s \circ s < \lambda$. Supose that $y_{mn} \xrightarrow{\mathcal{I}_2^{(\mu,\nu)}} \beta$ and $\mu(x_{mn} - y_{mn}, r) > 1 - s$ and $\nu(x_{mn} - y_{mn}, r) < s$ for all $m, n \in \mathbb{N}$. Then the set $P = \{(m,n) \in \mathbb{N} \times \mathbb{N} : \mu(y_{mn} - \beta, \varepsilon) \leq 1 - s$ or $\nu(y_{mn} - \beta, \varepsilon) \geq s\} \in \mathcal{I}_2$. Now for $(i,j) \in P^c$, we have $\mu(x_{ij} - \beta, r+\varepsilon) \geq \mu(x_{ij} - y_{ij}, r) \star \mu(y_{ij} - \beta, \varepsilon) > (1-s) \star (1-s) > 1 - \lambda$ and $\nu(x_{ij} - \beta, r+\varepsilon) \leq \nu(x_{ij} - y_{ij}, r) \circ \nu(y_{ij} - \beta, \varepsilon) < s \circ s < \lambda$. Therefore $\{(i,j) \in \mathbb{N} \times \mathbb{N} : \mu(x_{ij} - \beta, r+\varepsilon) \leq 1 - \lambda$ or $\nu(x_{ij} - \beta, r+\varepsilon) \geq \lambda\} \subset P$. Since $P \in \mathcal{I}_2$, $\{(i,j) \in \mathbb{N} \times \mathbb{N} : \mu(x_{ij} - \beta, r+\varepsilon) \leq 1 - \lambda$ or $\nu(x_{ij} - \beta, r+\varepsilon) \geq \lambda\} \in \mathcal{I}_2$. Therefore $\{x_{mn}\}$ is rough $\mathcal{I}_2$-convergent to β with respect to the probabilistic norm (μ, ν). This completes the proof. □

Theorem 3.6. *Let $\{x_{mn}\}$ be a double sequence in an IFNS $(X, \mu, \nu, \star, \circ)$. Then there do not exist $\beta_1, \beta_2 \in \mathcal{I}_2^{(\mu,\nu)}\text{-}LIM^r_{x_{mn}}$ for some $r > 0$ and every $\lambda \in (0,1)$ such that $\mu(\beta_1 - \beta_2, mr) \leq 1 - \lambda$ and $\nu(\beta_1 - \beta_2, mr) \geq \lambda$ for $m(\in \mathbb{R}) > 2$.*

Proof. We prove it by contradiction. If possible, let there exists $\beta_1, \beta_2 \in \mathcal{I}_2^{(\mu,\nu)}\text{-}LIM^r_{x_{mn}}$ such that $\mu(\beta_1 - \beta_2, mr) \leq 1 - \lambda$ and $\nu(\beta_1 - \beta_2, mr) \geq \lambda$ for $m(\in \mathbb{R}) > 2$. Now, for a given $\lambda \in (0,1)$ choose $s \in (0,1)$ such that $(1-s) \star (1-s) > 1-\lambda$ and $s \circ s < \lambda$. Then for every $\varepsilon > 0$, the sets $A, B \in \mathcal{I}_2$ where $A = \{(m,n) \in \mathbb{N} \times \mathbb{N} : \mu(x_{mn} - \beta_1, r + \frac{\varepsilon}{2}) \leq 1 - s$ or $\nu(x_{mn} - \beta_1, r + \frac{\varepsilon}{2}) \geq s\} \in \mathcal{I}_2$ and $B = \{(m,n) \in \mathbb{N} \times \mathbb{N} : \mu(x_{mn} - \beta_2, r + \frac{\varepsilon}{2}) \leq 1 - s$ or $\nu(x_{mn} - \beta_2, r + \frac{\varepsilon}{2}) \geq s\} \in \mathcal{I}_2$. Then $A^c \cap B^c \in \mathcal{F}(\mathcal{I}_2)$. Now for $(m,n) \in A^c \cap B^c$, we have $\mu(\beta_1 - \beta_2, 2r + \varepsilon) \geq \mu(x_{mn} - \beta_1, r + \frac{\varepsilon}{2}) \star \mu(x_{mn} - \beta_2, r + \frac{\varepsilon}{2}) > (1-s) \star (1-s) > 1 - \lambda$ and $\nu(\beta_1 - \beta_2, 2r + \varepsilon) \leq \nu(x_{mn} - \beta_1, r + \frac{\varepsilon}{2}) \circ \nu(x_{mn} - \beta_2, r + \frac{\varepsilon}{2}) < s \circ s < \lambda$. Therefore,

