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SOME NEW INEQUALITIES FOR (s, P)-FUNCTIONS

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Abstract. In this paper, we obtain some new Hermite-Hadamard type inequalities for functions whose first derivative in absolute value is (s, P)-function by using Hölder, power-mean and Hölder-İşcan integral inequalities. Then, the authors compare the results obtained with both Hölder and Hölder-İşcan integral inequalities and prove that the Hölder-İşcan integral inequality gives a better approximation than the Hölder integral inequality. Next, we point out some applications for certain inequalities related to special means of real numbers.
Keywords: (s, P)-function, Hermite-Hadamard inequality, Hölder-İşcan inequality.

1. Preliminaries

Let $f : I \to \mathbb{R}$ be a convex function. Then the following inequalities hold

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(u)du \le \frac{f(a)+f(b)}{2}$$

for all $a, b \in I$ with $a < b$. Both inequalities hold in the reversed direction if the function f is concave. This double inequality is well known as the Hermite-Hadamard inequality [8]. Note that some of the classical inequalities for means can

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be derived from Hermite-Hadamard integral inequalities for appropriate particular selections of the mapping f.

In [7], Dragomir et al. gave the following definition and related Hermite-Hadamard integral inequalities as follows:

Definition 1.1. *A nonnegative function $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is said to be P-function if the inequality*

$$f(tx + (1-t)y) \leq f(x) + f(y)$$

holds for all $x, y \in I$ and $t \in (0,1)$.

Theorem 1.1. *Let $f \in P(I)$, $a, b \in I$ with $a < b$ and $f \in L[a,b]$. Then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{2}{b-a} \int_a^b f(x)\,dx \leq 2\left[f(a) + f(b)\right]. \tag{1.1}$$

Definition 1.2. [15] Let $h : J \rightarrow \mathbb{R}$ be a non-negative function, $h \neq 0$. We say that $f : I \rightarrow \mathbb{R}$ is an h-convex function, or that f belongs to the class $SX(h, I)$, if f is non-negative and for all $x, y \in I$, $\alpha \in (0,1)$ we have

$$f(\alpha x + (1-\alpha)y) \leq h(\alpha)f(x) + h(1-\alpha)f(y).$$

If this inequality is reversed, then f is said to be h-concave, i.e. $f \in SV(h, I)$. It is clear that, if we choose $h(\alpha) = \alpha$ and $h(\alpha) = 1$, then the h-convexity reduces to convexity and definition of P-function, respectively.

Readers can look at [3, 15] for studies on h-convexity.

In [9], Hudzik and Maligranda considered among others the class of functions which are s-convex in the second sense.

Definition 1.3. A function $f : [0,\infty) \rightarrow \mathbb{R}$ is said to be $s-$convex in the second sense if

$$f(\alpha x + \beta y) \leq \alpha^s f(x) + \beta^s f(y)$$

for all $x, y \in [0,\infty)$, $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$ and for some fixed $s \in (0,1]$. This class of s-convex functions in the second sense is usually denoted by K_s^2.

It can be easily seen that for $s = 1$, s-convexity reduces to ordinary convexity of functions defined on $[0,\infty)$.

In [6], Dragomir and Fitzpatrick proved a variant of Hadamard's inequality which holds for s-convex functions in the second sense.

Theorem 1.2. *Suppose that $f:[0,\infty)\to[0,\infty)$ is an s-convex function in the second sense, where $s\in(0,1)$, and let $a,b\in[0,\infty)$, $a<b$.If $f\in L[a,b]$ then the following inequalities hold*

$$2^{s-1}f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int\limits_a^b f(x)dx\leq\frac{f(a)+f(b)}{s+1}. \tag{1.2}$$

Both inequalities hold in the reversed direction if f is s-concave. The constant $k=\frac{1}{s+1}$ is the best possible in the second inequality in (1.2).

In [14], Numan and İşcan gave the following definition and Hermite-Hadamard integral inequality for the (s,P)-functions:

Definition 1.4. [14] Let $s\in(0,1]$. A function $f:I\subset\mathbb{R}\to\mathbb{R}$ is called (s,P)-function if

$$f\left(tx+(1-t)y\right)\leq\left(t^s+(1-t)^s\right)\left[f(x)+f(y)\right] \tag{1.3}$$

for every $x,y\in I$ and $t\in[0,1]$.

