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LACUNARY STATISTICAL CONVERGENCE OF ORDER α IN PARTIAL METRIC SPACES

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Abstract. The present study introduces the notions of lacunary statistical convergence of order α and strongly $q-$ summability of order α in partial metric spaces. We examine the inclusion relations linked to these concepts.
Keywords: lacunary statistical convergence, partial metric spaces, inclusion relations.

1. Introduction

Partial metric spaces were introduced by Matthews [12] to denote a generalization of the usual metric spaces. The main difference between the partial metric and the standard metric is that the self-distances does not need to be equal to zero. The partial metric was first used in computer science and later applied to many other fields such as fixed point theory.

For the non-empty set X, the function $\rho : X \times X \to \mathbb{R}$ is called a partial metric, and the pair (X, ρ) is called a partial metric space if for all $x, y, z \in X$, the following assertions hold:

i. $0 \leq \rho(x,x) \leq \rho(x,y)$,

ii. If $\rho(x,x) = \rho(x,y) = \rho(y,y)$ then $x = y$,

iii. $\rho(x,y) = \rho(y,x)$,

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iv. $\rho(x,y) \leq \rho(x,z) + \rho(z,y) - \rho(z,z)$,

Let (x_n) be a sequence in the partial metric space (X,ρ). Then

i) (x_n) is bounded if there exists *a* real number $M > 0$ such that $\rho(x_n,x_m) \leq M$ for all $n,m \in \mathbb{N}$,

ii) (x_n) is called convergent to x in (X,ρ), written as $\lim\limits_{n\to\infty} x_n = x$, if

$$\lim_{n\to\infty} \rho(x_n,x) = \lim_{n\to\infty} \rho(x_n,x_n) = \rho(x,x).$$

Similarly, statistical convergence is a generalization of sequential convergence. The definition of statistical convergence (as almost convergence) was given by Zygmund [19] in the first edition of his monograph, published in Warsaw. Steinhaus [18] and Fast [8] and later Schoenberg [14] introduced the concept of statistical convergence independently. The concept of statistical convergence has been used in many areas of mathematics, such as number theory, probabilistic normed spaces, ergodic theory, Fourier analysis, measure theory, trigonometric series, and others, therefore, being a very active and intensively studied subject in many contexts. The concept of statistical convergence has been approached from various angles, and various definitions of convergence have been provided. For further reading on some recent relevant studies on lacunary statistical convergence, see [1], [2] and [16].

A sequence (x_k) of complex numbers is said to be statistically convergent to some number L if for every positive number ε,

$$\lim_{n\to\infty} \frac{1}{n} \left|\{k \leq n : |x_k - L| \geq \varepsilon\}\right| = 0.$$

The number L is called statistical limit of (x_k) and is written as $S - \lim x_k = L$ or $x_k \to L\,(S)$ and all statistically convergent sequences are denoted by S, i.e.

$$S = \left\{ x = (x_k)_{k\in\mathbb{N}} \in w : \text{for every} \varepsilon > 0, \lim_{n\to\infty} \frac{1}{n} \left|\{k \leq n : |x_k - L| \geq \varepsilon\}\right| = 0 \right\}$$

Similarly, a sequence (x_k) is statistically convergent of order α to some number L if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n^\alpha} \left|\{k \leq n : |x_k - L| \geq \varepsilon\}\right| = 0.$$

All statistically convergent sequences of order α are denoted by S^α. Some results on statistical convergence of order α can be found in Altın et al [3], Çolak[5] and Çolak et al.[6].

