FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 38, No 4 (2023), 731–740
https://doi.org/10.22190/FUMI230425047K
Original Scientific Paper

CHARACTERIZATION OF BI-NULL SLANT(BNS) HELICES OF (k, m)-TYPE IN R_1^3 AND R_2^5

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Abstract. The present study discusses bi-null slant helices of (k, m) type in R_2^5 and give the characterization for a curve to be certain (k, m) type bi null slant helix (BNS helix). The discussion includes the proofs for the non existence cases of (k, m) type bi null slant helices in R_2^5. Moreover certain characterizations and non existence have also been obtained for bi null slant helix to be (k, m) type using modified orthogonal frame.
Keywords: k-type Slant helix, Semi Euclidean space, Bi-null curves, Frenet Formulae.

1. Introduction

In 2004, Izumiya and Takeuchi [7] introduced the notion of slant helix which is defined as a curve ξ in R^3 where principal normal vector makes a constant angle with a fixed vector in R^3. Several geometers have studied slant helices [1, 8, 9] and gave characterizations for being such curves. In particular k-type slant helices have been one of the most interesting cases due to the rich geometric properties and applications in different branches of science and engineering [2, 6, 10]. Different varieties of k-type slant helices, k-type partially null and pseudo null helices etc. were further studied by Ergiiut et al [6] Ahmad T et.al.[2] and E. Nesovic et.al [10] respectively.

On the other hand, in 2012, bi null cartan curves were introduced and studied by M. Sakaki [12] in R_2^5 with concerned distinctive Frenet frame and the related

Received April 25, 2023, accepted: August 01, 2023
Communicated by Uday Chand De
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2010 *Mathematics Subject Classification.* Primary 53Z05; Secondary 53B50, 37C10, 57R25

curvatures called the Cartan curvatures. Proceeding on, in [13] and [14], some characterizations were proved for bi null Cartan curves to be k-type slant helices in semi Euclidean spaces R^6_3 and R^5_2 respectively.

Later in 2020, a class of slant helices called (k, m) type slant helices was considered in [3] which presented a study of (k, m) type slant helices for partially null and pseudo null curves in Minkowski space E^4_1.

The aim of this paper is to give characterization for bi null curve to be (k, m)-type slant helices in R^5_2 using the curvature function. Moreover, characterizations of bi-null curves to be (k, m) type slant helix have also been obtained in R^3_1 with modified orthogonal frame.

2. Preliminaries

Assume that R^5_2 is the 5-dimensional semi-Euclidean space with index 2. It is clear that if the standard co-ordinate system of R^5_2 is $\{x_1, x_2, x_3, x_4, x_5\}$, then the metric can be written as [13].

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2 - dx_4^2 - dx_5^2$$

The inner product on R^5_2 is denoted by $<,>$. We know that vector $X \in R^5_2 - \{0\}$ is called timelike if$< X, X >< 0$, spacelike if$< X, X >> 0$ and null (lightlike) if$< X, X >= 0$. If $X = 0$,then it will fall in the category of spacelike vectors. Also we have $||X|| = \sqrt{(| < X, X > |)}$. Here $||X||$ denotes the norm of a vector X. Two vectors X and Y are said to be orthogonal if $< X, Y >= 0$.

We now give a brief idea of modified orthogonal frame which in some sense generalizes Frenet frame in R^3_1.

Let ξ be a general analytic curve which can be re parameterized by its arc length s, where s∈I and I is a non empty open interval. Assuming that the curvature function has discrete zero points or k(s) is not identically zero, we have an orthogonal frame T, N, B defined as follows [4].

$$T = \frac{d\xi}{ds}, N = \frac{dT}{ds}, B = T \times N \tag{2.1}$$

where $T \times N$ is the vector product of T and N.The relationship between T, N and B and previous Frenet frame vectors at non zero points of k are

$$T = t, N = kn, B = \tau b$$

Thus from above equations we conclude that N=B=0, when k=0 and squares of length of N and B vary analytically in s. From equation 2.1, it is easy to calculate

$$\begin{bmatrix} T'(s) \\ N'(s) \\ B'(s) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -k^2 & \frac{k'}{k} & \tau \\ 0 & -\tau & \frac{k'}{k} \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{2.2}$$

where all the differentiation are done with respect to the arc length (s) and

$$\tau(s) = \frac{det(\xi', \xi'', \xi''')}{k^2}$$

is the torsion of ξ. From Frenet equation, we know that at any point, where $k^2 = 0$ is a removable singularity of τ. Let $<,>$ be the standard inner product of E^3, then T, N, B satisfies:

(2.3)
$$< T, T >= 1, < N, N >=< B, B >= k^2, < T, N >=< T, B >=< N, B >= 0$$

The orthogonal frame defined in 2.2 satisfying 2.3 is called as modified orthogonal frame.