$$\mu(\beta_1 - \beta_2, 2r + \varepsilon) > 1 - \lambda \text{ and } \nu(\beta_1 - \beta_2, 2r + \varepsilon) < \lambda \tag{3.1}$$

Now if we put $\varepsilon = mr - 2r, m > 2$ in Equation 3.1 then we have $\mu(\beta_1 - \beta_2, mr) > 1 - \lambda$ and $\nu(\beta_1 - \beta_2, mr) < \lambda$, which is a contradiction. This completes the proof. □

Now we define $\mathcal{I}_2$-cluster point analogue to Definition 2.7. Özcan and Or [34] defined rough statistical cluster point of double sequences in an IFNS and, here, we give its ideal version in the same sapce. Also, we prove an important result analogue to ([44], Theorem 4.7) in the same space which will be useful in the sequal.

Definition 3.3. Let $\{x_{mn}\}$ be a double sequence in an IFNS $(X, \mu, \nu, \star, \circ)$. Then a point $\zeta \in X$ is said to be $\mathcal{I}_2$-cluster point of $\{x_{mn}\}$ with respect to the intuitionistic fuzzy norm (μ, ν) if for every $\varepsilon > 0$ and $\lambda \in (0,1)$, $\{(m,n) \in \mathbb{N}\times\mathbb{N} : \mu(x_{mn}-\zeta,\varepsilon) > 1-\lambda$ and $\nu(x_{mn}-\zeta,\varepsilon) < \lambda\} \notin \mathcal{I}_2$.

We denote $\Gamma_{(x_{mn})}(\mathcal{I}_2^{(\mu,\nu)})$ to mean the set of all $\mathcal{I}_2$-cluster points of $\{x_{mn}\}$ with respect to the intuitionistic fuzzy norm (μ, ν) .

Definition 3.4. Let $\{x_{mn}\}$ be a double sequence in an IFNS $(X, \mu, \nu, \star, \circ)$ and $r \geq 0$. Then a point $\beta \in X$ is said to be rough $\mathcal{I}_2$-cluster point of $\{x_{mn}\}$ with respect to the intuitionistic fuzzy norm (μ, ν) if for every $\varepsilon > 0$ and $\lambda \in (0,1)$, $\{(m,n) \in \mathbb{N}\times\mathbb{N} : \mu(x_{mn}-\beta, r+\varepsilon) > 1-\lambda$ and $\nu(x_{mn}-\beta, r+\varepsilon) < \lambda\} \notin \mathcal{I}_2$.

We denote $\Gamma^r_{(x_{mn})}(\mathcal{I}_2^{(\mu,\nu)})$ to mean the set of all rough $\mathcal{I}_2$-cluster points of $\{x_{mn}\}$ with respect to the intuitionistic fuzzy norm (μ, ν) .

Remark 3.3. Now if we put $r = 0$ in Definition 3.4, then $\Gamma^r_{(x_{mn})}(\mathcal{I}_2^{(\mu,\nu)}) = \Gamma_{(x_{mn})}(\mathcal{I}_2^{(\mu,\nu)})$.

