FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 38, No 4 (2023), 771–791
https://doi.org/10.22190/FUMI230521050T
Original Scientific Paper

Γ-CONVERGENCE OF DOUBLE SEQUENCES OF FUNCTIONS, AND MINIMIZERS

Özer Talo[1] and Yurdal Sever[2]

[1]Manisa Celal Bayar Üniversitesi Küme evleri
Yunusemre, 45125 Manisa, Turkey
[2]Faculty of Art and Sciences, Department of Mathematics
Afyon Kocatepe University, 03200 Afyonkarahisar, Turkey

Abstract. In the present paper, we introduce the concept of Γ-convergence of a double sequence of functions defined from a metric space into real numbers. This convergence is useful as it is a convenient concept of convergence for approximating minimization problems in the field of mathematical optimization. First, we compare this convergence with pointwise and uniform convergence and obtain some properties of Γ-convergence. Later we deal with the problem of minimization. We prove that, under some additional assumptions, the Γ-convergence of a double sequence (f_{kl}) to a function f implies the convergence of the minimum values of f_{kl} to the minimum value of f. Moreover, we prove that each limit point of the double sequence of the minimizers of f_{kl} is a minimizer of f.
Keywords: Double sequence of functions, Pringsheim convergence, Set-valued function, Kuratowski convergence, Gamma-convergence, Minimizers.

1. Introduction

The pointwise convergence of sequence of functions, while useful in many ways, is insufficient for purposes such as setting up approximations of problems of optimization (see,[20]). First of all, it fails to preserve the lower semicontinuity of functions. Furthermore, it may not be able to keep the maximum or minimum values of the sequence of the functions.

Received May 21, 2023, accepted: October 21, 2023
Communicated by Dijana Mosić
Corresponding Author: Yurdal Sever (yurdalsever@hotmail.com, ysever@aku.edu.tr)
2010 *Mathematics Subject Classification.* Primary 49J53; Secondary 40B05, 40A05, 46N10.

In order to overcome these shortcomings, the type of convergence called infimal convergence was first studied by Wijsman [24, 25]. De Giorgi and Franzoni [8] called it Γ-convergence. De Giorgi [9], who deals with the variational compactness property of a class of integral functionals, examined the main ideas and techniques of Γ-convergence in function spaces. The main idea of using Γ-convergence is to obtain infimum values. Hence it provides the convenience of solving optimization problems. The notion of gamma convergence is a fundamental convergence theory for sequences of the lower semicontinuous functions in optimization theory, decision theory, homogenization problems, integral function theory, algorithmic procedures, and variational analysis. A noteworthy description of the theory was given by Wets [27] under the name of epi convergence in 1980 for the first time. The books "an introduction to gamma convergence" written by Dal Maso [15], and "Γ-convergence for Beginners" written by Braides [5] are remarkable comprehensive books on this subject. For more detailed information on Γ-convergence, we refer to [7, 6, 12].

Various types of convergence for double sequences can be defined (see [4, 13, 19]. The best known and well-studied convergence notion for double sequence spaces is Pringsheim convergence of double sequences is defined as the convergence of nets, where the set of indexes $\mathbb{N} \times \mathbb{N}$ is ordered in the natural way. In case of this convergence the row-index and the column-index tend independently to infinity. The main drawback of this convergence is that a convergent double sequence fails in general to be bounded.

In theory and applications, we may need to work with double sequences of functions. For example, uniform convergence of a double sequence of functions is a useful tool to obtain an approximation of the function of two variables [3, 10, 26, 21]. Minimization is preserved in this type of convergence, but the conditions are strong. For this reason, in this paper, we define Γ-convergence for a double sequence of functions and extend some fundamental results known concerning Γ-convergence in the literature to double sequences of functions. Furthermore, under some additional conditions, we prove that the limit of a double sequence of points selected from minimum points of f_{kl} is a minimum point of f when the double sequence (f_{kl}) Γ converges to the function f.

2. Definitions and notation

The well-known Pringsheim [19] convergence of double sequences is defined using the natural ordering of pairs of positive integers (there are several definitions of the limit of a double sequence that are not mutually identical). Since then, this concept of convergence has been studied in various areas of mathematics. Throughout the paper, the convergence of a double sequence means convergence in the Pringsheim sense.

A double sequence $x = (x_{jk})$ is said to be convergent to l, written $\lim_{j,k} x_{jk} = l$ or $x_{jk} \to l$, if for given $\varepsilon > 0$ there exists an integer n_0 such that $|x_{jk} - l| < \varepsilon$ whenever $j, k > n_0$. In this type of convergence, the indices j and k tend to be infinity independently of each other. A double sequence x is bounded if $\|x\| =$

$sup_{j,k}|x_{jk}| < \infty$. Double sequences have a variety of distinguishing characteristics as opposed to ordinary sequences. One of the reasons why double sequences theory presents difficulties not encountered in ordinary sequences theory is that a double sequence may converge without being a bounded sequence.

A double sequence x of points of a metric space (X, d) convergence to point $l \in X$ if $\lim_{j,k\to\infty} d(x_{jk}, l) = 0$. Patterson [17] gave the definition of subsequence and the Pringsheim limit point of a double sequence. A number $l \in X$ is said to be a Pringsheim limit point of a double sequence (x_{jk}) if there exist two strictly increasing sequences (j_i) and (k_i) such that $\lim_{i\to\infty} x_{j_i k_i} = l$. The set of all Pringsheim limit points of a double sequence (x_{jk}) will be denoted by $L_{(x_{jk})}$.

Lemma 2.1. [16] *Let $x = (x_{jk})$ be a double sequence of real numbers. Then the limit* $\lim_{j,k\to\infty} x_{jk} = l$ *exists if and only if for all increasing index sequences* $(j_i), (k_i)$ *such that, the ordinary limit* $\lim_{i\to\infty} x_{j_i k_i} = l$ *exists.*

Definition 2.1. [18] Let $x = (x_{jk})$ be a double sequence of real numbers and for each n, let $\alpha_n = \sup_n\{x_{jk} : k, l \geq n\}$. The Pringsheim limit superior of x is defined as follows:

(i) if $\alpha_n = +\infty$ for each n, then $\limsup_{j,k\to\infty} x_{jk} := +\infty$;

(ii) if $\alpha_n < +\infty$ for some n, then $\limsup_{j,k\to\infty} x_{jk} := \inf_n\{\alpha_n\}$.

Similarly, let $\beta_n = \inf_n\{x_{jk} : j, k \geq n\}$ then the Pringsheim limit inferior of $x = (x_{jk})$ is defined as follows:

(i) if $\beta_n = -\infty$ for each n, then $\liminf_{j,k\to\infty} x_{jk} := -\infty$;

(ii) if $\beta_n > -\infty$ for some n, then $\liminf_{j,k\to\infty} x_{jk} := \sup_n\{\beta_n\}$.

In all that follows, X will be a metric space equipped with the distance d, unless we explicitly state otherwise.

A double sequence (f_{kl}) of functions defined from X into $\mathbb{R}$ is said to be pointwise convergent to f on a set $S \subset X$, if for each point $x \in S$ and for each $\varepsilon > 0$, there exists a positive integer $N = N(x, \varepsilon)$ such that $|f_{kl}(x) - f(x)| < \varepsilon$ for all $k, l \geq N$. It is denoted by $\lim_{k,l\to\infty} f_{kl}(x) = f(x)$ on S.

A double sequence (f_{kl}) of functions defined from X into $\mathbb{R}$ is said to be uniformly convergent to the function f on a set $S \subset X$ if for each $\varepsilon > 0$, there exists a positive integer $N = N(\varepsilon)$ such that $k, l \geq N$ implies $|f_{kl}(x) - f(x)| < \varepsilon$, for all $x \in S$ (see [22]) .