Fridy and Orhan, in [9] and [10], defined the concepts of lacunary convergence and lacunary summability with lacunary sequences. An increasing sequence of integers $\theta = (k_r)_{r\in\mathbb{N}}$ is called lacunary sequence if $k = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. In our results, we use $I_r = (k_{r-1}, k_r]$ and $q_r = \frac{k_r}{k_{r-1}}$. A sequence (x_k) is lacunary statistically convergent and, respectively, lacunary statistically convergent of order α to some number L if for every $\varepsilon > 0$,

$$\lim_{r\to\infty} \frac{1}{h_r} \left|\{k \in I_r : |x_k - L| \geq \varepsilon\}\right| = 0$$

and, respectively,

$$\lim_{r\to\infty}\frac{1}{h_r^\alpha}\left|\{k\in I_r:|x_k-L|\geq\varepsilon\}\right|=0.$$

All lacunary statistically convergent sequences and all lacunary statistically convergent sequences of order α are denoted by S_θ and S_θ^α, respectively. Lacunary statistically convergent and strongly summable sequences of order α were studied by Şengül and Et in [15] and [7].

Some results on statistical convergence in usual metric spaces can be found in Bilalov [4] and Şengül et al.[17]. The concept of statistical convergence in partial metric spaces was firstly given by Nuray [13] as it follows: Let (x_k) be a sequence in partial metric space (X,ρ); the sequence (x_k) is said to be $\rho-$ statistically convergent to x if there exists a point $x\in X$ such that

$$\lim_{n\to\infty}\frac{1}{n}\left|\{k\leq n:|\rho(x_k,x)-\rho(x,x)|\geq\varepsilon\}\right|=0$$

for every $\varepsilon>0$. In addition, Nuray [13] also examined the relationship between the concept of statistical convergence in partial metric spaces and strong Cesàro summability.

All lacunary statistical convergence sequences in PMS are shown as

$$S_{\theta,\rho}=\Big\{x=(x_k)_{k\in\mathbb{N}}\in w:\ \text{there exists a } x\in X,\ \text{for every } \varepsilon>0,\ \lim_{r\to\infty}\frac{1}{h_r}\left|\{k\in I_r:|\rho(x_k,x)-\rho(x,x)|\geq\varepsilon\}\right|=0\Big\}.$$

The results on lacunary statistical convergence and lacunary summability in partial metric spaces are given in Gülle et al. [11]. It is natural to ask lacunary statistical convergence sequences of order α in PMS. We define such sequences as

$$S_{\theta,\rho}^\alpha=\Big\{x=(x_k)_{k\in\mathbb{N}}\in w:\text{there exists a } x\in X,\text{for every } \varepsilon>0,\ \lim_{r\to\infty}\frac{1}{h_r^\alpha}\left|\{k\in I_r:|\rho(x_k,x)-\rho(x,x)|\geq\varepsilon\}\right|=0\Big\}.$$

Our motivation for this study is prompted by Şengül and Et[15], [7] and Gülle et al. [11]. The results on lacunary statistical convergence and lacunary boundedness order α for usual metric spaces are also given in Şengül et al. [17].

2. Inclusion Theorems in PMS

For the $1<\alpha\in\mathbb{R}$, lacunary statistical convergence is not well defined (see Et, 2014 [15]). Therefore, when we consider, for example, the natural partial metric of real numbers, we generally arrive at the same conclusion for partial metric spaces. Also, taking $\alpha=1$, we get $S_{\theta,\rho}^\alpha(X)=S_{\theta,\rho}(X)$. Therefore, in the rest of article, we consider the case $\alpha \in (0,1)$ and, unless otherwise stated, $\theta = (k_r)_{r\in\mathbb{N}}$ is lacunary sequence, $I_r = (k_{r-1}, k_r]$ and $q_r = \frac{k_r}{k_{r-1}}$. For the sake of simplicity, it is considered that the sequence (x_k) and the element x, which we use in the proofs are chosen from the partial metric space (X,ρ), although we do not emphasize it every time. For the sake of brevity, we shall use PMS instead of partial metric space.

Definition 2.1. If there exists a point $x \in X$ such that

$$\lim_{r\to\infty} \frac{1}{h_r^\alpha} |\{k \in I_r : |\rho(x_k, x) - \rho(x,x)| \geq \varepsilon\}| = 0$$

for every $\varepsilon > 0$. In this case, it is stated that (x_k) is $\rho-$ lacunary statistically convergent of order α to x which is denoted by $S^\alpha_{\theta,\rho} - \lim x_k = x$ or $x_k \to x \left(S^\alpha_{\theta,\rho}(X)\right)$.