Theorem 3.7. *Let $\{x_{mn}\}$ be a double sequence in an IFNS $(X, \mu, \nu, \star, \circ)$ such that $x_{mn} \xrightarrow{\mathcal{I}_2^{(\mu,\nu)}} L$. Then $\Gamma_{(x_{mn})}(\mathcal{I}_2^{(\mu,\nu)}) = \{L\}$.*

Proof. If possible let $\Gamma_{(x_{mn})}(\mathcal{I}_2^{(\mu,\nu)}) = \{L, \mathcal{J}\}$, where $L \neq \mathcal{J}$. For a given $\lambda \in (0,1)$, choose $s \in (0,1)$ such that $(1-s)\star(1-s) > 1-\lambda$ and $(s)\circ(s) < \lambda$. Then for every $\varepsilon > 0$, $K_1 = \{(m,n) \in \mathbb{N}\times\mathbb{N} : \mu(x_{mn}-L, \frac{\varepsilon}{2}) > 1-s$ and $\nu(x_{mn}-L, \frac{\varepsilon}{2}) < s\} \notin \mathcal{I}_2$ and $K_2 = \{(m,n) \in \mathbb{N}\times\mathbb{N} : \mu(x_{mn}-\mathcal{J}, \frac{\varepsilon}{2}) > 1-s$ and $\nu(x_{mn}-\mathcal{J}, \frac{\varepsilon}{2}) < s\} \notin \mathcal{I}_2$. Clearly $K_1 \cap K_2 = \emptyset$, If not, let $(i,j) \in K_1 \cap K_2$. Then $\mu(L-\mathcal{J},\varepsilon) \geq \mu(x_{ij}-L,\frac{\varepsilon}{2}) \star \mu(x_{ij}-\mathcal{J},\frac{\varepsilon}{2}) > (1-s)\star(1-s) > 1-\lambda$ and $\nu(L-\mathcal{J},\varepsilon) \leq \nu(x_{ij}-L,\frac{\varepsilon}{2}) \circ \nu(x_{ij}-\mathcal{J},\frac{\varepsilon}{2}) < s \circ s < \lambda$. Since $\lambda \in (0,1)$ is arbitrary, $\mu(L-\mathcal{J},\varepsilon) = 1$, which gives $L = \mathcal{J}$ and $\nu(L-\mathcal{J},\varepsilon) = 0$, which gives $L = \mathcal{J}$ for all $\varepsilon > 0$. This yields to a contradiction. Therefore $K_2 \subset K_1^c$. Since $x_{mn} \xrightarrow{\mathcal{I}_2^{(\mu,\nu)}} L$, then $K_1^c = \{(m,n) \in \mathbb{N}\times\mathbb{N} : \mu(x_{mn}-L,\frac{\varepsilon}{2}) \leq 1-s$ or $\nu(x_{mn}-L,\frac{\varepsilon}{2}) \geq s\} \in \mathcal{I}_2$. Hence $K_2 \in \mathcal{I}_2$, which contradicts $K_2 \notin \mathcal{I}_2$. Therefore, $\Gamma_{(x_{mn})}(\mathcal{I}_2^{(\mu,\nu)}) = \{L\}$. This completes the proof. □

Theorem 3.8. *Let $\{x_{mn}\}$ be a double sequence in an IFNS $(X, \mu, \nu, \star, \circ)$. Then, for all $r > 0$ the set $\Gamma^r_{(x_{mn})}(\mathcal{I}_2^{(\mu,\nu)})$ is closed with respect to the intuitionistic fuzzy norm (μ, ν).*

Proof. The proof is almost similar to the proof of Theorem 3.3. So we omit details. □

Theorem 3.9. *Let $\{x_{mn}\}$ be a double sequence in an IFNS $(X, \mu, \nu, \star, \circ)$. Then for an arbitrary $x_1 \in \Gamma_{(x_{mn})}(\mathcal{I}_2^{(\mu,\nu)})$ and $\lambda \in (0,1)$ we have $\mu(x_2 - x_1, r) > 1 - \lambda$ and $\nu(x_2 - x_1, r) < \lambda$ for all $x_2 \in \Gamma^r_{(x_{mn})}(\mathcal{I}_2^{(\mu,\nu)})$.*