Denoted by $P_s(I)$ the class of all (s,P)-functions on interval I. Clearly, the definition of $(1,P)$-function is coincide with the definition of P-function.

We note that, every (s,P)-function is a h-convex function with the function $h(t)=t^s+(1-t)^s$.

Theorem 1.3. [14] *Let $s\in(0,1]$ and $f:[a,b]\to\mathbb{R}$ be a (s,P)-function. If $a<b$ and $f\in L[a,b]$, then the following Hermite-Hadamard type inequalities hold:*

$$2^{s-2}f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)dx\leq\frac{2}{s+1}\left[f(a)+f(b)\right].$$

Theorem 1.4. Hölder-İşcan integral inequality [10] *Let $p>1$ and $\frac{1}{p}+\frac{1}{q}=1$. If f and g are real functions defined on interval $[a,b]$ and if $|f|^p$, $|g|^q$ are integrable functions on $[a,b]$ then*

$$\begin{aligned}&\int_a^b|f(x)g(x)|\,dx\\ \leq\;&\frac{1}{b-a}\left\{\left(\int_a^b(b-x)\,|f(x)|^p\,dx\right)^{\frac{1}{p}}\left(\int_a^b(b-x)\,|g(x)|^q\,dx\right)^{\frac{1}{q}}\right.\\ &\left.+\left(\int_a^b(x-a)\,|f(x)|^p\,dx\right)^{\frac{1}{p}}\left(\int_a^b(x-a)\,|g(x)|^q\,dx\right)^{\frac{1}{q}}\right\}.\end{aligned}$$

Theorem 1.5. Improved power-mean integral inequality [13] *Let $q \geq 1$. If f and g are real functions defined on interval $[a,b]$ and if $|f|$, $|f|\,|g|^q$ are integrable functions on $[a,b]$, then*

$$\begin{aligned}
&\int_a^b |f(x)g(x)|\,dx \\
\leq\ & \frac{1}{b-a}\left\{\left(\int_a^b (b-x)\,|f(x)|\,dx\right)^{1-\frac{1}{q}}\left(\int_a^b (b-x)\,|f(x)|\,|g(x)|^q\,dx\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_a^b (x-a)\,|f(x)|\,dx\right)^{1-\frac{1}{q}}\left(\int_a^b (x-a)\,|f(x)|\,|g(x)|^q\,dx\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

The main purpose of this manuscript is to establish some new Hermite-Hadamard type inequality for the (s,P)-functions. In recent years many authors have studied error estimations of Hermite-Hadamard type inequalities; for refinements, counterparts, generalizations, for some related papers see [1,2,5,7,11,12].

2. Some new inequalities for the (s,P)-functions

The main purpose of this section is to establish new estimates that refine Hermite-Hadamard integral inequality for functions whose first derivative in absolute value is (s,P)-function and we will compare the results obtained with both Hölder, Hölder-İşcan integral inequalities and prove that the Hölder-İşcan integral inequality gives a better approximation than the Hölder integral inequality. Cerone and Dragomir [4] used the following lemma:

Lemma 2.1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° where $a,b \in I^\circ$ with $a<b$. If $f' \in L[a,b]$, then the following inequality holds:*

$$\begin{aligned}
&f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx \\
=\ & \frac{b-a}{4}\left[\int_0^1 tf'\left(t\frac{a+b}{2}+(1-t)a\right)dt+\int_0^1 (t-1)f'\left(tb+(1-t)\frac{a+b}{2}\right)dt\right].
\end{aligned}$$

Theorem 2.1. *Let $f : I \to \mathbb{R}$ be a differentiable mapping on I°, $a,b \in I^\circ$ with $a<b$ and assume that $f' \in L[a,b]$ and $s \in (0,1]$. If $|f'|$ is an (s,P)-function on interval $[a,b]$, then the following inequality holds*

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq\ & \frac{b-a}{2}\frac{1}{s+1}\left[A\left(|f'(a)|,|f'(b)|\right)+\left|f'\left(\frac{a+b}{2}\right)\right|\right],
\end{aligned} \tag{2.1}$$

where $A(.,.)$ is the arithmetic mean.