Remark 2.1. It can be easily seen that once we put k=1 in 2.3, the modified orthogonal frame coincides with Frenet frame.

Definition 2.1. [12] Any curve $\xi(t)$ in R^5_2 is a bi-null curve if span $\{\xi'(t), \xi''(t)\}$ is isotropic i.e $< \xi'(t), \xi'(t) >= 0$, $< \xi'(t), \xi''(t) >= 0$ and $< \xi''(t), \xi''(t) >= 0$, and $\{\xi'(t), \xi''(t)\}$ are linearly independent for all t.

We consider any bi null curve $\xi(t) \subset R^5_2$ with t as a parameter. Then for $\xi(t)$, there exist Frenet frame $\{T, N, B_1, B_2, B_3\}$ such that $\xi'(t) = T$ and

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \\ B_3' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & k_1 & 0 & 0 & 0 \\ -k_1 & 0 & -1 & 0 & k_0 \\ 0 & -k_0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \\ B_3 \end{bmatrix} \tag{2.4}$$

where B_1, B_2 are null and B_3 is a spacelike unit vector, $< T, B_1 >=< N, B_2 >= 1$. $Span\{T, B_1\}$, $span\{N, B_2\}$ and $span\{B_3\}$ are mutually orthogonal. The frame $\{T, N, B_1, B_2, B_3\}$ is a pseudo-orthonormal frame. Here the functions k_0 and k_1 are the curvatures.

Definition 2.2. [12, 5] Any bi null curve $\xi(t)$ in R^5_2 with $\{\xi^3(t), \xi^3(t)\} \neq 0$ is a bi-null Cartan curve if $\xi'(t), \xi''(t), \xi'''(t), \xi''''(t)$ are linearly independent for all t.

We now quote the following theorem which guarantees the existence of a unique bi null Cartan curve with Cartan frame [3, 11] for given curvatures $k_0(t)$ and $k_1(t)$.

Theorem 2.1. *Let $k_0(t)$ and $k_1(t)$ be the differentiable functions on $(t_0 - \epsilon, t_0 + \epsilon)$ Let p_0 be the point in R^5_2, and $\{T, N, B_1, B_2, B_3\}$ be a pseudo-orthonormal basis of R^5_2. Then there exists a unique bi-null Cartan curve $\xi(t)$ in R^5_2 with $\xi(t_0) = p_0$, bi-null arc parameter t and curvatures k_0, k_1, whose Cartan frame $\{T, N, B_1, B_2, B_3\}$ satisfies $T(t_0) = T$, $N(t_0) = N$, $B_1(t_0) = B_1$, $B_2(t_0) = B_2$, $B_3(t_0) = B_3$.*

3. Characterization of (k,m)-type BNS helics in R_1^3

First we give the definition of bi-null slant(BNS) helices of (k, m) type in R_1^3 .

Definition 3.1. [3] Let $\{\Gamma_1, \Gamma_2, \Gamma_3, \}$ be the frame for a bi null curve ξ in R_1^3. Then ξ is known as a bi-null slant helix of (k, m) type, if we are able to find a fixed vector $U \neq 0 \in R_1^3$ such that $< \Gamma_k, U >= \alpha$, and $< \Gamma_m, U >= \beta$,where α, β are constants for $1 \leq k \leq 3$ and $1 \leq m \leq 3$.

We can express U as U = $u_1 T + u_2 N + u_3 B_1$, where u_i's are differentiable functions of 't'. Here we write $\Gamma_1 = T$, $\Gamma_2 = N$, $\Gamma_3 = B_1$.

Theorem 3.1. $(1, 2)$ *and* $(1, 3)$ *type BNS helices in* R_1^3 *with modified orthogonal frame do not exist there.*

Proof. Let ξ represents a bi null slant helix of (1, 2) type in R_1^3. Then by definition, for any fixed vector U, we have

$$< T, U >= \alpha \quad and \quad < N, U >= \beta$$

where $\alpha \neq 0$ and $\beta \neq 0$ are constants. Differentiating with respect to t, we get

$$< T', U >= 0 \quad and \quad < N', U >= 0.$$

Now using equation (2.2), we get

$$< N, U >= 0$$

which contradicts our supposition. Hence there does not exist a BNS helix of (1, 2) type in R_1^3 with modified orthogonal frame. □

Similarly we can show that there does not exist BNS helix of (1, 3) type in R_1^3 with modified orthogonal frame.