Throughout this paper, $S^\alpha_\rho(X)$ denotes the class of sequences in partial metric space (X,ρ) which are $\rho-$ statistically convergent of order α.

Theorem 2.1. *For some reals α and β such that $0 < \alpha < \beta \leq 1$, the inclusion $S^\alpha_{\theta,\rho}(X) \subseteq S^\beta_{\theta,\rho}(X)$ holds and clearly $x_k \to x \left(S^\alpha_{\theta,\rho}(X)\right)$ implies $x_k \to x \left(S^\beta_{\theta,\rho}(X)\right)$.*

Proof. Suppose that $0 < \alpha < \beta \leq 1$. Then, the inequality $\frac{1}{h_r^\beta} \leq \frac{1}{h_r^\alpha}$ holds, so

$$\begin{aligned} & \lim_{r\to\infty} \frac{1}{h_r^\beta} |\{k \in I_r : |\rho(x_k, x) - \rho(x,x)| \geq \varepsilon\}| \\ \leq \ & \lim_{r\to\infty} \frac{1}{h_r^\alpha} |\{k \in I_r : |\rho(x_k, x) - \rho(x,x)| \geq \varepsilon\}| \end{aligned}$$

is provided for every $\varepsilon > 0$ and this clearly gives the desired inclusion $S^\alpha_{\theta,\rho}(X) \subseteq S^\beta_{\theta,\rho}(X)$. □

The inclusion may be strict and this can be seen by the following example.

Example 2.1. Let us consider the partial metric of real numbers defined $\rho : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \rho(x,y) = \max\{x,y\}$, the lacunary sequence $\theta = (2^r)$ and the sequence $(x_k) \subset \mathbb{R}$ such that

$$x_k = \begin{cases} 1, & k \text{ is a square} \\ 0, & \text{otherwise} \end{cases} .$$

Clearly, for $\varepsilon > 0$, $|\{k \in I_r : |\rho(x_k, 0) - \rho(0,0)| \geq \varepsilon\}| \leq \sqrt{n}$ holds. This means that $(x_k) \in S^\beta_{\theta,\rho}(X)$ for $\frac{1}{2} < \beta \leq 1$ but $(x_k) \notin S^\alpha_{\theta,\rho}(X)$ for $0 < \alpha \leq \frac{1}{2}$, that is $S^\alpha_{\theta,\rho}(X) \subset S^\beta_{\theta,\rho}(X)$.

As a consequence, from the last inequality above, we see $S^\alpha_{\theta,\rho}(X) \subseteq S_{\theta,\rho}(X)$.

Theorem 2.2. *For $\alpha \in (0,1)$ the inclusion*

$$S_\rho(X) \subseteq S^\alpha_{\theta,\rho}(X)$$

holds whenever $\liminf_{r\to\infty} \frac{h_r^\alpha}{k_r} > 0$ and $x_k \to x\,(S_\rho(X))$ implies $x_k \to x \left(S^\alpha_{\theta,\rho}(X)\right)$.

Proof. Under the hypohtesis, there exists a real $\delta > 0$ such that $\frac{h_r^\alpha}{k_r} \geqslant \delta$ for sufficiently large r. Hence $\frac{1}{k_r} \geqslant \delta \frac{1}{h_r^\alpha}$ and

$$\left|\{k \le k_r : |\rho(x_k, x) - \rho(x,x)| \ge \varepsilon\}\right| \ge \left|\{k \in I_r : |\rho(x_k, x) - \rho(x,x)| \ge \varepsilon\}\right|$$

hold. Therefore, we obtain

$$\begin{aligned} \frac{1}{k_r}\left|\{k \le k_r : |\rho(x_k, x) - \rho(x,x)| \ge \varepsilon\}\right| &\ge \frac{1}{k_r}\left|\{k \in I_r : |\rho(x_k, x) - \rho(x,x)| \ge \varepsilon\}\right| \\ &\ge \delta \frac{1}{h_r^\alpha}\left|\{k \in I_r : |\rho(x_k, x) - \rho(x,x)| \ge \varepsilon\}\right|. \end{aligned}$$