Proof. Using Lemma 2.1 and the inequalities

$$\begin{aligned}\left|f'\left(t\frac{a+b}{2}+(1-t)a\right)\right| &\leq \left(t^s+(1-t)^s\right)\left[\left|f'\left(\frac{a+b}{2}\right)\right|+\left|f'(a)\right|\right],\\ \left|f'\left(tb+(1-t)\frac{a+b}{2}\right)\right| &\leq \left(t^s+(1-t)^s\right)\left[\left|f'(b)\right|+\left|f'\left(\frac{a+b}{2}\right)\right|\right],\end{aligned}$$

we obtain

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|\\ \leq\ & \frac{b-a}{4}\left[\int_0^1 |t|\left|f'\left(t\frac{a+b}{2}+(1-t)a\right)\right|dt\right.\\ &\left.+\int_0^1 |1-t|\left|f'\left(tb+(1-t)\frac{a+b}{2}\right)\right|dt\right]\\ \leq\ & \frac{b-a}{4}\int_0^1 t\left(t^s+(1-t)^s\right)\left[\left|f'\left(\frac{a+b}{2}\right)\right|+\left|f'(a)\right|\right]dt\\ &+\frac{b-a}{4}\int_0^1 (1-t)\left(t^s+(1-t)^s\right)\left[\left|f'(b)\right|+\left|f'\left(\frac{a+b}{2}\right)\right|\right]dt\\ =\ & \frac{b-a}{2}\frac{1}{s+1}\left[A\left(\left|f'(a)\right|,\left|f'(b)\right|\right)+\left|f'\left(\frac{a+b}{2}\right)\right|\right],\end{aligned}$$

where

$$\int_0^1 t\left(t^s+(1-t)^s\right)dt=\int_0^1 (1-t)\left(t^s+(1-t)^s\right)dt=\frac{1}{s+1}.$$

This completes the proof of the theorem. □

Theorem 2.2. *Let $f: I \to \mathbb{R}$ be a differentiable mapping on I°, $a,b \in I^\circ$ with $a<b$ and assume that $f' \in L[a,b]$ and $s \in (0,1]$. If $\left|f'\right|^q$, $q>1$, is an (s,P)-function on interval $[a,b]$, then the following inequality holds*

$$\begin{aligned}(2.2)\qquad &\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|\\ \leq\ & 2^{\frac{1}{q}}\frac{b-a}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{2}{s+1}\right)^{\frac{1}{q}}\\ &\times\left[A^{\frac{1}{q}}\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q,\left|f'(a)\right|^q\right)+A^{\frac{1}{q}}\left(\left|f'(b)\right|^q,\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)\right],\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$ and $A(.,.)$ is the arithmetic mean.

Proof. Using Lemma 2.1, Hölder's integral inequality and the following inequalities

$$\begin{aligned}\left|f'\left(t\frac{a+b}{2}+(1-t)a\right)\right|^q &\leq (t^s+(1-t)^s)\left[\left|f'\left(\frac{a+b}{2}\right)\right|^q+|f'(a)|^q\right],\\ \left|f'\left(tb+(1-t)\frac{a+b}{2}\right)\right|^q &\leq (t^s+(1-t)^s)\left[|f'(b)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q\right],\end{aligned}$$