Proof. For given $\lambda \in (0,1)$, choose $s \in (0,1)$ such that $(1-s) \star (1-s) > 1 - \lambda$ and $s \circ s < \lambda$. Since $x_1 \in \Gamma_{(x_{mn})}(\mathcal{I}_2^{(\mu,\nu)})$, for every $\varepsilon > 0$, we get

$$\{(m,n) \in \mathbb{N} \times \mathbb{N} : \mu(x_{mn} - x_1, \varepsilon) > 1 - s \text{ and } \nu(x_{mn} - x_1, \varepsilon) < s\} \notin \mathcal{I}_2. \tag{3.2}$$

Now, let $(i,j) \in \{(m,n) \in \mathbb{N} \times \mathbb{N} : \mu(x_{mn} - x_1, \varepsilon) > 1 - s \text{ and } \nu(x_{mn} - x_1, \varepsilon) < s\}$. Then we have $\mu(x_{ij} - x_2, r + \varepsilon) \geq \mu(x_{ij} - x_1, \varepsilon) \star \mu(x_1 - x_2, r) > (1-s) \star (1-s) > 1 - \lambda$ and $\nu(x_{ij} - x_2, r + \varepsilon) \leq \nu(x_{ij} - x_1, \varepsilon) \circ \nu(x_2 - x_1, r) < s \circ s < \lambda$. Therefore $\{(m,n) \in \mathbb{N} \times \mathbb{N} : \mu(x_{mn} - x_1, \varepsilon) > 1 - s \text{ and } \nu(x_{mn} - x_1, \varepsilon) < s\} \subset \{(m,n) \in \mathbb{N} \times \mathbb{N} : \mu(x_{mn} - x_2, r + \varepsilon) > 1 - s \text{ and } \nu(x_{mn} - x_2, r + \varepsilon) < s\}$. So, from Equation 3.2, we get $\{(m,n) \in \mathbb{N} \times \mathbb{N} : \mu(x_{mn} - x_2, r + \varepsilon) > 1 - s \text{ and } \nu(x_{mn} - x_2, r + \varepsilon) < s\} \notin \mathcal{I}_2$. Hence $x_2 \in \Gamma^r_{(x_{mn})}(\mathcal{I}_2^{(\mu,\nu)})$. This completes the proof. □

Theorem 3.10. *Let $\{x_{mn}\}$ be a double sequence in an IFNS $(X, \mu, \nu, \star, \circ)$. Then for some $r > 0$, $\lambda \in (0,1)$ and fixed $x_0 \in X$ we have*

$$\Gamma^r_{(x_{mn})}(\mathcal{I}_2^{(\mu,\nu)}) = \bigcup_{x_0 \in \Gamma_{(x_{mn})}(\mathcal{I}_2^{(\mu,\nu)})} \overline{B(x_0, \lambda, r)}.$$