Taking limit as $r \to \infty$, we get $x_k \to x\,(S_\rho(X))$ implies $x_k \to x\left(S_{\theta,\rho}^\alpha(X)\right)$. □

Theorem 2.3. *For any lacunary sequence $\theta = (k_r)$ and $\alpha \in (0,1)$, if*

$$\lim_{r\to\infty} \sup \frac{(k_j - k_{j-1})^\alpha}{k_{r-1}^\alpha} < \infty,$$

$(j = 1, 2, ..., r)$ then the following inclusion holds

$$S_{\theta,\rho}^\alpha(X) \subset S_\rho(X)$$

and clearly $x_k \to x\left(S_{\theta,\rho}^\alpha(X)\right)$ implies $x_k \to x\,(S_\rho(X))$.

Proof. From the hypothesis there exists reals $H_j > 0$ such that $\frac{(k_j - k_{j-1})^\alpha}{k_{r-1}^\alpha} \le H_j$, $(j = 1, 2, ..., r)$ for all r. Choose a sequence $x = (x_k) \in S_{\theta,\rho}^\alpha(X)$ such that $x_k \to x\left(S_{\theta,\rho}^\alpha(X)\right)$ which means that $\lim_{r\to\infty} \frac{1}{h_r^\alpha}\left|\{k \in I_r : |\rho(x_k, x) - \rho(x,x)| \ge \varepsilon\}\right| = 0$. Then there exists a $r_0 \in \mathbb{N}$ such that for given $\varepsilon > 0$

$$\frac{1}{n}\left|\{k \le n : |\rho(x_k, x) - \rho(x,x)| \ge \varepsilon\}\right| \le \frac{1}{h_r^\alpha}\left|\{k \in I_r : |\rho(x_k, x) - \rho(x,x)| \ge \varepsilon\}\right| < \varepsilon$$

hold for all $r > r_0$. Similarly, the fact that the being $x_k \to x\,(S_\rho(X))$ can be followed from (Gülle et all, Theorem 10 [11]). □

Combining the last two results and Theorem 9 and 10 in [11], we can easily see the following results.

Corollary 2.1. *For any lacunary sequence $\theta = (k_r)$ and $\alpha \in (0,1)$, if $\lim_{r\to\infty}\inf \frac{h_r^\alpha}{k_r} > 0$ and $\lim_{r\to\infty} \sup \frac{(k_j - k_{j-1})^\alpha}{k_{r-1}^\alpha} < \infty$, $(j = 1, 2, ..., r)$ then the equality*

$$S_{\theta,\rho}^\alpha(X) = S_\rho(X)$$

holds.

3. Lacunary Summability Of Order α in PMS

In this section, the definition of strongly $q-$lacunary summability of order α in partial metric spaces and some properties of the set $\left[N_{\theta,\rho}^{\alpha}\right]^q$ related to this are given according to differenet conditions. Also, some relationships between $\left[N_{\theta,\rho}^{\alpha}\right]^q-$summability and $S_{\theta,\rho}^{\beta}-$convergence are given.

Definition 3.1. The sequence (x_k) in a partial metric space (X,ρ) is called strongly $q-$lacunary summable of order α if there exists an element $x \in X$ such that

$$\lim_{r\to\infty} \frac{1}{h_r^{\alpha}} \sum_{k\in I_r} \left|\rho\left(x_k,x\right)-\rho\left(x,x\right)\right|^q = 0.$$

In this case, we say that the sequence (x_k) is strong $q-$lacunary summable to x of order α and we show that $x_k \to x\left(\left[N_{\theta,\rho}^{\alpha}\right]^q\right)$. From now on $\left[N_{\theta,\rho}^{\alpha}\right]^q$ denotes the class of all sequences in the partial metric space (X,ρ) which are strong $q-$lacunary summable of order α.