which is the (s,P)-function of $|f'|^q$, we have

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\\ \leq\ & \frac{b-a}{4}\left[\int_0^1 |t|\left|f'\left(t\frac{a+b}{2}+(1-t)a\right)\right|dt\right.\\ &\left.+\int_0^1 |1-t|\left|f'\left(tb+(1-t)\frac{a+b}{2}\right)\right|dt\right]\\ \leq\ & \frac{b-a}{4}\left(\int_0^1 t^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(t\frac{a+b}{2}+(1-t)a\right)\right|^q dt\right)^{\frac{1}{q}}\\ &+\frac{b-a}{4}\left(\int_0^1 (1-t)^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(tb+(1-t)\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}}\\ \leq\ & \frac{b-a}{4}\left(\int_0^1 t^p dt\right)^{\frac{1}{p}}\left(\int_0^1 (t^s+(1-t)^s)\left[\left|f'\left(\frac{a+b}{2}\right)\right|^q+|f'(a)|^q\right]dt\right)^{\frac{1}{q}}\\ &+\frac{b-a}{4}\left(\int_0^1 (1-t)^p dt\right)^{\frac{1}{p}}\left(\int_0^1 (t^s+(1-t)^s)\left[|f'(b)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q\right]dt\right)^{\frac{1}{q}}\\ =\ & \frac{b-a}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{2}{s+1}\left[\left|f'\left(\frac{a+b}{2}\right)\right|^q+|f'(a)|^q\right]\right)^{\frac{1}{q}}\\ &+\frac{b-a}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{2}{s+1}\left[|f'(b)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q\right]\right)^{\frac{1}{q}}\\ =\ & 2^{\frac{2}{q}}\frac{b-a}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\\ &\times\left[A^{\frac{1}{q}}\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q,|f'(a)|^q\right)+A^{\frac{1}{q}}\left(|f'(b)|^q,\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)\right],\end{aligned}$$

where

$$\begin{aligned}\int_0^1 |t|^p\,dt &= \int_0^1 |1-t|^p\,dt=\frac{1}{p+1},\\ \int_0^1 (t^s+(1-t)^s)\,dt &= \frac{2}{s+1}.\end{aligned}$$

This completes the proof of the theorem. □

Theorem 2.3. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^{\circ}$ with $a < b$ and assume that $f' \in L[a, b]$ and $s \in (0, 1]$ If $|f'|^{q}$, $q \geq 1$, is an (s, P)-function on the interval $[a, b]$, then the following inequality holds*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx \right| \leq \frac{b-a}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}} \times \left\{ \left[\left| f'\left(\frac{a+b}{2}\right)\right|^{q} + |f'(a)|^{q} \right] + \left[|f'(b)|^{q} + \left| f'\left(\frac{a+b}{2}\right)\right|^{q} \right]^{\frac{1}{q}} \right\}. \tag{2.3}$$

Proof. From Lemma 2.1, well known power-mean integral inequality and the property of the (s, P)-function of the function $|f'|^{q}$, we get

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx \right| \\
\leq\ & \frac{b-a}{4}\left[\int_0^1 |t| \left| f'\left(t\frac{a+b}{2} + (1-t)a\right)\right| dt \right. \\
&\left. + \int_0^1 |1-t| \left| f'\left(tb + (1-t)\frac{a+b}{2}\right)\right| dt \right] \\
\leq\ & \frac{b-a}{4}\left(\int_0^1 tdt\right)^{1-\frac{1}{q}} \left(\int_0^1 t\left|f'\left(ta + (1-t)b\right)\right|^{q} dt\right)^{\frac{1}{q}} \\
&+ \frac{b-a}{4}\left(\int_0^1 (1-t)dt\right)^{1-\frac{1}{q}} \left(\int_0^1 (1-t)\left|f'\left(ta + (1-t)b\right)\right|^{q} dt\right)^{\frac{1}{q}} \\
\leq\ & \frac{b-a}{4}\left(\int_0^1 tdt\right)^{1-\frac{1}{q}} \left(\int_0^1 t\left(t^{s} + (1-t)^{s}\right)\left[\left|f'\left(\frac{a+b}{2}\right)\right|^{q} + |f'(a)|^{q}\right] dt\right)^{\frac{1}{q}} \\
&+ \frac{b-a}{4}\left(\int_0^1 (1-t)dt\right)^{1-\frac{1}{q}} \\
&\times \left(\int_0^1 (1-t)\left(t^{s} + (1-t)^{s}\right)\left[|f'(b)|^{q} + \left|f'\left(\frac{a+b}{2}\right)\right|^{q}\right] dt\right)^{\frac{1}{q}} \\
=\ & \frac{b-a}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}} \\
&\times \left\{ \left[\left| f'\left(\frac{a+b}{2}\right)\right|^{q} + |f'(a)|^{q} \right] + \left[|f'(b)|^{q} + \left| f'\left(\frac{a+b}{2}\right)\right|^{q} \right]^{\frac{1}{q}} \right\},
\end{aligned}$$

where

$$\begin{aligned}
\int_0^1 tdt &= \int_0^1 (1-t)dt = \frac{1}{2}, \\
\int_0^1 t\left(t^{s} + (1-t)^{s}\right)dt &= \int_0^1 (1-t)\left(t^{s} + (1-t)^{s}\right)dt = \frac{1}{s+1}.
\end{aligned}$$