Proof. For a given $\lambda \in (0,1)$, choose $s \in (0,1)$ such that $(1-s) \star (1-s) > 1 - \lambda$ and $s \circ s < \lambda$. Let $y_0 \in \bigcup_{x_0 \in \Gamma_{(x_{mn})}(\mathcal{I}_2^{(\mu,\nu)})} \overline{B(x_0, \lambda, r)}$. Then there is $x_0 \in \Gamma_{(x_{mn})}(\mathcal{I}_2^{(\mu,\nu)})$ such that $\mu(x_0 - y_0, r) > 1 - s$ and $\nu(x_0 - y_0, r) < s$. Now, since $x_0 \in \Gamma_{(x_{mn})}(\mathcal{I}_2^{(\mu,\nu)})$, for every $\varepsilon > 0$ there exists a set $M = \{(m,n) \in \mathbb{N} \times \mathbb{N} : \mu(x_{mn} - x_0, \varepsilon) > 1 - s \text{ and } \nu(x_{mn} - x_0, \varepsilon) < s\}$ with $M \notin \mathcal{I}_2$. Let $(i,j) \in M$. Now we have $\mu(x_{ij} - y_0, r + \varepsilon) \geq \mu(x_{ij} - x_0, \varepsilon) \star \mu(x_0 - y_0, r) > (1-s) \star (1-s) > 1 - \lambda$ and $\nu(x_{ij} - y_0, r + \varepsilon) \leq \nu(x_{ij} - x_0, \varepsilon) \circ \nu(x_0 - y_0, r) < s \circ s < \lambda$. Therefore $M \subset \{(i,j) \in \mathbb{N} \times \mathbb{N} : \mu(x_{ij} - y_0, r + \varepsilon) > 1 - \lambda \text{ and } \nu(x_{ij} - y_0, r + \varepsilon) < \lambda\}$. Since $M \notin \mathcal{I}_2$, $\{(i,j) \in \mathbb{N} \times \mathbb{N} : \mu(x_{ij} - y_0, r + \varepsilon) > 1 - \lambda \text{ and } \nu(x_{ij} - y_0, r + \varepsilon) < \lambda\} \notin \mathcal{I}_2$. Hence $y_0 \in \Gamma^r_{(x_{mn})}(\mathcal{I}_2^{(\mu,\nu)})$. Therefore $\bigcup_{x_0 \in \Gamma_{(x_{mn})}(\mathcal{I}_2^{(\mu,\nu)})} \overline{B(x_0, \lambda, r)} \subseteq \Gamma^r_{(x_{mn})}(\mathcal{I}_2^{(\mu,\nu)})$.

Conversely, suppose that $y_* \in \Gamma^r_{(x_{mn})}(\mathcal{I}_2^{(\mu,\nu)})$. We shall show that $y_* \in \bigcup_{x_0 \in \Gamma_{(x_{mn})}(\mathcal{I}_2^{(\mu,\nu)})} \overline{B(x_0, \lambda, r)}$. If possible, let $y_* \notin \bigcup_{x_0 \in \Gamma_{(x_{mn})}(\mathcal{I}_2^{(\mu,\nu)})} \overline{B(x_0, \lambda, r)}$. So, $\mu(x_0 - y_*, r) \leq 1 - \lambda$ and $\nu(x_0 - y_*, r) \geq \lambda$ for every $x_0 \in \Gamma_{(x_{mn})}(\mathcal{I}_2^{(\mu,\nu)})$. Now, by Theorem 3.9, we have $\mu(x_0 - y_*, r) > 1 - \lambda$ and $\nu(x_0 - y_*, r) < \lambda$, which is a contradiction. Therefore, $\Gamma^r_{(x_{mn})}(\mathcal{I}_2^{(\mu,\nu)}) \subseteq \bigcup_{x_0 \in \Gamma_{(x_{mn})}(\mathcal{I}_2^{(\mu,\nu)})} \overline{B(x_0, \lambda, r)}$. Hence $\Gamma^r_{(x_{mn})}(\mathcal{I}_2^{(\mu,\nu)}) = \bigcup_{x_0 \in \Gamma_{(x_{mn})}(\mathcal{I}_2^{(\mu,\nu)})} \overline{B(x_0, \lambda, r)}$. This completes the proof. □

Theorem 3.11. *Let $\{x_{mn}\}$ be a double sequence in an IFNS $(X, \mu, \nu, \star, \circ)$. Then for any $\lambda \in (0,1)$ the following statements hold:*

1. *If* $x_0 \in \Gamma_{(x_{mn})}(\mathcal{I}_2^{(\mu,\nu)})$ *then* $\mathcal{I}_2^{(\mu,\nu)}$*-*$LIM^r_{x_{mn}} \subseteq \overline{B(x_0,\lambda,r)}$.