First, we compare different order summable sequence spaces.

Theorem 3.1. *For some reals α and β such that $0<\alpha\leq\beta\leq 1$, the inclusion $\left[N_{\theta,\rho}^{\alpha}\right]^q \subseteq \left[N_{\theta,\rho}^{\beta}\right]^q$ holds and clearly $x_k \to x\left(\left[N_{\theta,\rho}^{\alpha}\right]^q\right)$ implies $x_k \to x\left(\left[N_{\theta,\rho}^{\beta}\right]^q\right)$.*

Proof. For an arbitrarily chosen sequence $(x_k) \in \left[N_{\theta,\rho}^{\alpha}\right]^q$, the desired result can be easily seen with the help of the following inequality:

$$\frac{1}{h_r^{\beta}} \sum_{k\in I_r} \left|\rho\left(x_k,x\right)-\rho\left(x,x\right)\right|^q \leq \frac{1}{h_r^{\alpha}} \sum_{k\in I_r} \left|\rho\left(x_k,x\right)-\rho\left(x,x\right)\right|^q.$$

□

The result points to a relation between lacunary statistically convergence of order α and strong $q-$lacunary summability of order α.

Theorem 3.2. *For some reals α and β such that $0<\alpha\leq\beta\leq 1$, the inclusion $\left[N_{\theta,\rho}^{\alpha}\right]^q \subseteq S_{\theta,\rho}^{\beta}$ holds and clearly $x_k \to x\left(\left[N_{\theta,\rho}^{\alpha}\right]^q\right)$ implies $x_k \to x\left(S_{\theta,\rho}^{\beta}\right)$.*

Proof. Assume that $(x_k) \in \left[N_{\theta,\rho}^{\alpha}\right]^q$ and $\varepsilon>0$. Then, following inequalities

$$\begin{aligned}
\sum_{k\in I_r} \left|\rho\left(x_k,x\right)-\rho\left(x,x\right)\right|^q &= \sum_{\substack{k\in I_r \\ \left|\rho(x_k,x)-\rho(x,x)\right|^q\geq\varepsilon}} \left|\rho\left(x_k,x\right)-\rho\left(x,x\right)\right|^q \\
&\quad+ \sum_{\substack{k\in I_r \\ \left|\rho(x_k,x)-\rho(x,x)\right|^q\leq\varepsilon}} \left|\rho\left(x_k,x\right)-\rho\left(x,x\right)\right|^q \\
&\geq \sum_{\substack{k\in I_r \\ \left|\rho(x_k,x)-\rho(x,x)\right|^q\geq\varepsilon}} \left|\rho\left(x_k,x\right)-\rho\left(x,x\right)\right|^q \\
&\geq \varepsilon^q \left|\left\{k\in I_r : \left|\rho\left(x_k,x\right)-\rho\left(x,x\right)\right|\geq\varepsilon\right\}\right|
\end{aligned}$$

hold and hence we obtain

$$\varepsilon^q \frac{1}{h_r^\beta} \left|\{k \in I_r : |\rho(x_k, x) - \rho(x, x)| \geq \varepsilon\}\right| \leq \frac{1}{h_r^\alpha} \sum_{k \in I_r} |\rho(x_k, x) - \rho(x, x)|^q .$$

Taking limit as $r \to \infty$, we have $(x_k) \in S_{\theta,\rho}^\beta$. □

The next two results are inclusion results obtained by taking different lacunary sequences.