Theorem 3.2. ξ *is a bi null slant helix of* $(2, 3)$ *type in* R_1^3 *with modified orthogonal frame parameterized by arclength 't' with* $k_0, k_1 \neq 0$ *if and only if*

$$\alpha^2 k^2 + (\alpha^2 + \beta^2) d(\frac{k'}{k}) = 0$$

Proof. Let ξ represents a bi null slant helix of (2, 3) type in R_1^3 with modified orthogonal frame.

$$< N, U >= \alpha \quad and \quad < B, U >= \beta$$

where $\alpha \neq 0$ and $\beta \neq 0$ are constants. Then we can write

$$U = u_1 T + \alpha N + \beta B.$$

Differentiating with respect to 't', we get

$$u_1 T' + u_1' T + \alpha N' + \beta B' = 0.$$

Using equation (2.2), we get

$$u_1' T + u_1 N + \alpha(-k^2 T + \frac{k'}{k} N + \tau B) + \beta(-\tau N + \frac{k'}{k} B) = 0.$$

On simplification we get

$$u_1' - \alpha k^2 = 0, u_1 + \alpha \frac{k'}{k} - \beta\tau = 0, \alpha\tau + \beta \frac{k'}{k} = 0. \tag{3.1}$$

Solving 2nd and 3rd equation of equation 3.1, we have

$$u_1 \alpha + \frac{k'}{k}(\alpha^2 + \beta^2) = 0. \tag{3.2}$$

Now differentiating equation 3.2 and using 1st equation of 3.1, we arrive at

$$\alpha^2 k^2 + (\alpha^2 + \beta^2) d(\frac{k'}{k}) = 0.$$

Conversely choose $u_1 = \beta\tau - \alpha\frac{k'}{k}$ such that

$$U = (\beta\tau - \alpha \frac{k'}{k}) T + \alpha N + \beta B.$$

Differentiating above equation, we obtain

$$U' = (\beta\tau - \alpha \frac{k'}{k}) T' + (\beta\tau' - \alpha \frac{kk' - k'^2}{k^2}) T + \alpha N' + \beta B'.$$

Using equation (2.2), we get

$$U' = (\beta\tau - \alpha \frac{k'}{k}) N + (\beta\tau' - \alpha \frac{kk' - k'^2}{k^2}) T + \alpha(-k^2 T + \frac{k'}{k} N + \tau B) + \beta(-\tau N + \frac{k'}{k} B).$$

Finally, using equation (3.1) in above equation we get $U' = 0$. Hence proved. □

As a whole, we conclude the results of this section in the form of the following table.

Table 3.1: Existence and non-existence of BNS helix in modified orthogonal frame in R_1^3

Type of BNS helix	Existence/Non-existence
(1,2)-type	does not exist
(1,3)-type	does not exist
(2,3)-type	exists iff $\alpha^2 k^2 + (\alpha^2+\beta^2)d(\frac{k'}{k}) = 0$

4. Characterization of (k,m)-type BNS helics in R_2^5

First we give the definition of bi-null slant(BNS) helices of (k, m) type in R_2^5 .

Definition 4.1. [3] Let $\{\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5\}$ be the frame for a bi null curve ξ in R_2^5. Then ξ is known as a bi-null slant helix of (k, m) type, if we are able to find a fixed vector $U \neq 0 \in R_2^5$ such that $< \Gamma_k, U >= \alpha$, and $< \Gamma_m, U >= \beta$,where α, β are constants for $1 \leq k \leq 5$ and $1 \leq m \leq 5$.

We can express U as U = $u_1T + u_2N + u_3B_1 + u_4B_2 + u_5B_3$, where u_i's are differentiable functions of 't'. Here we write $\Gamma_1 = T, \Gamma_2 = N, \Gamma_3 = B_1, \Gamma_4 = B_2, \Gamma_5 = B_3$.

Theorem 4.1. *There does not exist* $(1, 2)$ *type BNS helix in* R_2^5*.*

Proof. Let ξ represents a bi null slant helix of (1, 2) type in R_2^5. Then by definition, for any fixed vector U, we have

$$< T, U >= \alpha \quad and \quad < N, U >= \beta$$

where $\alpha \neq 0$ and $\beta \neq 0$ are constants. Differentiating with respect to t, we find

$$< T', U >= 0 \quad and \quad < N', U >= 0.$$

Using equation (2.4), we get

$$< N, U >= 0 \quad and \quad < B_3, U >= 0$$

which contradicts our supposition. Hence there does not exist a BNS helix of (1, 2) type in R_2^5. □

Similarly we can prove the non existence of (1,5), (2,3), (2,5), (3,4), (3,5) and (4,5) type bi null slant helices.