Let $A \subset X$ and $x \in X$. Then the distance from a point x to A is given by

$$d(x, A) := \inf_{a\in A} d(x, a),$$

where we set $d(x, \emptyset) := \infty$. As long as A is closed, having $d(x, A) = 0$ is equivalent to having $x \in A$.

The open ball with center x and radius $\varepsilon > 0$ in X is denoted by $B(x, \varepsilon)$, i.e.,

$$B(x, \varepsilon) = \{y \in X \mid d(x, y) < \varepsilon\}.$$

There are different types of convergence for sequences of sets that are important for certain applications. The most studied of them are Kuratowski [14], Wijsman convergence [24, 25]. See also [1, 2, 11]. Sever et al. [23] have presented these types of convergence for double sequences of sets and investigated the relations between these types of convergence. We now recall the definition of Kuratowski convergence for a double sequence of sets.

Definition 2.2. [23] Let $(A_{kl})_{k,l\in\mathbb{N}}$ be a double sequence of subsets of X. We say that the subset

$$\limsup_{k,l\to\infty} A_{kl} := \left\{x \in X \mid \forall\varepsilon > 0,\ \ \forall n \in \mathbb{N},\ \ \exists k, l \geq n : B(x, \varepsilon) \cap A_{kl} \neq \emptyset\right\}$$

is the upper limit of the double sequence (A_{kl}) and that the subset

$$\liminf_{k,l\to\infty} A_{kl} := \left\{x \in X \mid \forall\varepsilon > 0,\ \ \exists n \in \mathbb{N} : B(x, \varepsilon) \cap A_{kl} \neq \emptyset,\ \ \forall k, l \geq n\right\}$$

is its lower limit. If there exists a set $A \subseteq X$ such that

$$A = \liminf_{k,l\to\infty} A_{kl} = \limsup_{k,l\to\infty} A_{kl}$$

then the double sequence set (A_{kl}) converges to A, denoted $\lim_{k,l\to\infty} A_{kl} = A$, in the sense of Kuratowski.

By Proposition 3.5 in [23], the sets $\liminf_{k,l\to\infty} A_{kl}$ and $\limsup_{k,l\to\infty} A_{kl}$ are closed in X.

Also, the upper and lower limits of a double sequence of sets are characterized by the following various ways.

Proposition 2.1. [23] *Let $(A_{kl})_{k,l\in\mathbb{N}}$ be a double sequence of closed subsets of X. Then*

(i) $\limsup_{k,l\to\infty} A_{kl} := \{x \in X : \liminf_{k,l\to\infty} d(x, A_{kl}) = 0\}$,

(ii) $\liminf_{k,l\to\infty} A_{kl} := \{x \in X : \lim_{k,l\to\infty} d(x, A_{kl}) = 0\}$.

Proposition 2.2. [23] *If $(A_{kl})_{k,l\in\mathbb{N}}$ is a double sequence of closed sets in X, then*

$$\liminf_{k,l\to\infty} A_{kl} = \Big\{x \mid \text{ there exists a double sequence } (y_{kl}),\ y_{kl} \in A_{kl} \text{ for any } k, l \in \mathbb{N}, \text{ with } \lim_{k,l\to\infty} y_{kl} = x\Big\}.$$

Proposition 2.3. [23] *If $(A_{kl})_{k,l\in\mathbb{N}}$ is a double sequence of closed sets in X, then*

$$\limsup_{k,l\to\infty} A_{kl} = \Big\{x \mid \textit{there exist two increasing sequences } k_i, l_i,\ y_{k_il_i}\in A_{k_il_i} \textit{ for any } i\in\mathbb{N},\ \textit{with } \lim_{i\to\infty} y_{k_il_i} = x\Big\}.$$

The double sequence set (A_{kl}) is said to be Pringsheim bounded if there exist a compact set K and $n \in \mathbb{N}$ such that $A_{kl} \subseteq K$ whenever $k, l \geq n$.

Let $f : X \to \overline{\mathbb{R}}$, where $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$. Epigraph and level set of the function f on X are defined by

$$epif := \{(x,\alpha) \in X \times \mathbb{R} \mid f(x) \leq \alpha\}$$

and for all $\alpha \in \mathbb{R}$

$$\{f \leq \alpha\} := \{x \in X \mid f(x) \leq \alpha\},$$

respectively.

The function f is level bounded if for each $\alpha \in \mathbb{R}$ there exists a compact set B_α such that $\{f \leq \alpha\} \subseteq B_\alpha$.

The function whose epigraph is equal to the closure of the epigraph of f is called the closure of f and is denoted by clf, that is

$$cl(epif) = epi(clf).$$

For every $x \in X$

$$(clf)(x) = \sup_{U\in\mathcal{N}(x)} \inf_{y\in U} f(y).$$

The function f is said to be lower semicontinuous at $x \in X$ if $f(x) = (clf)(x)$. The sets of points of x where the minimum of f over X is regarded as being attained

$$\text{argmin} f = \big\{x \in X : f(x) = \inf_{x\in X} f(x)\big\}.$$

Theorem 2.1. [15] *Assume that the function $f : X \to \overline{\mathbb{R}}$ is level bounded and lower semicontinuous. Then f has a minimum point in X.*

For every $a, b \in \overline{\mathbb{R}}$ we set $a \vee b = \max\{a, b\}$. For $\varepsilon > 0$, the set $\varepsilon - \text{argmin} f$ is the set of all points $x \in X$ satisfying

$$f(x) \leq \left(\inf_{x\in X} f(x) + \varepsilon\right) \vee \left(-\frac{1}{\varepsilon}\right)$$

3. Γ-convergence of Double Sequences of Functions

In this section, we state the definition of the Γ-limit of a double sequence of functions with the help of equality of two semi-limits, upper and lower Γ-limits, which are always defined and which can be studied separately.

From now on, let X be a metric space,$\mathcal{N}(x)$ be the family of all open neighborhoods of $x \in X$ and (f_{kl}) be a double sequence of functions from X into $\overline{\mathbb{R}}$.

Definition 3.1. The lower Γ-limit of the double sequence of functions (f_{kl}) is defined by

$$\big(\Gamma - \liminf_{k,l\to\infty} f_{kl}\big)(x) := \sup_{U\in\mathcal{N}(x)} \liminf_{k,l\to\infty} \inf_{y\in U} f_{kl}(y) \tag{3.1}$$

and the upper Γ-limit of the double sequence of functions (f_{kl}) is defined by

$$\big(\Gamma - \limsup_{k,l\to\infty} f_{kl}\big)(x) := \sup_{U\in\mathcal{N}(x)} \limsup_{k,l\to\infty} \inf_{y\in U} f_{kl}(y). \tag{3.2}$$

Moreover, a double sequence of functions (f_{kl}) Γ-converges if and only if

$$\Gamma - \liminf_{k,l\to\infty} f_{kl} = \Gamma - \limsup_{k,l\to\infty} f_{kl}$$

and the Γ-limit is common function of $\Gamma - \liminf$ and $\Gamma - \limsup$. In this case we denote $f := \Gamma - \lim_{k,l\to\infty} f_{kl}$.

Remark 3.1. If the functions $f_{kl}(x)$ are independent of x, i.e., for every $k, l \in \mathbb{N}$ there exists a constant $a_{kl} \in \overline{\mathbb{R}}$ such that $f_{kl}(x) = a_{kl}$ for every $x \in X$, then

$$(\Gamma - \liminf_{k,l\to\infty} f_{kl})(x) = \liminf_{k,l\to\infty} a_{kl} \quad \text{and} \quad (\Gamma - \limsup_{k,l\to\infty} f_{kl})(x) = \limsup_{k,l\to\infty} a_{kl}.$$

If the functions $f_{kl}(x)$ are independent both of k and l, i.e., there exists $f : X \to \overline{\mathbb{R}}$ such that $f_{kl}(x) = f(x)$ for every $k, l \in \mathbb{N}$ then

$$\Gamma - \liminf_{k,l\to\infty} f_{kl} = \Gamma - \limsup_{k,l\to\infty} f_{kl} = clf.$$

First, we compare Γ-convergence with pointwise convergence. Γ-convergence neither implies nor is implied by pointwise convergence. The following examples show how a double sequence of functions can have both a Γ-limit and a pointwise limit but are different in general.