Theorem 3.3. *Assume that $\theta = (k_r)$ and $\theta' = (s_r)$ are two lacunary sequence such that $I_r \subseteq J_r$ for all $r \in \mathbb{N}$ where $I_r = (k_{r-1}, k_r]$ and $J_r = (s_{r-1}, s_r]$. For some reals α and β such that $0 < \alpha \leq \beta \leq 1$, if*

$$\liminf_{r\to\infty} \frac{h_r^\alpha}{l_r^\beta} > 0$$

then the inclusion $\left[N_{\theta',\rho}^\beta\right]^q \subseteq \left[N_{\theta,\rho}^\alpha\right]^q$ holds and $x_k \to x\left(\left[N_{\theta,\rho}^\beta\right]^q\right)$ implies $x_k \to x\left(\left[N_{\theta,\rho}^\alpha\right]^q\right)$.

Proof. For the sequence $(x_k) \in \left[N_{\theta',\rho}^\beta\right]^q$, we have the inequalities:

$$\begin{aligned} \frac{1}{h_r^\alpha} \sum_{k \in I_r} |\rho(x_k, x) - \rho(x, x)|^q &\leq \tfrac{1}{h_r^\alpha} \textstyle\sum_{k \in J_r} |\rho(x_k, x) - \rho(x, x)|^q \\ &\leq \tfrac{l_r^\beta}{h_r^\alpha} \tfrac{1}{l_r^\beta} \textstyle\sum_{k \in J_r} |\rho(x_k, x) - \rho(x, x)|^q . \end{aligned}$$

Taking limit as $r \to \infty$, under the assumption of the theorem, we have $(x_k) \in \left[N_{\theta,\rho}^\alpha\right]^q$. □

Theorem 3.4. *Assume that $\theta = (k_r)$ and $\theta' = (s_r)$ are two lacunary sequence such that $I_r \subseteq J_r$ for all $r \in \mathbb{N}$ where $I_r = (k_{r-1}, k_r]$ and $J_r = (s_{r-1}, s_r]$. For some reals α and β such that $0 < \alpha \leq \beta \leq 1$, if*

$$\liminf_{r\to\infty} \frac{h_r^\alpha}{l_r^\beta} > 0$$

then the inclusion $\left[N_{\theta',\rho}^\beta\right]^q \subseteq S_{\theta,\rho}^\alpha$ holds and clearly $x_k \to x\left(\left[N_{\theta,\rho}^\beta\right]^q\right)$ implies $x_k \to x\left(S_{\theta,\rho}^\alpha\right)$.

Proof. Assume that $(x_k) \in \left[N_{\theta',\rho}^{\beta}\right]^q$ and $\varepsilon > 0$. Then, following inequalities

$$\begin{aligned}
\sum_{k\in J_r} |\rho(x_k,x)-\rho(x,x)|^q &= \sum_{\substack{k\in J_r \\ |\rho(x_k,x)-\rho(x,x)|^q\geq\varepsilon}} |\rho(x_k,x)-\rho(x,x)|^q \\
&+ \sum_{\substack{k\in J_r \\ |\rho(x_k,x)-\rho(x,x)|^q\leq\varepsilon}} |\rho(x_k,x)-\rho(x,x)|^q \\
&\geq \sum_{\substack{k\in J_r \\ |\rho(x_k,x)-\rho(x,x)|^q\geq\varepsilon}} |\rho(x_k,x)-\rho(x,x)|^q \\
&\geq \varepsilon^q \left|\{k\in J_r : |\rho(x_k,x)-\rho(x,x)|\geq\varepsilon\}\right| \\
&\geq \varepsilon^q \left|\{k\in I_r : |\rho(x_k,x)-\rho(x,x)|\geq\varepsilon\}\right|
\end{aligned}$$

hold and hence we obtain

$$\begin{aligned}
& \varepsilon^q \frac{h_r^\alpha}{l_r^\beta}\frac{1}{h_r^\alpha} \left|\{k\in I_r : |\rho(x_k,x)-\rho(x,x)|\geq\varepsilon\}\right| \\
\leq\ & \frac{1}{l_r^\beta}\varepsilon^q \left|\{k\in J_r : |\rho(x_k,x)-\rho(x,x)|\geq\varepsilon\}\right| \\
\leq\ & \frac{1}{l_r^\beta}\sum_{k\in J_r} |\rho(x_k,x)-\rho(x,x)|^q .
\end{aligned}$$