2. $\mathcal{I}_2^{(\mu,\nu)}$-$LIM^r_{x_{mn}} = \bigcap_{x_0 \in \Gamma_{(x_{mn})}(\mathcal{I}_2^{(\mu,\nu)})} \overline{B(x_0,\lambda,r)} = \{\eta \in X : \Gamma_{(x_{mn})}(\mathcal{I}_2^{(\mu,\nu)}) \subseteq \overline{B(\eta,\lambda,r)}\}$.

Proof. 1. For a given $\lambda \in (0,1)$, choose $s_1, s_2 \in (0,1)$ such that $(1-s_1)\star(1-s_2) > 1-\lambda$ and $s_1 \circ s_2 < \lambda$. If possible, we suppose that there exists an element $x_0 \in \Gamma_{(x_{mn})}(\mathcal{I}_2^{(\mu,\nu)})$ and $\gamma \in \mathcal{I}_2^{(\mu,\nu)}$-$LIM^r_{x_{mn}}$ such that $\gamma \notin \overline{B(x_0,\lambda,r)}$ i.e., $\mu(\gamma - x_0, r) < 1-\lambda$ and $\nu(\gamma - x_0, r) > \lambda$. Let $\varepsilon > 0$ be given. Since $\gamma \in \mathcal{I}_2^{(\mu,\nu)}$-$LIM^r_{x_{mn}}$, the sets $M_1 = \{(m,n) \in \mathbb{N}\times\mathbb{N} : \mu(x_{mn} - x_0, \varepsilon) > 1-s_1$ and $\nu(x_{mn}-x_0,\varepsilon) < s_1\} \notin \mathcal{I}_2$ and $M_2 = \{(m,n) \in \mathbb{N}\times\mathbb{N} : \mu(x_{mn} - \gamma, r+\varepsilon) \le 1-s_2$ or $\nu(x_{mn}-\gamma, r+\varepsilon) \ge s_2\} \in \mathcal{I}_2$. Now for $(i,j) \in M_1 \cap M_2^c$, we have $\mu(\gamma - x_0, r) \ge \mu(x_{ij}-x_0,\varepsilon)\star\mu(x_{ij}-\gamma, r+\varepsilon) > (1-s_1)\star(1-s_2) > 1-\lambda$ and $\nu(\gamma - x_0, r) \le \nu(x_{ij}-x_0,\varepsilon)\circ\nu(x_{ij}-\gamma,r+\varepsilon) < s_1 \circ s_2 < \lambda$, which is a contradiction. Therefore $\gamma \in \overline{B(x_0,\lambda,r)}$. Hence $\mathcal{I}_2^{(\mu,\nu)}$-$LIM^r_{x_{mn}} \subseteq \overline{B(x_0,\lambda,r)}$.

2. Using Part 1, we get

$$\mathcal{I}_2^{(\mu,\nu)}\text{-}LIM^r_{x_{mn}} \subseteq \bigcap_{x_0 \in \Gamma_{(x_{mn})}(\mathcal{I}_2^{(\mu,\nu)})} \overline{B(x_0,\lambda,r)}. \tag{3.3}$$

Now, let $\beta \in \bigcap_{x_0 \in \Gamma_{(x_{mn})}(\mathcal{I}_2^{(\mu,\nu)})} \overline{B(x_0,\lambda,r)}$. So, we have $\mu(\beta - x_0, r) \ge 1-\lambda$ and $\nu(\beta - x_0, r) \le \lambda$ for $x_0 \in \Gamma_{(x_{mn})}(\mathcal{I}_2^{(\mu,\nu)})$ and, therefore $\Gamma_{(x_{mn})}(\mathcal{I}_2^{(\mu,\nu)}) \subseteq \overline{B(\beta,\lambda,r)}$ i.e., we can write

$$\bigcap_{x_0 \in \Gamma_{(x_{mn})}(\mathcal{I}_2^{(\mu,\nu)})} \overline{B(x_0,\lambda,r)} \subseteq \{\eta \in X : \Gamma_{(x_{mn})}(\mathcal{I}_2^{(\mu,\nu)}) \subseteq \overline{B(\eta,\lambda,r)}\} \tag{3.4}$$

