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ON THE EXISTENCE AND EXAMPLES OF HOMOGENEOUS GEODESICS IN GENERALIZED m-KROPINA SPACE

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Abstract. In this paper, we find a necessary and sufficient condition for a non-zero vector to be a geodesic vector in homogeneous generalized m-Kropina space. Further, we prove the existence of at least one homogeneous geodesic. However, it is conjectured that the outcomes and proofs in the case of Finsler geometry are not ideal, since gen-eral Finsler metrics are non-reversible. In Finsler geometry, the trajectory of unique homogeneous geodesic should be regarded as two geodesics with initial vectors X and $-X$. Hence, we construct an $(n+1)$-dimensional and a 4-dimensional space to find homogeneous geodesics explicitly.
Keywords: generalized m-Kropina space, Finsler geometry, homogeneous geodesic.

1. Introduction

A geodesic can be thought of literally as a curve that reduces the distance between two places. Homogeneous geodesics have gained attention in both Riemannian and Finsler geometry recently. A geodesic $\gamma(t) : \mathbb{R} \to M$ in a Finsler manifold

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(M, F) is said to be homogeneous geodesic, if there exists one-parameter group of isometries $\phi : \mathbb{R} \times M \to M$ such that

$$\gamma(t) = \phi(t, \gamma(0)), t \in \mathbb{R}.$$

Geodesics are treated similar to relative equilibria in mechanics and physics. The qualitative description of the behavior of the related mechanical system with symmetries depends on the description of such relative equilibria. Geodesics have always been exciting to find and study, and this has been true since geometry's inception. Due to the numerous uses of geodesics and homogeneous geodesics in physics [22, 5, 6, 23] and other mathematics disciplines, there has been an interest in their study recently.

There is a lot of literature in mechanics devoted to the investigation of relative equilibria. In [1], author extended Euler's theory of rigid-motions while studying left invariant Riemannian metrics on Lie groups. In [20], the author discussed that in homogeneous space with an invariant metric, geodesic flow can be seen as framework of Smale's mechanical system with symmetries. T$\acute{o}$th [26] studied the paths that were orbits of one-parameter symmetry group G. In fact, he discovered the conditions for solutions of Euler-Lagrange or Hamiltonian equations to coincide with the orbit of a one-parameter subgroup of a symmetry group.

Kajzer has studied the existence of homogeneous geodesics in [14]. In this study, the authors showed that in Lie groups with left invariant metrics, at least one homogeneous geodesic element can travel through the identity element. Kowalski and Szenthe [17] also showed that every homogeneous Riemannian manifold has at least one homogeneous geodesic across each point.

Additionally, Kowalski and Vl$\acute{a}\check{s}$ek [18] established a few examples of homogeneous Riemannian manifolds of any dim $n \geq 4$ with precisely one homogeneous geodesic. Latiffi [21] proposed the term 'geodesic vector' in homogeneous Finsler space and proved that any vector in every connected Lie group with a bi-invariant Finsler metric is a geodesic vector.

Recently, the existence of homogeneous geodesic for infinite series metric and exponential metric have been discussed in [15]. Also, some important results related to homogeneous Finsler spaces have been established in [25]. In homogeneous Kropina spaces, the existence of homogeneous geodesic through any arbitrary point have been discussed in [13] and it is also proved that under some conditions result holds for any (α, β)-homogeneous space. In this paper, homogeneous geodesics of 3-dimensional non-unimodular real Lie groups equipped with a left invariant Randers metric of Douglas type are also discussed as an example. In [10], author has showed the examples of homogeneous Randers manifold admitting just two homogeneous geodesic. In [2], authors have extended the study of left-invariant (α, β)-metrics on 4-dimensional Lie groups.

2. Preliminaries

In this section, we discuss basic definitions and notations of Finsler geometry. For more elaborate concepts of Finsler geometry and homogeneous Finsler geometry, refer [3, 4, 7]. Let V be an n-dimensional real vector space endowed with smooth norm F on $V\backslash\{0\}$, which is non-negative i.e., $F(u) \geq 0 \ \forall\ u \ \in V$, positively homogeneous i.e., $F(\lambda u) = \lambda F(u) \ \forall \ \lambda > 0$, and strongly convex i.e., if $\{u_1, u_2, ..., u_n\}$ be the basis of V such that $y = y^1 u_1 + y^2 u_2 + ... + y^n u_n$, then the Hessian matrix $(g_{ij}) := \left(\left[\frac{1}{2}F^2\right]_{y^i y^j}\right)$, is positive definite at every point of $V\backslash\{0\}$. The pair (V, F) is called Minkowski space and F is called Minkowski norm.
Let M be a connected (smooth) manifold. A Finsler metric on M is a function $F : TM \to [0, \infty)$ which satisfies:

1. F is smooth on slit tangent bundle $TM\backslash\{0\}$,
2. The restriction of F to any $T_xM, x \in M$ is a Minkowski norm.