Example 3.1. Let $f_{kl} : \mathbb{R} \to \mathbb{R}$ defined by $f_{kl}(x) = klxe^{-2k^2l^2x^2}$. Then the double sequence of function (f_{kl}) Γ-converges in $\mathbb{R}$ to the function

$$f(x) = \begin{cases} -\frac{1}{2}e^{-\frac{1}{2}} & , \quad x = 0 \\ 0 & , \quad x \neq 0 \end{cases}$$

where as (f_{kl}) converges pointwise to function $g(x) = 0$ in $\mathbb{R}$.

Example 3.2. Define $f_{kl} : \mathbb{R} \to \mathbb{R}$ by

$$f_{kl}(x) = \begin{cases} klxe^{-2k^2l^2x^2} & , \quad k+l \text{ is even} \\ 2klxe^{-2k^2l^2x^2} & , \quad k+l \text{ is odd.} \end{cases}$$

Then the double sequence of function (f_{kl}) converges pointwise to function $f(x) = 0$ but (f_{kl}) does not Γ-converge in $\mathbb{R}$. In fact

$$\Big(\Gamma - \liminf_{k,l\to\infty} f_{kl}\Big)(x) = \begin{cases} -e^{-\frac{1}{2}} & , \quad x = 0 \\ 0 & , \quad x \neq 0 \end{cases}$$

and

$$(\Gamma - \limsup_{k,l\to\infty} f_{kl})(x) = \begin{cases} -\frac{1}{2}e^{-\frac{1}{2}} & , \quad x = 0 \\ 0 & , \quad x \neq 0. \end{cases}$$

Example 3.3. Define $f_{kl} : \mathbb{R} \to \mathbb{R}$ by $f_{kl}(x) = \sin(k+l)x$ then the double sequence of functions (f_{kl}) Γ-converges in $\mathbb{R}$ to the function $f(x) = -1$ but (f_{kl}) does not converge pointwise.

Note that even if the double sequence (f_{kl}) converges pointwise to f, the sequence $(\inf_{x\in X} f_{kl}(x))$ cannot converge to $\min_{x\in X} f(x)$. We can see this fact in the following example.

Example 3.4. Define $f_{kl} : \mathbb{R} \to \mathbb{R}$ by

$$f_{kl}(x) = \begin{cases} 0 & , \quad x = \frac{1}{k+l} \\ 1 & , \quad \text{otherwise.} \end{cases}$$

Then the double sequence of function (f_{kl}) Γ-converges in $\mathbb{R}$ to the function

$$f(x) = \begin{cases} 0 & , \quad x = 0 \\ 1 & , \quad \text{otherwise} \end{cases}$$

where as (f_{kl}) converges pointwise to the function $g(x) = 1$ in $\mathbb{R}$. On the other hand, If we examine the limit of the minimum values of the functions and the minimum value of the limit function, it will be seen that they are different from each other.

$$\lim_{k,l\to\infty} \inf_{x\in\mathbb{R}} f_{kl}(x) = \inf_{x\in\mathbb{R}} f(x) = 0,$$

but

$$\lim_{k,l\to\infty} \inf_{x\in\mathbb{R}} f_{kl}(x) \neq \inf_{x\in\mathbb{R}} g(x) = 1.$$

In the next example we show that the iterated $\Gamma-$limits and $\Gamma-$limit of a double sequence of (f_{kl}) are unrelated.

Example 3.5. Define $f_{kl} : \mathbb{R} \to \mathbb{R}$ by $f_{kl}(x) = \frac{l}{k+l}x^2$. Then $\Gamma - \lim_k(\Gamma - \lim_l f_{kl}) = x^2$ and $\Gamma - \lim_l(\Gamma - \lim_k f_{kl}) = 0$. However, $\Gamma - \lim_{k,l\to\infty} f_{kl}$ does not exist. On the other hand define $g_{kl}(x) = \frac{(-1)^{k+l}}{\min\{k,l\}}x^2$. Then $\Gamma - \lim_{k,l\to\infty} g_{kl} = 0$. But either of iterated $\Gamma-$limits does not exist.

We next give the relationship between uniform convergence and Γ-convergence of a double sequence of functions.

Proposition 3.1. *If a double sequence of functions (f_{kl}) converges to the function f uniformly, then the double sequence (f_{kl}) Γ-converges to $cl f$.*

Proof. Suppose that double sequence of functions (f_{kl}) converges to f uniformly. For every open subset U of X we have

$$\lim_{k,l\to\infty} \inf_{y\in U} f_{kl}(y) = \inf_{y\in U} f(y),$$

hence for every $x \in X$

$$\sup_{U\in\mathcal{N}(x)} \lim_{k,l\to\infty} \inf_{y\in U} f_{kl}(y) = \sup_{U\in\mathcal{N}(x)} \inf_{y\in U} f(y),$$

which yields that the sequence (f_{kl}) Γ-converges to $cl f$. $\square$

We showed above that Γ-convergence and pointwise convergence of a double sequence of functions are independent, in general. Now we investigate the relationship between them and some conditions under which Γ-convergence and pointwise convergence of a double sequence of functions are equivalent.

Proposition 3.2. *Let (f_{kl}) be a double sequence of functions from X into $\overline{\mathbb{R}}$. Then*

$$\Gamma - \liminf_{k,l\to\infty} f_{kl} \leq \liminf_{k,l\to\infty} f_{kl} \quad \text{and} \quad \Gamma - \limsup_{k,l\to\infty} f_{kl} \leq \limsup_{k,l\to\infty} f_{kl}. \tag{3.3}$$

In particular, if (f_{kl}) Γ-converges to f and converges to pointwise to g then $f \leq g$.

Proof. For every $x \in X$ and for every $U \in \mathcal{N}(x)$ we obtain $\inf_{y\in U} f_{kl}(y) \leq f_{kl}(x)$. Consequently,

$$\liminf_{k,l\to\infty} \inf_{y\in U} f_{kl}(y) \leq \liminf_{k,l\to\infty} f_{kl}(x),$$
$$\limsup_{k,l\to\infty} \inf_{y\in U} f_{kl}(y) \leq \limsup_{k,l\to\infty} f_{kl}(x).$$

The result is obtained by taking supremum over all $U \in \mathcal{N}(x)$. $\square$

To obtain the reverse inequalities of (3.3) we give the concept of Pringsheim equi-lower semicontinuous for a double sequence of functions.

Definition 3.2. A double sequence function (f_{kl}) is said to be Pringsheim equi-lower semicontinuous at a point $x \in X$ if for every $\varepsilon > 0$ there exist $U \in \mathcal{N}(x)$ and $n \in \mathbb{N}$ such that $f_{kl}(y) \geq f_{kl}(x) - \varepsilon$ for every $y \in U$ and for every $k, l \geq n$. (f_{kl}) is said to be Pringsheim equi-lower semicontinuous on X if (f_{kl}) is Pringsheim equi-lower semicontinuous at each point $x \in X$.

Theorem 3.1. *Let a double sequence function (f_{kl}) be Pringsheim equi-lower semicontinuous at a point $x \in X$. Then*

$$\big(\Gamma - \liminf_{k,l\to\infty} f_{kl}\big)(x) = \liminf_{k,l\to\infty} f_{kl}(x) \quad \textit{and} \quad \big(\Gamma - \limsup_{k,l\to\infty} f_{kl}\big)(x) = \limsup_{k,l\to\infty} f_{kl}(x).$$

In particular, if (f_{kl}) $\Gamma-$ converges to f if and only if (f_{kl}) converges to f pointwise in X.