Now we shall show that $\{\eta \in X : \Gamma_{(x_{mn})}(\mathcal{I}_2^{(\mu,\nu)}) \subseteq \overline{B(\eta,\lambda,r)}\} \subseteq \mathcal{I}_2^{(\mu,\nu)}$-$LIM^r_{x_{mn}}$. Let $\beta \notin \mathcal{I}_2^{(\mu,\nu)}$-$LIM^r_{x_{mn}}$. Then for $\varepsilon > 0$, $\{(m,n) \in \mathbb{N}\times\mathbb{N} : \mu(x_{mn} - \beta, r+\varepsilon) \le 1-\lambda$ or $\nu(x_{mn}-\beta, r+\varepsilon) \ge \lambda\} \notin \mathcal{I}_2$, which gives there exists an $\mathcal{I}_2$-cluster point x_0 for the double sequence $\{x_{mn}\}$ with $\mu(\beta - x_0, r+\varepsilon) \le 1-\lambda$ and $\beta - x_0, r+\varepsilon) \ge \lambda$. Hence $\Gamma_{(x_{mn})}(\mathcal{I}_2^{(\mu,\nu)}) \nsubseteq \overline{B(\beta,\lambda,r)}$ and so, $\beta \notin \{\eta \in X : \Gamma_{(x_{mn})}(\mathcal{I}_2^{(\mu,\nu)}) \subseteq \overline{B(\eta,\lambda,r)}\}$. So,

$$\{\eta \in X : \Gamma_{(x_{mn})}(\mathcal{I}_2^{(\mu,\nu)}) \subseteq \overline{B(\eta,\lambda,r)}\} \subseteq \mathcal{I}_2^{(\mu,\nu)}\text{-}LIM^r_{x_{mn}}. \tag{3.5}$$

Therefore from Equations 3.3, 3.4 and 3.5, we have $\mathcal{I}_2^{(\mu,\nu)}$-$LIM^r_{x_{mn}} = \bigcap_{x_0 \in \Gamma_{(x_{mn})}(\mathcal{I}_2^{(\mu,\nu)})} \overline{B(x_0,\lambda,r)} = \{\eta \in X : \Gamma_{(x_{mn})}(\mathcal{I}_2^{(\mu,\nu)}) \subseteq \overline{B(\eta,\lambda,r)}\}$. This completes the proof.

□

Theorem 3.12. *Let $\{x_{mn}\}$ be a double sequence in an IFNS $(X, \mu, \nu, \star, \circ)$ such that $x_{mn} \xrightarrow{\mathcal{I}_2^{(\mu,\nu)}} y_*$. Then, for any $\lambda \in (0,1)$ and $r > 0$, we have $\mathcal{I}_2^{(\mu,\nu)}$-$LIM^r_{x_{mn}} = \overline{B(y_*, \lambda, r)}$.*