The space (M, F) is called Finsler space. Let $\gamma : [0, 1] \to M$ be a C^1-curve. Then Finsler length $L(\gamma)$ of γ is defined as

$$L(\gamma) = \int_0^1 F(\gamma(t), \gamma^{'}(t))dt.$$

Further, Finsler distance $d_F(p, q)$ between two points $p, q \in M$ is defined as

$$d_F(p, q) = inf_\gamma L(\gamma),$$

where infimum is taken over all piecewise C^1-curves joining p and q.

Definition 2.1. Let $F = \alpha\phi(s); s = \beta/\alpha$, where ϕ is a smooth function on an open interval $(-b_0, b_0)$, $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric, $\beta = b_i(x)y^i$ is a 1-form on an n-dimensional manifold with $||\beta|| < b_0$. Then, F is Finsler metric if and only if following conditions are satisfied:

$$\phi(s) > 0, \quad \phi(s) - s\phi^{'}(s) + (b^2 - s^2)\phi^{''}(s) > 0 \ \ \forall \ |s| \leq b < b_0. \tag{2.1}$$

An (α, β)-metric is said to be singular Finsler metric, if either $\phi(0)$ is not defined or $\phi(s)$ does not satisfy 2.1. In this paper, we study generalized m-Kropina spaces, which form a special class of (α, β)-metric. Kropina metric is a type of non-regular (α, β)-metric where $\phi(s) = \frac{1}{s}$, i.e., $F = \frac{\alpha^2}{\beta}$. The concept is proposed by Russian physicist V. K. Kropina [19]. Despite having singularities ($\beta = 0$), it is useful in the Lagrangian function's representation of the general dynamic system. Hence, due to the physical and applied importance of Kropina metric, we here investigate geodesics for generalized m-Kropina metric. Generalized m-Kropina metric is an important class of (α, β)-metric defined as

$$F(\alpha,\beta) = \frac{\alpha^{m+1}(x,y)}{\beta^m(x,y)},\ (m \neq 0,1).$$

Consider the inner product $\langle , \rangle$ on tangent space $T_xM,\ x \in M$ defined as

$$\langle u, v\rangle = a_{ij}u^i v^j,\ u, v \in T_xM,$$

where a_{ij} is a Riemannian metric.
Using the above defined inner product we induce an inner product on the cotangent space, T_x^*M, of M at x,

$$\langle dx^i, dx^j\rangle = a^{ij}.$$

Using this inner product, a linear isomorphism can be defined between T_xM and T_x^*M [9]. Hence, 1-form β corresponds to smooth vector field X on M given by

$$X|_x = b^i \frac{\partial}{\partial x^i},\ b^i = a^{ij}b_j,$$

which further implies

$$\langle X|_x, y\rangle = \langle b^i \frac{\partial}{\partial x^i}, y^j \frac{\partial}{\partial x^j}\rangle = b^i y^j a_{ij} = b_j y^j = \beta(y).$$

Also, $||\beta|| = \alpha(X|_x) < 1$. On the basis of above discussion, w can conclude the following Lemma:

Lemma 2.1. *Let (M, α) be a Riemannian space. Then the generalized m-Kropina space, (M, F) where $F = \frac{\alpha^{m+1}}{\beta^m}$, $(m \neq -1, 0, 1)$ $\beta = b_i y^i$, a 1-form with $||\beta|| = \sqrt{b_i b^i}$, consists of Riemannian metric α along with a smooth vector field X on M, which satisfies $\alpha(X|_x) < 1\ \forall\ x \in M$, i.e.,*

$$F(x,y) = \frac{\alpha(x,y)^{m+1}}{\langle X|_x, y\rangle^m},$$

where $\langle , \rangle$ is the inner product on T_xM induced by the Riemannian metric α.