Proof. We prove only the first equality, the proof of the other one being similar. By Proposition 3.2 it is sufficient to show that

$$\liminf_{k,l\to\infty} f_{kl}(x) \leq \big(\Gamma - \liminf_{k,l\to\infty} f_{kl}\big)(x).$$

Since (f_{kl}) is Pringsheim equi-lower semicontinuous at x, for every $\varepsilon > 0$ there exist $U \in \mathcal{N}(x)$ and $n \in \mathbb{N}$ such that $f_{kl}(y) \geq f_{kl}(x) - \varepsilon$ for every $y \in U$ and for every $k, l \geq n$. Hence

$$f_{kl}(x) \leq \inf_{y\in U} f_{kl}(y) + \varepsilon, \quad \text{whenever } k, l \geq n.$$

We deduce that

$$\liminf_{k,l\to\infty} f_{kl}(x) \leq \liminf_{k,l\to\infty} \inf_{y\in U} f_{kl}(y) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary and $U \in \mathcal{N}(x)$, we get

$$\liminf_{k,l\to\infty} f_{kl}(x) \leq \sup_{U\in\mathcal{N}(x)} \liminf_{k,l\to\infty} \inf_{y\in U} f_{kl}(y),$$

which is the desired conclusion. □

The following theorem shows the relation between Γ-convergence of a double sequence of functions and Kuratowski convergence of their epigraphs. This is the reason why Γ-convergence is sometimes called epi-convergence.

Theorem 3.2. *Let (f_{kl}) be a double sequence of functions defined from X into $\overline{\mathbb{R}}$. Then*

$$epi(\Gamma - \liminf_{k,l\to\infty} f_{kl}) = \limsup_{k,l\to\infty}(epi f_{kl}), \tag{3.4}$$

and

$$epi(\Gamma - \limsup_{k,l\to\infty} f_{kl}) = \liminf_{k,l\to\infty}(epi f_{kl}). \tag{3.5}$$

Proof. Since the proofs of equality (3.4) and (3.5) are quite similar, we only prove the first one. For the convenience, set $\Gamma - \liminf_{k,l\to\infty} f_{kl} = h$. For this we must show inclusion $epi\, h \subset \limsup_{k,l\to\infty}(epi f_{kl})$ and vice versa. Let $(x,\alpha) \in epi\, h$ if and only if $h(x) \leq \alpha$. By the definition of the lower Γ-limit of the double sequence of functions (f_{kl}), for every $\varepsilon > 0$, and for every $U \in \mathcal{N}(x)$ we have

$$\liminf_{k,l\to\infty} \inf_{y\in U} f_{kl}(y) < \alpha + \varepsilon,$$

and which is equivalent to say that for every $n \in \mathbb{N}$ there exist $k, l \geq n$ such that $\inf_{y\in U} f_{kl}(y) < \alpha + \varepsilon$. This gives us

$$epi f_{kl} \cap \big(U \times (\alpha + \varepsilon, \alpha - \varepsilon)\big) \neq \emptyset.$$

Since $U \times (\alpha + \varepsilon, \alpha - \varepsilon)$ is an arbitrary neighborhood of (x, α),

$$(x,\alpha) \in \limsup_{k,l\to\infty}(epi f_{kl}).$$

Consequently, we have proved that $(x, \alpha) \in epi\, h$ if and only if

$$(x,\alpha) \in \limsup_{k,l\to\infty}(epi f_{kl}),$$

thus we have completed the proof. $\square$

Remark 3.2. We know that the lower limit and upper limit of the double sequence $(epi f_{kl})$ are closed. Therefore, from (3.4) and (3.5) the functions $\Gamma - \liminf_{k,l\to\infty} f_{kl}$ and $\Gamma - \limsup_{k,l\to\infty} f_{kl}$ are lower semicontinuous on X.

The following theorem shows the relationship between Γ-convergence of a double sequence of functions and Kuratowski-convergence of their level sets.

Theorem 3.3. *Let (f_{kl}) be a double sequence of functions defined from X into $\overline{\mathbb{R}}$,*

$$\Gamma - \liminf_{k,l\to\infty} f_{kl} = h \quad \text{and} \quad \Gamma - \limsup_{k,l\to\infty} f_{kl} = g.$$

Then for every $s \in \mathbb{R}$ we have

$$\{h \leq s\} = \bigcap_{t>s} \limsup_{k,l\to\infty}\{f_{kl} \leq t\}, \tag{3.6}$$

$$\{g \leq s\} = \bigcap_{t>s} \liminf_{k,l\to\infty}\{f_{kl} \leq t\}. \tag{3.7}$$

In particular, (f_{kl}) Γ-converges to f if and only if

$$\{f \leq s\} = \bigcap_{t>s} \limsup_{k,l\to\infty}\{f_{kl} \leq t\} = \bigcap_{t>s} \liminf_{k,l\to\infty}\{f_{kl} \leq t\}$$

for every $s \in \mathbb{R}$.

Proof. We shall prove only (3.6), the proof of (3.7) being analogous. Let us take an arbitrary point $x \in X$ belongs to the set $\{h \leq s\}$. Then for each $t > s$, $h(x) < t$. By definition lower Γ-limit for every $U \in \mathcal{N}(x)$ we obtain

$$\liminf_{k,l\to\infty} \inf_{y\in U} f_{kl}(y) < t.$$

Therefore, for every $n \in \mathbb{N}$ there exist $k, l \geq n$ such that $\inf_{y\in U} f_{kl}(y) < t$. The equivalent of this inequality is $\{f_{kl} < t\} \cap U \neq \emptyset$. This yields $x \in \limsup_{k,l\to\infty}\{f_{kl} \leq t\}$. Since t is arbitrary number grater than s we have

$$x \in \bigcap_{t>s} \limsup_{k,l\to\infty}\{f_{kl} \leq t\}.$$

We now assume that $x \in X$ belongs to the intersection set on the right hand side of (3.6). Then for each $t > s$, $x \in \limsup_{k,l\to\infty}\{f_{kl} \leq t\}$. From (3.2), for every $U \in \mathcal{N}(x)$ and for every $n \in \mathbb{N}$ there exist $k, l \geq n$ such that $\{f_{kl} < t\} \cap U \neq \emptyset$. Since this is equivalent to $\inf_{y\in U} f_{kl}(y) < t$, we have

$$\liminf_{k,l\to\infty} \inf_{y\in U} f_{kl}(y) \leq t.$$

Since $U \in \mathcal{N}(x)$ is arbitrary, we obtain $h(x) \leq t$. As a consequence, $t > s$ implies $t \geq h(x)$. This gives us $s \geq h(x)$, that is $x \in \{h \leq s\}$, and the proof is complete. □

Remark 3.3. In general the equalities

$$\{\Gamma - \liminf_{k,l\to\infty} f_{kl} \leq s\} = \limsup_{k,l\to\infty}\{f_{kl} \leq s\}, \quad \text{and} \quad \{\Gamma - \limsup_{k,l\to\infty} f_{kl} \leq s\} = \liminf_{k,l\to\infty}\{f_{kl} \leq s\}$$

do not hold, even if (f_{kl}) Γ-converges to f. For instance if $X = \mathbb{R}$, $f_{kl}(x) = \frac{1}{kl}$, and $f(x) = 0$, then (f_{kl}) Γ-converges to f but

$$\{f \leq 0\} = \mathbb{R} \neq \emptyset = \lim_{k,l\to\infty}\{f_{kl} \leq 0\}.$$

4. Convergence of Minima and of Minimizers

In this section we focus on a minimization problem and investigate conditions ensuring that the double sequence of the minimum values of f_{kl} converges to the minimum value of f.