This completes the proof of the theorem. □

Corollary 2.1. *Under the assumption of Theorem 2.3, If we take $q=1$ in the inequality (2.3), then we get the following inequality:*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{b-a}{2}\frac{1}{s+1}\left[A\left(\left|f'(a)\right|,\left|f'(b)\right|\right)+\left|f'\left(\frac{a+b}{2}\right)\right|\right].$$

This inequality coincides with the inequality (2.1).

Theorem 2.4. *Let $f:I\to\mathbb{R}$ be a differentiable mapping on I°, $a,b\in I^{\circ}$ with $a<b$ and assume that $f'\in L[a,b]$ and $s\in(0,1]$. If $|f'|^q$, $q>1$, is an (s,P)-function on interval $[a,b]$, then the following inequality holds*

$$\begin{aligned}(2.4)\qquad &\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|\\ \leq\ & 2^{\frac{1}{q}}\frac{b-a}{4}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\left[\left(\frac{1}{(p+1)(p+2)}\right)^{\frac{1}{p}}+\left(\frac{1}{p+2}\right)^{\frac{1}{p}}\right]\\ &\times\left[A^{\frac{1}{q}}\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q,\left|f'(a)\right|^q\right)+A^{\frac{1}{q}}\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q,\left|f'(b)\right|^q\right)\right],\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$ and $A(.,.)$ is the arithmetic mean.

Proof. Using Lemma 2.1, Hölder-İşcan integral inequality and the following inequalities

$$\begin{aligned}\left|f'\left(t\frac{a+b}{2}+(1-t)a\right)\right|^q &\leq (t^s+(1-t)^s)\left[\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'(a)\right|^q\right]\\ \left|f'\left(tb+(1-t)\frac{a+b}{2}\right)\right|^q &\leq (t^s+(1-t)^s)\left[\left|f'(b)\right|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q\right],\end{aligned}$$

we have

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|\\ \leq\ &\frac{b-a}{4}\left[\int_0^1|t|\left|f'\left(t\frac{a+b}{2}+(1-t)a\right)\right|dt\right.\\ &\left.+\int_0^1|1-t|\left|f'\left(tb+(1-t)\frac{a+b}{2}\right)\right|dt\right]\\ \leq\ &\frac{b-a}{4}\left(\int_0^1(1-t)t^pdt\right)^{\frac{1}{p}}\left(\int_0^1(1-t)\left|f'\left(t\frac{a+b}{2}+(1-t)a\right)\right|^q dt\right)^{\frac{1}{q}}\end{aligned}$$