Theorem 4.2. *ξ is a bi null slant helix of* $(1, 3)$ *type in* R_2^5 *parameterized by arc length 't' with* $k_0, k_1 \neq 0$ *if and onlt if* $k_1 \neq 0$ *is a constant.*

Proof. Let ξ represents a bi null slant helix of (1, 3) type in R_2^5. Let U a fixed vector, then by definition we have

$$< T, U >= \alpha \quad and \quad < B_1, U >= \beta$$

where $\alpha \neq 0$ and $\beta \neq 0$ are constants. Differentiating with respect to t, we get

$$< T', U >= 0 \quad and \quad < B_1', U >= 0.$$

Using equation 2.4, we conclude

$$< N, U >= 0 \quad and \quad k_1 < N, U >= 0.$$

Differentiating first part with respect to 't', we get

$$< N', U >= 0.$$

Again by using equation (2.4) in the above equation , we obtain

$$< B_3, U >= 0.$$

Differentiating the above equation with respect to' t' and using equation (2.4), we get

$$-k_0 < N, U > - < B_2, U >= 0$$

OR

$$< B_2, U >= 0.$$

Therefore we can write

$$U = \alpha T + \beta B_1$$

Differentiating with respect to 't', we get

$$\alpha T' + \beta B_1' = 0.$$

Now putting equation 2.4 in the above equation, which implies

$$k_1 = -\frac{\alpha}{\beta} = constant.$$

Conversely, assume that k_1 is a constant. For $\beta \neq 0$, choose the vector U as

$$U = -\beta k_1 T + \beta B_1.$$

On differentiating this with respect to t we get

$$U' = 0$$

and hence

$$< T, U >= constant \quad and \quad < B_1, U >= constant.$$

Hence ξ is a bi null slant helix of (1, 3) type in R_2^5. $\square$

Theorem 4.3. *ξ is a bi null slant helix of $(1,4)$ type in R_2^5 if and only if $k_1 = 0$ and $\beta k_0' - \alpha = 0$, where 't' is a parameter.*

Proof. Let ξ represents a bi null slant helix of (1, 4) type in R_2^5.

$$< T, U >= \alpha \quad and \quad < B_2, U >= \beta$$

where $\alpha \neq 0$ and $\beta \neq 0$ are constants. Then we can write

$$U = \alpha T + u_2 N + u_3 B_1 + \beta B_2 + u_5 B_3.$$

Differentiating with respect to 't', we get

$$\alpha T' + u_2 N' + u_2' N + u_3 B_1' + u_3' B_1 + \beta B_2' + u_5 B_3' + u_5' B_3 = 0.$$

Using equation 2.4, we arrive at

$$\alpha N + u_2 B_3 + u_2' N + u_3 k_1 N + u_3' B_1 + \beta(-k_1 T - B_1 + k_0 B_3) + u_5(-k_0 N - B_2) + u_5' B_3 = 0$$

which implies that

$$k_1 = 0, \quad \beta k_0' - \alpha = 0.$$

Conversely choose the vector U as

$$U = -\alpha T - k_0 \beta N + (\beta t + A) B_1 + \beta B_2.$$

On differentiating with respect to 't' we get

$$U' = 0$$

which gives

$$< T, U >= constant \quad and \quad < B_2, U >= constant.$$

Hence ξ is a bi null slant helix of (1, 4) type in R_2^5. □

Theorem 4.4. *ξ is a bi null slant helix of $(2,4)$ type in R_2^5 if $\int k_1 dt + k_1 t + C = 0$, where C is a constant of integration.*

Proof. Let ξ represents a bi null slant helix of (2, 4) type in R_2^5. Then for a fixed vector U, we have

$$< T, U >= \alpha \quad and \quad < B_2, U >= \beta$$

where $\alpha \neq 0$ and $\beta \neq 0$ are constants. Differentiating with respect to t, we get

$$< N', U >= 0 \quad and \quad < B_2', U >= 0.$$

Using equation 2.4, we find

$$< B_3, U >= 0$$

which gives

$$U = u_1 T + \alpha N + u_3 B_1 + \beta B_2.$$

Differentiating and using equation (2.4)

$$u_1 N + u_1' T + \alpha B_3 + u_3 k_1 N + u_3' B_1 + \beta(-k_1 T - B_1 + k_0 B_3) = 0.$$

Which, on simplification implies that

$$\int k_1 dt + k_1 t + C = 0$$

where C is constant of integration. Hence proved. $\square$

As a whole, the results of this section can be tabulated as follows:

Table 4.1: Existence and non-existence of BNS helix

Type of BNS helix	Existence/Non-existence
(1,2)-type	does not exist
(1,3)-type	exists iff $k_1 \neq 0$ is a constant
(1,4)-type	exists iff $k_1 = 0$ and $\beta k_0' - \alpha = 0$,where 't'is a parameter
(1,5)-type	does not exist
(2,3)-type	does not exist
(2,4)-type	exists if $\int k_1 dt + k_1 t + C = 0$
(2,5)-type	does not exists
(3,4)-type	does not exists
(3,5)-type	does not exists
(4,5)-type	does not exists

Acknowledgement

The authors would like to thank Prof. Mohammad Hasan Shahid D/o Mathematics Jamia Millia Islamia, New Delhi for his careful reading and valuable suggestions to improve the manuscript.

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