Proposition 4.1. *Let (f_{kl}) be a double sequence of functions from X into $\overline{\mathbb{R}}$ and Let U be an open subset of X. If $\Gamma - \liminf_{k,l\to\infty} f_{kl} = h$ and $\Gamma - \limsup_{k,l\to\infty} f_{kl} = g$, then the following inequalities hold.*

$$\begin{aligned} \inf_{x\in U} h(x) &\geq \liminf_{k,l\to\infty} \inf_{x\in U} f_{kl}(x), \\ \inf_{x\in U} g(x) &\geq \limsup_{k,l\to\infty} \inf_{x\in U} f_{kl}(x). \end{aligned}$$

Proof. Let U be an open subset of X. For any $x \in U$, we have $U \in \mathcal{N}(x)$. By the definition of Γ-lower limit of double sequences of functions, the inequality

$$h(x) \geq \liminf_{k,l\to\infty} \inf_{y\in U} f_{kl}(y)$$

is obtained. Consequently,

$$\inf_{x\in U} h(x) \geq \liminf_{k,l\to\infty} \inf_{y\in U} f_{kl}(y).$$

In the same manner, we can see that

$$\inf_{x\in U} g(x) \geq \limsup_{k,l\to\infty} \inf_{y\in U} f_{kl}(y)$$

and the proof is achieved. □

In particular, taking $U = X$, we obtain

$$\inf_{x\in X} h(x) \geq \liminf_{k,l\to\infty} \inf_{x\in X} f_{kl}(x) \quad \text{and} \quad \inf_{x\in X} g(x) \geq \limsup_{k,l\to\infty} \inf_{x\in X} f_{kl}(x).$$

Proposition 4.2. *Let K be a compact subset of X and $\Gamma - \liminf_{k,l\to\infty} f_{kl} = h$. Then*

$$\min_{x\in K} h(x) \leq \liminf_{k,l\to\infty} \inf_{x\in K} f_{kl}(x).$$

Proof. First of all, let us note that the h takes the minimum value on K since $\Gamma - \liminf_{k,l\to\infty} f_{kl}$ is lower semicontinuous on K. Assume that

$$\liminf_{k,l\to\infty} \inf_{x\in K} f_{kl}(x) < \infty$$

and $\alpha \in \mathbb{R}$ is an arbitrary number satisfying

$$\liminf_{k,l\to\infty} \inf_{x\in K} f_{kl}(x) < \alpha.$$

Then for every $n \in \mathbb{N}$ there exist $k, l \geq n$ such that $\inf_{x\in K} f_{kl}(x) \leq \alpha$. So there exists a $x_{kl} \in K$ such that $f_{kl}(x_{kl}) \leq \alpha$. Since K is compact, so there exist two increasing index sequence k_i, l_i such that $\lim_{i\to\infty} x_{k_i l_i} = x_0 \in K$. Therefore for every $U \in \mathcal{N}(x_0)$ there exists a i_0 such that $x_{k_i l_i} \in U$ for all $i \geq i_0$. Hence $\inf_{x\in U} f_{k_i l_i}(x) \leq f_{k_i l_i}(x_{k_i l_i}) \leq \alpha$ and we obtain

$$\liminf_{k,l\to\infty} \inf_{x\in U} f_{kl}(x) \leq \alpha.$$

Since $U \in \mathcal{N}(x_0)$ is arbitrary

$$h(x_0) = \sup_{U\in\mathcal{N}(x_0)} \liminf_{k,l\to\infty} \inf_{x\in U} f_{kl}(x) \leq \alpha.$$

Consequently, we have $\min_{x\in K} h(x) \leq h(x_0) \leq \alpha$. This implies that

$$\min_{x\in K} h(x) \leq \liminf_{k,l\to\infty} \inf_{x\in K} f_{kl}(x),$$

which completes the proof. □

The following example shows that, when the double sequence (f_{kl}) is not Γ-convergent, the inequality

$$\min_{x\in K}\big(\Gamma-\limsup_{k,l\to\infty} f_{kl}\big)(x) \le \limsup_{k,l\to\infty}\inf_{x\in K} f_{kl}(x) \tag{4.1}$$

may not be true for some compact subset K of X.

Example 4.1. Let $X=\mathbb{R}$, $f_{kl}(x)=\Big(x-(-1)^{k+l}\Big)^2$, and let $K=[-1,1]$. Then we can evaluate

$$\begin{aligned}\Big(\Gamma-\liminf_{k,l\to\infty} f_{kl}\Big)(x) &= \min\Big\{(x-1)^2,(x+1)^2\Big\},\\ \Big(\Gamma-\limsup_{k,l\to\infty} f_{kl}\Big)(x) &= \max\Big\{(x-1)^2,(x+1)^2\Big\}.\end{aligned}$$

Since $\min_{x\in\mathbb{R}}\Big\{(\Gamma-\limsup_{k,l\to\infty} f_{kl})(x)\Big\}=\min_{x\in K}\Big\{(\Gamma-\limsup_{k,l\to\infty} f_{kl})(x)\Big\}=1$ and $\min_{x\in\mathbb{R}} f_{kl}(x)=\min_{x\in K} f_{kl}(x)=0$, for all $k,l\in\mathbb{N}$, the condition (4.1) is not satisfied.

Theorem 4.1. *Let $\Gamma-\lim_{k,l\to\infty} f_{kl}=f$. Suppose that there exists a compact subset K of X and $n_0\in\mathbb{N}$ such that*

$$\inf_{x\in X} f_{kl}(x)=\inf_{x\in K} f_{kl}(x) \quad \textit{whenever} \quad k,l\ge n_0. \tag{4.2}$$

Then f attains its minimum on X and

$$\min_{x\in X} f(x)=\lim_{k,l\to\infty}\inf_{x\in X} f_{kl}(x). \tag{4.3}$$

Proof. By Propositions 4.1 and 4.2 we have the following inequality

$$\begin{aligned}\inf_{x\in X} f(x) &\le \min_{x\in K} f(x)\\ &\le \liminf_{k,l\to\infty}\inf_{x\in K} f_{kl}(x)\\ &= \liminf_{k,l\to\infty}\inf_{x\in X} f_{kl}(x)\\ &\le \limsup_{k,l\to\infty}\inf_{x\in X} f_{kl}(x)\\ &\le \inf_{x\in X} f_{kl}(x).\end{aligned}$$

Consequently, we obtain equality (4.3). □

Example 4.2. Define $f_{kl}:\mathbb{R}\to\mathbb{R}$ by

$$f_{kl}(x)=\begin{cases}\frac{(x+k)^2}{k^2} & , \quad l=1, \text{ for all } k,\\ 1+\frac{x^2}{k} & , \quad l>1, \text{ for all } k.\end{cases}$$

Then $\Gamma-\lim_{k,l\to\infty} f_{kl}=1$. We can not find a compact set K such that for all $k,l\in\mathbb{N}$,

$$\inf_{x\in X} f_{kl}(x)=\inf_{x\in K} f_{kl}(x).$$

However, $K=[-1,1]$ and for all $k,l>1$ we have condition (4.2). By Theorem 4.1, equality (4.3) holds.

To provide a practical criterion for verifying condition (4.2), we appeal to the concept of Pringsheim level boundedness.

Definition 4.1. A double sequence of functions (f_{kl}) from X to $\overline{\mathbb{R}}$ is Pringsheim level bounded if for each $\alpha \in \mathbb{R}$ there exists a compact set B_α along with $n \in \mathbb{N}$ such that $\{f_{kl} \leq \alpha\} \subseteq B_\alpha$, whenever $k, l \geq n$.