$$
\begin{aligned}
&+\frac{b-a}{4}\left(\int_0^1 tt^p dt\right)^{\frac{1}{p}}\left(\int_0^1 t\left|f'\left(t\frac{a+b}{2}+(1-t)a\right)\right|^q dt\right)^{\frac{1}{q}}\\
&+\frac{b-a}{4}\left(\int_0^1 (1-t)(1-t)^p dt\right)^{\frac{1}{p}}\left(\int_0^1 (1-t)\left|f'\left(tb+(1-t)\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}}\\
&+\frac{b-a}{4}\left(\int_0^1 t(1-t)^p dt\right)^{\frac{1}{p}}\left(\int_0^1 t\left|f'\left(tb+(1-t)\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}}\\
= \;&\frac{b-a}{4}\left(\frac{1}{(p+1)(p+2)}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\left[\left|f'\left(\frac{a+b}{2}\right)\right|^q+|f'(a)|^q\right]\right)^{\frac{1}{q}}\\
&+\frac{b-a}{4}\left(\frac{1}{p+2}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\left[\left|f'\left(\frac{a+b}{2}\right)\right|^q+|f'(a)|^q\right]\right)^{\frac{1}{q}}\\
&+\frac{b-a}{4}\left(\frac{1}{p+2}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\left[|f'(b)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q\right]\right)^{\frac{1}{q}}\\
&+\frac{b-a}{4}\left(\frac{1}{(p+1)(p+2)}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\left|f'\left(tb+(1-t)\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\\
= \;&\frac{b-a}{4}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\left[\left(\frac{1}{(p+1)(p+2)}\right)^{\frac{1}{p}}+\left(\frac{1}{p+2}\right)^{\frac{1}{p}}\right]\left[\left|f'\left(\frac{a+b}{2}\right)\right|^q+|f'(a)|^q\right]^{\frac{1}{q}}\\
&+\frac{b-a}{4}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\left[\left(\frac{1}{(p+1)(p+2)}\right)^{\frac{1}{p}}+\left(\frac{1}{p+2}\right)^{\frac{1}{p}}\right]\left[|f'(b)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q\right]^{\frac{1}{q}}\\
= \;&2^{\frac{1}{q}}\frac{b-a}{4}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\left[\left(\frac{1}{(p+1)(p+2)}\right)^{\frac{1}{p}}+\left(\frac{1}{p+2}\right)^{\frac{1}{p}}\right]\\
&\times\left[A^{\frac{1}{q}}\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q,|f'(a)|^q\right)+A^{\frac{1}{q}}\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q,|f'(b)|^q\right)\right],
\end{aligned}
$$

where

$$
\begin{aligned}
\int_0^1 (1-t)t^p dt &= \int_0^1 t(1-t)^p dt = \frac{1}{(p+1)(p+2)},\\
\int_0^1 t^{p+1} dt &= \int_0^1 (1-t)^{p+1} dt = \frac{1}{p+2}, \frac{1}{s+1},\\
\int_0^1 (1-t)\left(t^s+(1-t)^s\right)dt &= \int_0^1 t\left(t^s+(1-t)^s\right)dt = \frac{1}{s+1}.
\end{aligned}
$$

This completes the proof of the theorem. □

Remark 2.1. The inequality (2.4) gives better result than the inequality (2.2). Let us show that

$$
\left(\frac{1}{(p+1)(p+2)}\right)^{\frac{1}{p}}+\left(\frac{1}{p+2}\right)^{\frac{1}{p}}\le 2^{\frac{1}{q}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}
$$

If we use the concavity of the function $h:[0,\infty)\to\mathbb{R}$, $h(x)=x^{\lambda}, 0<\lambda\leq 1$, we get

$$\begin{aligned}\left(\frac{1}{(p+1)(p+2)}\right)^{\frac{1}{p}}+\left(\frac{1}{p+2}\right)^{\frac{1}{p}} &\leq 2\left[\frac{1}{2}\left(\frac{1}{(p+1)(p+2)}\right)^{\frac{1}{p}}+\frac{1}{2}\left(\frac{1}{p+2}\right)^{\frac{1}{p}}\right]\\ &= 2.2^{-\frac{1}{p}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\\ &= 2^{\frac{1}{q}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}},\end{aligned}$$

which completes the proof of remark.

Theorem 2.5. *Let $f:I\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I°, $a,b\in I^{\circ}$ with $a<b$ and assume that $f'\in L[a,b]$ and $s\in(0,1]$ If $|f'|^{q}, q\geq 1$, is an (s,P)-function on the interval $[a,b]$, then the following inequality holds*

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \qquad (2.5)\\ \leq\ &\frac{b-a}{4}\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\left[\left(\frac{2}{(s+2)(s+3)}\right)^{\frac{1}{q}}\right]\left[\left|f'\left(\frac{a+b}{2}\right)\right|^{q}+|f'(a)|^{q}\right]^{\frac{1}{q}}\\ &+\frac{b-a}{4}\left(\frac{1}{3}\right)^{1-\frac{1}{q}}\left(\frac{s^2+3s+4}{(s+1)(s+2)(s+3)}\right)^{\frac{1}{q}}\left[\left|f'\left(\frac{a+b}{2}\right)\right|^{q}+|f'(a)|^{q}\right]^{\frac{1}{q}}\\ &+\frac{b-a}{4}\left(\frac{1}{3}\right)^{1-\frac{1}{q}}\left[\left(\frac{s^2+3s+4}{(s+1)(s+2)(s+3)}\right)^{\frac{1}{q}}\right]\left[|f'(b)|^{q}+\left|f'\left(\frac{a+b}{2}\right)\right|^{q}\right]^{\frac{1}{q}}\\ &+\frac{b-a}{4}\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\left(\frac{2}{(s+2)(s+3)}\right)^{\frac{1}{q}}\left[|f'(b)|^{q}+\left|f'\left(\frac{a+b}{2}\right)\right|^{q}\right]^{\frac{1}{q}}.\end{aligned}$$