Proof. Let $\lambda_2 \in (0,1)$. Choose $\lambda_1 \in (0,1)$ such that $\lambda_1 \star \lambda > \lambda_2$ and $\lambda_1 \circ \lambda < \lambda_2$. Let $\varepsilon > 0$ be given. Since $x_{mn} \xrightarrow{\mathcal{I}_2^{(\mu,\nu)}} y_*$, the set $P = \{(m,n) \in \mathbb{N} \times \mathbb{N} : \mu(x_{mn} - y_*, \varepsilon) \leq 1 - \lambda_1$ or $\nu(x_{mn} - y_*, \varepsilon) \geq \lambda_1\} \in \mathcal{I}_2$. Now, let $\xi \in \overline{B(y_*, \lambda, r)}$. Then $\mu(\xi - y_*, r) \geq 1 - \lambda$ and $\nu(\xi - y_*, r) \leq \lambda$. Now for $(m,n) \in P^c$, we have $\mu(x_{mn} - \xi, r + \varepsilon) \geq \mu(x_{mn} - y_*, \varepsilon) \star \mu(\xi - y_*, r) > (1 - \lambda_1) \star (1 - \lambda) > 1 - \lambda_2$ and $\nu(x_{mn} - \xi, r + \varepsilon) \leq \nu(x_{mn} - y_*, \varepsilon) \circ \nu(\xi - y_*, r) < \lambda_1 \circ \lambda < \lambda_2$. Therefore $\{(m,n) \in \mathbb{N} \times \mathbb{N} : \mu(x_{mn} - \xi, r + \varepsilon) \leq 1 - \lambda_2$ or $\nu(x_{mn} - \xi, r + \varepsilon) \geq \lambda_2\} \subset P$. Hence $\xi \in \mathcal{I}_2^{(\mu,\nu)}$-$LIM^r_{x_{mn}}$. Consequently, $\overline{B(y_*, \lambda, r)} \subseteq \mathcal{I}_2^{(\mu,\nu)}$-$LIM^r_{x_{mn}}$. Now, using Theorem 3.7 and 3.11, we have $\mathcal{I}_2^{(\mu,\nu)}$-$LIM^r_{x_{mn}} \subseteq \overline{B(y_*, \lambda, r)}$. Therefore $\mathcal{I}_2^{(\mu,\nu)}$-$LIM^r_{x_{mn}} = \overline{B(y_*, \lambda, r)}$. This completes the proof. □

Theorem 3.13. *Let $\{x_{mn}\}$ be a double sequence in an IFNS $(X, \mu, \nu, \star, \circ)$ such that $x_{mn} \xrightarrow{\mathcal{I}_2^{(\mu,\nu)}} L$. Then, for any $\lambda \in (0,1)$, $\Gamma^r_{(x_{mn})}(\mathcal{I}_2^{(\mu,\nu)}) = \mathcal{I}_2^{(\mu,\nu)}$-$LIM^r_{x_{mn}}$ for some $r > 0$.*

Proof. Since $x_{mn} \xrightarrow{\mathcal{I}_2^{(\mu,\nu)}} L$, therefore from Theorem 3.7, $\Gamma_{(x_{mn})}(\mathcal{I}_2^{(\mu,\nu)}) = \{L\}$. Again from Theorem 3.10, $\Gamma^r_{(x_{mn})}(\mathcal{I}_2^{(\mu,\nu)}) = \overline{B(L, \lambda, r)}$. And, from Theorem 3.12, $\overline{B(L, \lambda, r)} = \mathcal{I}_2^{(\mu,\nu)}$-$LIM^r_{x_{mn}}$. Therefore $\Gamma^r_{(x_{mn})}(\mathcal{I}_2^{(\mu,\nu)}) = \mathcal{I}_2^{(\mu,\nu)}$-$LIM^r_{x_{mn}}$. This completes the proof. □

Conclusion

We have discussed the notions of rough ideal convergence and rough ideal cluster points of double sequences in intuitionistic fuzzy normed spaces. Some of the basic properties have been discussed in an intuitionistic fuzzy normed space, remembering the structruc of this space. Most importantly, it has been verified that $\{x_{mn}\}$ be a double sequence in an IFNS $(X, \mu, \nu, \star, \circ)$. Then for all $r > 0$, the set $\mathcal{I}_2^{(\mu,\nu)}$-$LIM^r_{x_{mn}}$ is bounded and convex. Some new properties can be generated by the structure of intuitionistic fuzzy normed spaces, which can serve as the motivation for a new study.

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