Lemma 4.1. *Let (f_{kl}) be Pringsheim level bounded. If $\Gamma - \lim_{k,l\to\infty} f_{kl} = f$, then f is level bounded.*

Proof. Let s be a real number and $t > s$. Since (f_{kl}) is Pringsheim level bounded, there exist $n \in \mathbb{N}$ and compact set B_t such that $\{f_{kl} \leq t\} \subseteq B_t$ whenever $k, l \geq n$. So $\liminf_{k,l\to\infty}\{f_{kl} \leq t\} \subseteq B_t$. By Theorem 3.3 we obtain

$$\{f \leq s\} \subseteq \bigcap_{t>s} B_t.$$

Since $\bigcap_{t>s} B_t$ is compact, f is level bounded. □

Theorem 4.2. *Suppose that a double sequence of functions (f_{kl}) is Pringsheim level bounded and $\Gamma - \lim_{k,l\to\infty} f_{kl} = f$. Then equality (4.3) holds.*

Proof. Since (f_{kl}) is Pringsheim level bounded and by Lemma 4.1, f is level bounded. Therefore f attains minimum on X. Hence $\inf_{x\in X} f(x) = \min_{x\in X} f(x)$. On the other hand, by Proposition 4.1,

$$\limsup_{k,l\to\infty} \inf_{x\in X} f_{kl}(x) \leq \min_{x\in X} f(x) < \infty.$$

Hence there exists an α such that

$$\limsup_{k,l\to\infty} \inf_{x\in X} f_{kl}(x) \leq \alpha.$$

Thus there exists $n_1 \in \mathbb{N}$ such that $\inf_{x\in X} f_{kl}(x) \leq \alpha$, whenever $k, l \geq n_1$. Since (f_{kl}) is Pringsheim level bounded, there exist a compact set B_α and $n_2 \in \mathbb{N}$ such that $\{f_{kl} \leq \alpha\} \subseteq B_\alpha$, whenever $k, l \geq n_2$. Let $n_0 = \max\{n_1, n_2\}$, then for every $k, l \geq n_0$

$$\inf_{x\in X} f_{kl}(x) = \inf_{x\in B_\alpha} f_{kl}(x).$$

So condition (4.2) is satisfied. By Theorem 4.1 we have equality (4.3). □

Example 4.3. Define $f_{kl} : \mathbb{R} \to \mathbb{R}$ by

$$f_{kl}(x) = \begin{cases} -1 & , \quad l = 1, \text{ for all } k, \\ 1 + x^2 & , \quad l > 1, \text{ for all } k. \end{cases}$$

Then, (f_{kl}) is Pringsheim level bounded and $\Gamma - \lim_{k,l\to\infty} f_{kl} = 1 = f(x)$. By Theorem 4.2, we have $\min_{x\in X} f(x) = \lim_{k,l\to\infty} \inf_{x\in X} f_{kl}(x)$.

Remark 4.1. The single sequence version of Theorem 4.2 says that a single sequence of functions (f_k) need to be eventually level bounded. That is, for each $\alpha \in \mathbb{R}$ there exists a compact set B_α along with $n \in \mathbb{N}$ such that $\{f_k \leq \alpha\} \subseteq B_\alpha$, whenever $k \geq n$ (see Theorem 7.8 in [15], and Theorem 7.33 in [20]). However, in Examples 4.2 and 4.3 we see that infinitely many terms of (f_{kl}) can be level unbounded, but Theorem 4.2 still is valid. The number of level unbounded functions leads to one of the main difference between single sequence and double sequence in $\Gamma-$convergence.

In the rest of the paper, we suggest the procedure to find the minimizers of a function.

Lemma 4.2. *Assume that* $\Gamma - \lim_{k,l\to\infty} f_{kl} = f$ *and* (x_{kl}) *be a double sequence in* X.

(i) If x *is a Pringsheim limit point of the double sequence* (x_{kl}), *then*

$$f(x) \leq \limsup_{k,l\to\infty} f_{kl}(x_{kl}). \tag{4.4}$$

(ii) If $\lim_{k,l\to\infty} x_{kl} = x \in X$, *then*

$$f(x) \leq \liminf_{k,l\to\infty} f_{kl}(x_{kl}). \tag{4.5}$$

Proof. First, we prove (i). Assume that x is a Pringsheim limit point of (x_{kl}). Then there exist two strictly increasing sequences (k_i) and (l_i) such that $\lim_{i\to\infty} x_{k_i l_i} = x$. Let $U \in \mathcal{N}(x)$. For every $n \in \mathbb{N}$ we can find $k_i, l_i \geq n$ such that $x_{k_i l_i} \in U$ and thus

$$\inf_{k,l\geq n} \inf_{x\in U} f_{kl}(x) \leq f_{k_i l_i}(x_{k_i l_i}) \leq \sup_{k,l\geq n} f_{kl}(x_{kl})$$

holds. Letting $n \to \infty$ the above inequality yields

$$\liminf_{k,l\to\infty} \inf_{x\in U} f_{kl}(x) \leq \limsup_{k,l\to\infty} f_{kl}(x_{kl}).$$

Since $U \in \mathcal{N}(x)$ is arbitrary we have (4.4).

Now we prove (ii). Assume that $\lim_{kl\to\infty} x_{kl} = x$. For each $U \in \mathcal{N}(x)$ there exists $n_0 \in \mathbb{N}$ such that for every $k, l \geq n_0$, $x_{kl} \in U$ and thus $\inf_{x\in U} f_{kl}(x) \leq f_{kl}(x_{kl})$. This implies

$$\liminf_{k,l\to\infty} \inf_{x\in U} f_{kl}(x) \leq \liminf_{k,l\to\infty} f_{kl}(x_{kl}).$$

Since $U \in \mathcal{N}(x_0)$ is arbitrary, we have (4.5). □

Theorem 4.3. *Assume that* $\Gamma - \lim_{k,l\to\infty} f_{kl} = f$. *Then*

$$\limsup_{k,l\to\infty}(\text{argmin} f_{kl}) \subseteq \bigcap_{\varepsilon>0} \limsup_{k,l\to\infty}(\varepsilon - \text{argmin} f_{kl}) \subseteq \text{argmin} f. \tag{4.6}$$

If

$$\bigcap_{\varepsilon>0} \limsup_{k,l\to\infty}(\varepsilon - \text{argmin } f_{kl}) \neq \emptyset, \tag{4.7}$$

then argmin $f \neq \emptyset$ *and*

$$\min_{y\in X} f(y) = \limsup_{k,l\to\infty} \inf_{y\in X} f_{kl}(y). \tag{4.8}$$

Proof. For every $\varepsilon > 0$ and $k, l \in \mathbb{N}$, since argmin $f_{kl} \subseteq \varepsilon -$ argmin f_{kl}, we have the first inclusion in (4.6).

We now prove for each $\varepsilon > 0$

$$\limsup_{k,l\to\infty}(\varepsilon - \text{argmin } f_{kl}) \subseteq \varepsilon - \text{argmin} f. \tag{4.9}$$

Let ε be an arbitrary but fixed positive number and $x \in \limsup_{k,l\to\infty}(\varepsilon - \text{argmin} f_{kl})$. Then there exists a double sequence $(x_{kl}) \subseteq \varepsilon - \text{argmin} f_{kl}$ such that $x \in L_{(x_{kl})}$. By Lemma 4.2,

$$f(x) \leq \limsup_{k,l\to\infty} f_{kl}(x_{kl}).$$

Since $x_{kl} \in \varepsilon - \text{argmin} f_{kl}$ we have

$$f_{kl}(x_{kl}) \leq (\inf_{y\in X} f_{kl}(y) + \varepsilon) \vee (-\frac{1}{\varepsilon}).$$

Therefore,

$$\begin{aligned} f(x) &\leq \limsup_{k,l\to\infty} f_{kl}(x_{kl}) \\ &\leq (\limsup_{k,l\to\infty} \inf_{y\in X} f_{kl}(y) + \varepsilon) \vee (-\frac{1}{\varepsilon}) \\ &\leq (\inf_{y\in X} f(y) + \varepsilon) \vee (-\frac{1}{\varepsilon}). \end{aligned}$$

This implies that $x \in \varepsilon -$ argmin f. Therefore inclusion (4.9) holds. Since for every $\varepsilon > 0$ inclusion (4.9) holds, we have

$$\bigcap_{\varepsilon>0} \limsup_{k,l\to\infty}(\varepsilon - \text{argmin } f_{kl}) \subseteq \bigcap_{\varepsilon>0}(\varepsilon - \text{argmin} f) = \text{argmin} f.$$

This completes the proof of (4.6).