Proof. From Lemma 2.1, improved power-mean integral inequality and the property of the (s,P)-function of the function $|f'|^{q}$, we get

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|\\ \leq\ &\frac{b-a}{4}\left[\int_0^1 |t|\left|f'\left(t\frac{a+b}{2}+(1-t)a\right)\right|dt\right.\\ &\left.+\int_0^1 |1-t|\left|f'\left(tb+(1-t)\frac{a+b}{2}\right)\right|dt\right]\\ \leq\ &\frac{b-a}{4}\left(\int_0^1(1-t)tdt\right)^{1-\frac{1}{q}}\left(\int_0^1(1-t)t\left|f'\left(t\frac{a+b}{2}+(1-t)a\right)\right|^{q}dt\right)^{\frac{1}{q}}\\ &+\frac{b-a}{4}\left(\int_0^1 t^2dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^2\left|f'\left(t\frac{a+b}{2}+(1-t)a\right)\right|^{q}dt\right)^{\frac{1}{q}}\end{aligned}$$

$$
\begin{aligned}
&+\frac{b-a}{4}\left(\int_0^1 (1-t)^2dt\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-t)^2\left|f'\left(tb+(1-t)\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}}\\
&+\frac{b-a}{4}\left(\int_0^1 t(1-t)dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t(1-t)\left|f'\left(tb+(1-t)\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}}\\
\leq\ & \frac{b-a}{4}\left(\int_0^1 (1-t)tdt\right)^{1-\frac{1}{q}}\\
&\times\left(\left[\left|f'\left(\frac{a+b}{2}\right)\right|^q+|f'(a)|^q\right]\int_0^1 (1-t)t\left(t^s+(1-t)^s\right)dt\right)^{\frac{1}{q}}\\
&+\frac{b-a}{4}\left(\int_0^1 t^2dt\right)^{1-\frac{1}{q}}\left(\left[\left|f'\left(\frac{a+b}{2}\right)\right|^q+|f'(a)|^q\right]\int_0^1 t^2\left(t^s+(1-t)^s\right)dt\right)^{\frac{1}{q}}\\
&+\frac{b-a}{4}\left(\int_0^1 (1-t)^2dt\right)^{1-\frac{1}{q}}\\
&\times\left(\left[|f'(b)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q\right]\int_0^1 (1-t)^2\left(t^s+(1-t)^s\right)dt\right)^{\frac{1}{q}}\\
&+\frac{b-a}{4}\left(\int_0^1 t(1-t)dt\right)^{1-\frac{1}{q}}\\
&\times\left(\left[|f'(b)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q\right]\int_0^1 t(1-t)\left(t^s+(1-t)^s\right)dt\right)^{\frac{1}{q}}\\
=\ & \frac{b-a}{4}\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\left[\left(\frac{2}{(s+2)(s+3)}\right)^{\frac{1}{q}}\right]\left[\left|f'\left(\frac{a+b}{2}\right)\right|^q+|f'(a)|^q\right]^{\frac{1}{q}}\\
&+\frac{b-a}{4}\left(\frac{1}{3}\right)^{1-\frac{1}{q}}\left(\frac{s^2+3s+4}{(s+1)(s+2)(s+3)}\right)^{\frac{1}{q}}\left[\left|f'\left(\frac{a+b}{2}\right)\right|^q+|f'(a)|^q\right]^{\frac{1}{q}}\\
&+\frac{b-a}{4}\left(\frac{1}{3}\right)^{1-\frac{1}{q}}\left[\left(\frac{s^2+3s+4}{(s+1)(s+2)(s+3)}\right)^{\frac{1}{q}}\right]\left[|f'(b)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q\right]^{\frac{1}{q}}\\
&+\frac{b-a}{4}\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\left(\frac{2}{(s+2)(s+3)}\right)^{\frac{1}{q}}\left[|f'(b)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q\right]^{\frac{1}{q}}
\end{aligned}
$$