Let (M, F) be a Finsler space. A diffeomorphism of M onto itself is said to be isometry, if it preserves the Finsler function, i.e., $F(\phi(p), d\phi_p(X)) = F(p, X)$ for any $p \in M$ and $X \in T_pM$. Let G be a Lie group and M a smooth manifold. If G has smooth action on M, then G is called Lie transformation group of M. A connected Finsler space (M, F) is said to be homogeneous Finsler space, if action of group of isometries of (M, F), denoted by $I(M, F)$ is transitive on M.
Let $G \subset I(M, F)$ be a connected Lie group acting transitively on Finsler space (M, F), and at a fixed point $p \in M$, let H be its isotropy group. Then M can be written as coset space G/H, with a G-invariant Finsler metric F. It is evident to

see that H is comapct, since action of H leaves invariant unit sphere in T_pM. Hence, we obtain reductive decomposition of $\mathfrak{g}$, Lie algebra of G as

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m},$$

where $\mathfrak{g}$ and $\mathfrak{h}$ are Lie algebras of G and H respectively and $\mathfrak{m} \subset \mathfrak{g}$ is a vector subspace such that $Ad(\mathfrak{h})(\mathfrak{m}) \subset \mathfrak{m}$, where Ad denotes Adjoint representation of G.

Remark 2.1. [7] A homogeneous Finsler manifold $M = G/H$ is reductive homogeneous space.

Next proposition shows that G-invariant Finsler metrics on G/H can be identified with Minkowski norm F as follows:

Proposition 2.1. *[8] Let G/H be a reductive homogeneous manifold satisfying*

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m}.$$

Then there exists a one-to-one correspondence between the G-invariant Finsler metrics on G/H and the Minkowski norms F on $\mathfrak{m}$ which satisfy

$$F(Ad(h)x) = F(x),\ \forall\ h \in H, x \in \mathfrak{m}.$$

A regular smooth curve γ with velocity vector $T = \dot{\gamma}$, is said to be Finslerian geodesic, if it satisfies

$$D_T\left(\frac{T}{F(T)}\right) = 0,$$

with reference vector T. Here, D is defined from Chern connection, which is torsion free and almost metric compatible. A geodesic $\gamma(t)$ passing through origin $eH \in M = G/H$ is said to be homogeneous if it is one-parameter subgroup of G, i.e., $\gamma(t) = exp(tZ)(eH)$, $t \in \mathbb{R}$ and Z is a non zero vector in Lie algbera of G. A non-zero vector $X \in \mathfrak{g}$ is said to be a geodesic vector, if the curve $exp(tX)(eH)$ is constant speed geodesic of (M, F). If all the geodesics of a Riemannian manifold M are homogeneous, then M is callled g.o.(geodesic orbit) space.
A Finsler space (M, F) is called a Finsler g.o. space, if every geodesic of (M, F) is the orbit of a one-parameter subgroup of $G = I(M, F)$, i.e., if $\phi : \mathbb{R} \to M$ is a geodesic, then there exists a non-zero vector $Z \in \mathfrak{g} = Lie(G)$ and $p \in M$ such that $\phi(t) = exp(tZ).p$.
More precisely, a Finsler space (M, F) is called Finsler g.o.(geodesic orbit) space, if and only if the projections of all the geodesic vectors cover the set $T_{eH}(G/H) - \{0\}$.
A Finsler g.o. space has vanishing S-curvature for Busemann volume form [21, 7].
Further, every Finsler g.o. space is homogeneous [7].
The following result provides criterion to study geodesic vector in Lie algbera level and hence provide a useful tool to study homogeneous geodesic.

Lemma 2.2. *[21] Suppose $(G/H, F)$ is a homogeneous Finsler space with a reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$. A non-zero vector $Y \in \mathfrak{g}$ is a geodesic vector if and only if it satisfies*

$$g_{Y_{\mathfrak{m}}}(Y_{\mathfrak{m}}, [Y, Z]_{\mathfrak{m}}) = 0, \ \forall Z \in \mathfrak{g},$$

where the subscript $\mathfrak{m}$ denotes the projection of a vector from $\mathfrak{g}$ to $\mathfrak{m}$.

3. Necessary and sufficient condition

In this section, we discuss homogeneous geodesic in homogeneous generalized m-Kropina space. We provide some necessary and sufficient condition for a non-zero vector to be geodesic vector in homogeneous generalized m-Kropina space.