If condition (4.7) holds, then argmin $f \neq \emptyset$. Let

$$x \in \bigcap_{\varepsilon>0} \limsup_{k,l\to\infty}(\varepsilon - \text{argmin} f_{kl}).$$

For every $\varepsilon > 0$, $x \in \limsup_{k,l\to\infty}(\varepsilon - \text{argmin } f_{kl})$. From first part of the proof

$$\inf_{y\in X} f(y) \le f(x) \le (\limsup_{k,l\to\infty} \inf_{y\in X} f_{kl}(y) + \varepsilon) \vee (-\frac{1}{\varepsilon}).$$

Since ε is arbitrary, we obtain

$$\inf_{y\in X} f(y) \le f(x) \le \limsup_{k,l\to\infty} \inf_{y\in X} f_{kl}(y) \le \inf_{y\in X} f(y).$$

Hence, x is a minimizer of f and (4.8) holds. $\square$

In the following example, we show the advantage of approaching a function by using a double sequence in Γ-convergence.

Example 4.4. Let $X = \mathbf{R}$ and let $f_{kl}(x) = \big(x^2 - 1\big) \vee \frac{1}{k+l}\big(x - (-1)^{k+l}\big)^2$. Then the double sequence f_{kl} Γ-converges to the function $f(x) = \big(x^2 - 1\big) \vee 0$. . On the other hand we take $k = l$ the sequence $g_k = f_{kk}(x) = \big(x^2 - 1\big) \vee \frac{1}{2k}(x - 1)^2$ is Γ-convergent to the function $f(x) = \big(x^2 - 1\big) \vee 0$.

$$\text{argmin} f_{kl} = \begin{cases} \{-1\}, & \text{if } k + l \text{ is odd} \\ \{1\}, & \text{if } k + l \text{ is even} \end{cases} \quad \text{and argmin} g_k = \{1\}.$$

So $\limsup_{k,l\to\infty} \text{argmin} f_{kl} = \{-1, 1\}$ and $\limsup_{k\to\infty} \text{argmin} g_k = \{1\}$. By Theorem 4.3, both -1 and 1 are in argmin$f$. Therefore, it can find more points that minimize the function f if we use a double sequence instead of a single sequence in Γ-convergence.

Theorem 4.4. *Assume that* $\Gamma - \lim_{k,l\to\infty} f_{kl} = f$ *and*

$$\bigcap_{\varepsilon>0} \liminf_{k,l\to\infty} (\varepsilon - \text{argmin} f_{kl}) \neq \emptyset. \tag{4.10}$$

Then argmin $f \neq \emptyset$ *and*

$$\min_{y\in X} f(y) = \lim_{k,l\to\infty} \inf_{y\in X} f_{kl}(y). \tag{4.11}$$

Proof. If condition (4.10) holds, then from (4.7) we have argmin$f \neq \emptyset$. Let

$$x \in \bigcap_{\varepsilon>0} \liminf_{k,l\to\infty} (\varepsilon - \text{argmin} f_{kl}). \tag{4.12}$$

Then for every $\varepsilon > 0$, $x \in \liminf_{k,l\to\infty}(\varepsilon - \text{argmin} f_{kl})$. By Proposition 2.2 there exists a double sequence (x_{kl}) in $\varepsilon -$ argmin f_{kl} such that $x_{kl} \to x$. By Lemma 4.2

$$f(x) \le \liminf_{k,l\to\infty} f_{kl}(x_{kl}) \le (\liminf_{k,l\to\infty} \inf_{y\in X} f_{kl}(y) + \varepsilon) \vee (-\frac{1}{\varepsilon}).$$

Since $\varepsilon > 0$ is arbitrary,

$$\begin{aligned}\inf_{y\in X} f(y) \leq f(x) &\leq \liminf_{k,l\to\infty} \inf_{y\in X} f_{kl}(y)\\ &\leq \limsup_{k,l\to\infty} \inf_{y\in X} f_{kl}(y)\\ &\leq \inf_{y\in X} f_{kl}(y).\end{aligned}$$

Hence, we have (4.11). □

Corollary 4.1. *Let* $\Gamma - \lim_{k,l\to\infty} f_{kl} = f$.

(i) If there exists a double sequence (ε_{kl}) *of positive numbers converging to* 0 *such that* $\limsup_{k,l\to\infty}(\varepsilon_{kl} - \text{argmin } f_{kl}) \neq \emptyset$, *then* argmin $f \neq \emptyset$ *and* (4.8) *holds.*

(ii) If there exists a double sequence (ε_{kl}) *of positive numbers converging to* 0 *such that* $\liminf_{k,l\to\infty}(\varepsilon_{kl} - \text{argmin } f_{kl}) \neq \emptyset$, *then* argmin $f \neq \emptyset$ *and* (4.11) *holds.*

Proof. Since (ε_{kl}) is convergent to 0, for arbitrary fixed $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $\varepsilon_{kl} < \varepsilon$ whenever $k, l \geq n_0$. Therefore $\varepsilon_{kl} - \text{argmin } f_{kl} \subseteq \varepsilon - \text{argmin } f_{kl}$. This implies

$$\begin{aligned}\limsup_{k,l\to\infty}(\varepsilon_{kl} - \text{argmin} f_{kl}) &\subseteq \limsup_{k,l\to\infty}(\varepsilon - \text{argmin} f_{kl}),\\ \liminf_{k,l\to\infty}(\varepsilon_{kl} - \text{argmin} f_{kl}) &\subseteq \liminf_{k,l\to\infty}(\varepsilon - \text{argmin} f_{kl}).\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, (4.7) and (4.10) are satisfied. This completes the proof. □

Corollary 4.2. *Let* $\Gamma - \lim_{k,l\to\infty} f_{kl} = f$. *For every* $k, l \in \mathbb{N}$, *let* $x_{kl} \in (\varepsilon_{kl} -$ argmin $f_{kl})$ *where* $\varepsilon_{kl} \searrow 0$.

(i) If x *is a Pringsheim limit point of* x_{kl}, *then* x *is a minimizer of* f *and*

$$\min_{y\in X} f(y) = f(x) = \limsup_{k,l\to\infty} f_{kl}(x_{kl})$$

(ii) $x_{kl} \to x$, *then*

$$\min_{y\in X} f(y) = f(x) = \lim_{k,l\to\infty} f_{kl}(x_{kl})$$

Proof. First, we will prove (i). Suppose that x is a Pringsheim limit point of x_{kl}. Then $x \in \limsup_{k,l\to\infty}(\varepsilon_{kl} - \text{argmin} f_{kl})$. So (4.7) holds. Therefore x is a minimizer of f and since $x_{kl} \in (\varepsilon_{kl} - \text{argmin} f_{kl})$, we have

$$\limsup_{k,l\to\infty} f_{kl}(x_{kl}) = \limsup_{k,l\to\infty} \inf_{y\in X} f_{kl}(y).$$

From (4.8), we have

$$f(x) = \min_{y \in X} f(y) = \limsup_{k,l\to\infty} f_{kl}(x_{kl}).$$

Now, we will prove (ii). Assume that $x_{kl} \to x$. Then $x \in \liminf_{k,l\to\infty}(\varepsilon_{kl} - \mathrm{argmin} f_{kl})$. So (4.10) holds. By $x_{kl} \in (\varepsilon_{kl} - \mathrm{argmin} f_{kl})$ and (4.11), we have

$$\lim_{k,l\to\infty} f_{kl}(x_{kl}) = \lim_{k,l\to\infty} \inf_{y\in X} f_{kl}(y) = f(x) = \min_{y\in X} f(y),$$

which completes the proof. □

In the next theorem, we will give a necessary and sufficient condition for the convergence of minimum values.