where

$$
\begin{aligned}
\int_0^1 (1-t)tdt &= \frac{1}{6},\quad \int_0^1 t^2dt=\int_0^1 (1-t)^2dt=\frac{1}{3},\\
\int_0^1 (1-t)t\left(t^s+(1-t)^s\right)dt &= \frac{2}{(s+2)(s+3)},\\
\int_0^1 t^2\left(t^s+(1-t)^s\right)dt &= \int_0^1 (1-t)^2\left(t^s+(1-t)^s\right)dt=\frac{s^2+3s+4}{(s+1)(s+2)(s+3)}.
\end{aligned}
$$

This completes the proof of the theorem. □

Corollary 2.2. *Under the assumption of Theorem 2.5, If we take $q = 1$ in the inequality (2.5), then we get the following inequality:*

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|\\ \le\ &\frac{b-a}{4}\left(\frac{s+6}{(s+2)(s+3)}\right)\left(\left[\left|f'\left(\frac{a+b}{2}\right)\right|+|f'(a)|\right]+\left[|f'(b)|+\left|f'\left(\frac{a+b}{2}\right)\right|\right]\right).\end{aligned}$$

3. Applications for special means for the (s, P)-functions

Throughout this section, for shortness, the following notations will be used for special means of two nonnegative numbers a, b with $b > a$:

1. The arithmetic mean

$$A := A(a,b) = \frac{a+b}{2}, \quad a, b \ge 0.$$

2. The geometric mean

$$G := G(a,b) = \sqrt{ab}, \quad a, b \ge 0.$$

3. The harmonic mean

$$H := H(a,b) = \frac{2ab}{a+b}, \quad a, b > 0.$$

4. The logarithmic mean

$$L := L(a,b) = \begin{cases} \frac{b-a}{\ln b - \ln a}, & a \ne b, \\ a, & a = b \end{cases} ; \quad a, b > 0.$$

5. The p-logarithmic mean

$$L_p := L_p(a,b) = \begin{cases} \left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}}, & a \ne b, p \in \mathbb{R}\setminus\{-1, 0\} \\ a, & a = b \end{cases} ; \quad a, b > 0.$$

6.The identric mean

$$I := I(a,b) = \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}, \quad a, b > 0.$$

These means are often used in numerical approximation and in other areas. However, the following simple relationships are known in the literature:

$$H \le G \le L \le I \le A.$$

It is also known that L_p is monotonically increasing over $p \in \mathbb{R}$, denoting $L_0 = I$ and $L_{-1} = L$.

Proposition 3.1. *Let $a,b \in [0,\infty)$ with $a < b$, $s \in (0,1]$ and $n \geq 2$. Then, the following inequalities are obtained:*

$$|A^n - L_n^n| \leq \frac{n(b-a)}{2(s+1)} \left[A\left(a^{n-1}, b^{n-1}\right) + A^{n-1}(a,b) \right].$$

Proof. The assertion follows from the inequalities (2.1) for the function

$$f(x) = x^n, \quad x \in [0,\infty).$$

□

Proposition 3.2. *Let $a,b \in (0,\infty)$ with $a < b$ and $s \in (0,1]$. Then, the following inequalities are obtained:*

$$\left|A^{-1} - L^{-1}\right| \leq \frac{b-a}{2(s+1)} \left[H^{-1}\left(a^2, b^2\right) + A^{-2}(a,b) \right].$$

Proof. The assertion follows from the inequalities (2.1) for the function

$$f(x) = x^{-1}, \quad x \in (0,\infty).$$

□

4. Conclusion

In this paper, some new Hermite-Hadamard type inequalities are obtained for functions whose first derivative in absolute value is the (s,P)-function by using the Hölder, power-mean and Hölder-İşcan integral inequalities. In addition, better approaches have been obtained for such functions. The method applied in this study can be applied to different types of convex functions.

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