Taking limit as $r\to\infty$, under the assumption of the theorem, we have $(x_k) \in S_{\theta,\rho}^{\alpha}$. □

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REFERENCES

1. N.D. Aral and H.Şengül Kandemir: *I−Lacunary statistical convergence of order β of difference sequences of fractional order*, Facta Universitatis (NIS) Ser. Math. Inform., **36**(1), (2021), 43-55.
2. N.D. Aral, H.Şengül Kandemir and M. Et: *Strongly lacunary convergence of order β of difference sequences of Fractional Order in Neutrosophic Normed Spaces*, Filomat, **37**(19), (2023), 6443-6451.
3. Y. Altin, H. Altinok and R. Çolak: *Statistical convergence of order α for difference sequences*, Quaestiones Mathematicae, **38**(4), (2015), 505-514. doi: 10.2989/16073606.2014.981685.
4. B. Bilalov and T. Nazarova: *On statistical convergence in metric space*, Journal of Mathematics Research, **7**(1), (2015), 37-43. doi: 10.5539/jmr.v7n1p37.
5. R. Çolak: *Statistical convergence of order α*, Modern Methods in Analysis and Its Applications, edited by M. Mursaleen, 121-129. New Delhi, India: Anamaya Publication, 2010.

6. R. Çolak and Ç.A. Bektaş: *λ-statistical convergence of order α*, Acta Mathematica Scientia, **31**(3), (2011), 953–959. doi: 10.1016/S0252-9602(11)60288-9.
7. M. Et and H. Şengül: *Some cesaro-type summability spaces of order α and lacunary statistical convergence of order α*. Filomat **28**(8), (2014), 1593-1602.
8. H. Fast: *Sur la convergence statistique*, Colloquium Mathematicae, **2**, 3-4, (1951), 241-244. doi: 10.4064/cm-2-3-4-241-244.
9. J. A. Fridy and C. Orhan: *Lacunary statistical convergence*. Pacific Journal of Mathematics **160**(1), (1993), 43-51.
10. J. A. Fridy and C. Orhan: *Lacunary statistical summability*. Journal of Mathematical Analysis and Applications **173**(2), (1993), 497-504. doi.org/10.1006/jmaa.1993.1082
11. E. Gülle, E. Dündar and U. Ulusu: *Summability anad lacunary statistical convergence concepts in partial metric*. Preprint, 2023. doi.org/10.1006/jmaa.1993.1082
12. S.G. Matthews: *Partial metric topology*, Annals of the New York Academy of Sciences, **728**, (1994), 183-197. doi:10.1111/j.1749-6632.1994.tb44144.x.
13. F. Nuray: *Statistical convergence in partial metric spaces*, Korean Journal of Mathematics, **30**(1), (2022), 155-160. doi: 10.11568/kjm.2022.30.1.155.
14. I. J. Schoenberg: *The integrability of certain functions and related summability methods*, The American Mathematical Monthly, **66**, (1959), 361-375. doi:10.1080/00029890.1959.11989303.
15. H. Şengül and M. Et: *On lacunary statistical convergence of order α*, Acta Mathematica Scientia, **34**(2), (2014), 473-482. doi.org/10.1016/S0252-9602(14)60021-7
16. H. Şengül and M. Et: *On $I-$ lacunary statistical convergence of order α of sequences of sets*, Filomat, **31**(8), (2017), 2403-2412.
17. H.Şengül, M. Et and H. Çakalli: *Lacunary d-statistical boundedness of order α in metric spaces*. Azerbaijan Journal of Mathematics, **12**(1), (2022), 98-108.
18. H. Steinhaus: *Sur la convergence ordinaire et la convergence asymptotique*, Colloquium Mathematicum, **2**, (1951), 73-74.
19. A. Zygmund: *Trigonometric Series*, Cambridge: Cambridge University Press, 1979.