Theorem 4.5. *Assume that* $\Gamma - \lim_{k,l\to\infty} f_{kl} = f$ *and* $\mathrm{argmin} f \neq \emptyset$. *Then*

$$\bigcap_{\varepsilon>0} \liminf_{k,l\to\infty}(\varepsilon - \mathrm{argmin} f_{kl}) = \mathrm{argmin} f. \tag{4.13}$$

if and only if (4.11) *holds.*

Proof. Assume that (4.13) holds. Since $\mathrm{argmin} f \neq \emptyset$, (4.10) holds. By Theorem 4.4, we have (4.11).

Now assume that (4.11) is satisfied. Let $x \in \mathrm{argmin} f$. Then $f(x) = \min_{y\in X} f(y)$ and

$$f(x) = \lim_{k,l\to\infty} \inf_{y\in X} f_{kl}(y).$$

Given $\varepsilon > 0$. Then there exists $n_0 \in \mathbb{N}$ such that for every $k, l \geq n_0$ we have

$$f(x) - \frac{\varepsilon}{2} < \inf_{y\in X} f_{kl}(y).$$

On the other hand, since $\Gamma - \lim_{k,l\to\infty} f_{kl} = f$, for each $U \in \mathcal{N}(x)$

$$\limsup_{k,l\to\infty} \inf_{y\in U} f_{kl}(y) \leq f(x).$$

Thus, there exists $n_1 \in \mathbb{N}$ such that for every $k, l \geq n_1$ we have

$$\inf_{y\in U} f_{kl}(y) < f(x) + \frac{\varepsilon}{2}.$$

Let $n_2 = \max\{n_0, n_1\}$. Then for every $k, l \geq n_2$ we obtain

$$\inf_{y\in U} f_{kl}(y) < \inf_{y\in X} f_{kl}(y) + \varepsilon.$$

This implies that $U \cap (\varepsilon - \text{argmin} f_{kl}) \neq \emptyset$. It follows that $x \in \liminf_{k,l\to\infty}(\varepsilon - \text{argmin} f_{kl})$. Since ε is arbitrary,

$$x \in \bigcap_{\varepsilon>0} \liminf_{k,l\to\infty}(\varepsilon - \text{argmin} f_{kl}).$$

Applying this result together with Theorem 4.3 we obtain

$$\begin{aligned} \text{argmin} f &\subseteq \bigcap_{\varepsilon>0} \liminf_{k,l\to\infty}(\varepsilon - \text{argmin} f_{kl}) \\ &\subseteq \bigcap_{\varepsilon>0} \limsup_{k,l\to\infty}(\varepsilon - \text{argmin} f_{kl}) \\ &\subseteq \text{argmin} f \end{aligned}$$

which completes the proof. □

5. Conclusion

In the present paper, we defined the concept of Γ-convergence of a double sequence of functions. First, we showed that Γ-convergence and pointwise convergence of a double sequence (f_{kl}) of functions are independent, and proved uniform convergence of a double sequence of functions implies Γ-convergence. Later, we proved Γ-convergence of a double sequence of functions coincides with Kuratowski convergence of their epigraphs. Finally, we gave some conditions ensuring that the Γ-convergence of a double sequence (f_{kl}) to a function f implies the convergence of the minimum values of f_{kl} to the minimum value of f and proved if the double sequence (f_{kl}) Γ converges to the function f, then the limit of a double sequence of points selected from minimum points of f_{kl} is a minimum point of f

In [6, 9] the notion of Γ-limit is extended from the case of functions with values in $\mathbb{R}$ to the case of those with values in an arbitrary complete lattice. So, the definition and results in this paper can be also given for double sequences of functions with values in an arbitrary complete lattice in this framework.

REFERENCES

1. J. P. Aubin and H. Frankowska: *Set-Valued Analysis*. Birkhauser, Boston, 1990.
2. G. Beer: *Topologies on Closed and Closed Convex Sets*. Kluwer Academic, Dordrecht, 1993.
3. C. Belen, M. Mursaleen and M. Yildirim: *Statistical A-summability of double sequences and a Korovkin type approximation theorem*. Bull. Korean Math. Soc. **49(4)** (2012), 851–861.
4. J. Boos, T. Leiger and K. Zeller: *Consistency theory for SM-methods*, *Acta Math. Hungar.*, **76** (1997), 83–116.
5. A. Braides: *Γ-convergence for Beginners*. Oxford University Press, Oxford, 2002.

6. G. Buttazzo: *Su una definizione generale dei* Γ*-limiti*. Boll. Un. Mat. Ital. **5(14-B)** (1977), 722–744.
7. G. Buttazzo and G. Dal Maso: Γ*-convergence and optimal control problems*. J. Optim. Theory Appl. **38** (1982), 385–407.
8. E. De Giorgi and T. Franzoni: *Su un tipo di convergenza variazionale*. Atti Accad. Naz. Lincei Rend. Cl. Sci. Mat. **8(58)** (1975), 842–850.
9. E. De Giorgi: Γ*-convergenza e G-convergenza*, Bollettino dell' Unione Matematica Italiana. **14-A** (1977), 213–220.
10. F. Dirik and K. Demirci: *Korovkin-type approximation theorem for functions of two variables in statistical sense*. Turkish J. Math. **34(1)** (2010), 73–83.
11. A. L. Dontchev and R. T. Rockafellar: *Implicit Functions and Solution Mappings, A view from variational analysis*. Springer, 2009.
12. M. Focardi: Γ*-convergence: a tool to investigate physical phenomena across scales*. Math. Methods Appl. Sci. **35(14)** (2012), 1613–1658.
13. G. H. Hardy: *On the convergence of certain multiple series*, *Proc. Cambridge Philos. Soc.*, **19** (1917), 86–95.
14. C. Kuratowski: *Topologie*, vol.I, PWN, Warszawa, 1958.
15. G. Dal Maso : *An Introduction to* Γ*-convergence*. Boston, 1993.
16. F. Moricz and B. E. Rohedes: *Some characterization almost convergence for single sequence and double sequences*, Publications De L'Institut Mathematique. **48(60)** (1990), 61–68.
17. R. F. Patterson: *Analogues of some fundamental theorems of summability theory. Internet*. J. Math. Math. Sci. **23** (2000), 1–9.
18. R. F. Patterson: *Double sequence core teorems*, Int. J. Math. Math. Sci. **22(4)** (1999), 785–793.
19. A. Pringsheim: *Elementare Theorie der unendliche Doppelreihen. Sitsungs berichte der Math*. Akad. der Wissenschafftenzu Münch. Ber. **7** (1898), 101–153.
20. R. T. Rockafellar and A. J. -B. Wets : *Variational Analysis*. Springer, Berlin, 1998.
21. D. Stancu: *A method for obtaining polynomials of Bernstein type of two variables*. Amer. Math. Monthly. **70(3)** (1963), 260–264.
22. Y. Sever and E. Dündar: *On convergence of double sequences of functions*. Electronic Journal of Mathematical Analysis and Applications. **2(2)** (2014), 67–72.
23. Y. Sever, Ö. Talo and B. Altay: *On convergence of double sequences of closed sets*. Contemporary Analysis and Applied Mathematics. **3(1)** (2015), 30–49.
24. R. A. Wijsman: *Convergence of sequences of convex sets, cones and functions*. Bull. Amer. Math. Soc. **70** (1964), 186–188.
25. R. A. Wijsman: *Convergence of sequences of convex sets, cones and functions II*. Trans. Amer. Math. Soc. **123(1)** (1966), 32–45.
26. V. I. Volkov: *On the convergence of sequences of linear positive operators in the space of continuous functions of two variables*. Dokl. Akad. Nauk SSSR (N.S.) **115** (1957), 17–19.
27. A. J. -B. Wets: *Convergence of Convex Functions, Variational Inequalities and Convex Optimization Problems*. New York, 1980.