Corollary 3.1. *Let $(G/H, F)$ be a homogeneous Finsler space equipped with generalized m- Kropina metric arising from an invariant Riemannian metric $\langle , \rangle$ and an invariant vector field $\tilde{X}$, such that $X = \tilde{X}(H)$. Then necessary and sufficient condition for a non-zero vector $Y \in \mathfrak{g}$ to be a geodesic vector is*

(3.1)
$$\frac{\langle Y_{\mathfrak{m}}, Y_{\mathfrak{m}} \rangle^m}{\langle X, Y_{\mathfrak{m}} \rangle^{2m+1}} \left[(m+1) \langle X, Y_{\mathfrak{m}} \rangle \langle Y_{\mathfrak{m}}, [Y, Z]_{\mathfrak{m}} \rangle - m \langle Y_{\mathfrak{m}}, Y_{\mathfrak{m}} \rangle \langle X, [Y, Z]_{\mathfrak{m}} \rangle \right] = 0, \quad \forall Z \in \mathfrak{m}.$$

Proof. Using the formula (2.7) for (α, β)-metric from [24], we get the following corollary directly by taking $\phi(s) = \frac{1}{s^m}$. $\square$

Further, we use Theorem 2.2 of [24] to get the following remark:

Remark 3.1. Let $(G/H, F)$ be a homogeneous generalized m-Kropina space with assumptions same as taken in Theorem 3.1. Then the vector X is a geodesic vector of $(G/H, \langle , \rangle)$ if and only if it is a geodesic vector of $(G/H, F)$. In other words, a non zero vector is a geodesic vector of generalized m-Kropina metric if and only if it is a geodesic of its base Riemannian metric.

Also, as direct consequence of Corollary 3.1, we can conclude the following corollary:

Corollary 3.2. *Let $(G/H, F)$ be a homogeneous generalized m-Kropina space with assumptions same as taken in Theorem 3.1. Let $Y \in \mathfrak{g}$ be a non-zero vector such that $\langle X, [Y, Z]_{\mathfrak{m}} \rangle = 0 \ \forall \ Z \in \mathfrak{m}$. Then Y is a geodesic vector of $(G/H, \langle , \rangle)$ if and only if it is a geodesic vector of $(G/H, F)$.*

4. Existence

In this section, we prove the existence of atleast one homogeneous geodesic on homogeneous generalized m-Kropina space passing through origin.

Proposition 4.1. *Let $(G/H, F)$ be homogeneous generalized m-Kropina space. Then there exists atleast one homogeneous geodesic arising from each origin.*

Proof. Suppose $G \subset I(M, F)$ be connected Lie group acting transitively on (M, F). Let H be isotropy group at $\{eH\} \in G/H$. Let $\mathcal{K}$ be killing form and $rad\mathcal{K}$ be its null space.
Firstly, let us suppose $rad\mathcal{K} = \mathfrak{m}$. In [17], it is proved that Lie algebra $\mathfrak{g}$ has reductive decomposition $\mathfrak{m} + \mathfrak{h}$ such that $\mathfrak{m}$-projection $[\mathfrak{g}, \mathfrak{g}]$ is a proper subspace of $\mathfrak{m}$. Consider $Y \in [\mathfrak{g}, \mathfrak{g}]_{\mathfrak{m}}$ be a non-zero vector which satisfies $\langle Y, Y\rangle = 1$. Let $W = X \in [\mathfrak{g}, \mathfrak{g}]_{\mathfrak{m}}^{\perp}$. We use Theorem 3.1 to check that W is a geodesic vector. Since, equation 3.1 with respect to W can be written as:

$$\frac{\langle W_{\mathfrak{m}}, W_{\mathfrak{m}}\rangle^m}{\langle X, W_{\mathfrak{m}}\rangle^{2m+1}}\left[\langle X, W_{\mathfrak{m}}\rangle\langle (m+1)W_{\mathfrak{m}}, [W, Z]_{\mathfrak{m}}\rangle - \langle W_{\mathfrak{m}}, W_{\mathfrak{m}}\rangle\langle mX, [W, Z]_{\mathfrak{m}}\rangle\right] = 0,$$

which implies that

$$\frac{\langle W_{\mathfrak{m}}, W_{\mathfrak{m}}\rangle^m}{\langle X, W_{\mathfrak{m}}\rangle^{2m+1}}\left[\langle X, [W, Z]_{\mathfrak{m}}\rangle\right] = 0.$$

This proves the existence of atleast one geodesic through origin.
Secondly, we suppose $rad\mathcal{K} \subsetneq \mathfrak{m}$. If $rad\mathcal{K}$ is a proper subset of $\mathfrak{m}$, then from [27], it is proved that $\mathfrak{m}$ can be decomposed into eigenspaces as $\mathfrak{m} = V_0 + V_1 + ...V_r$ with respect to $\mathcal{K}$-symmetric endomorphism defined as $K(X, Y) = \langle\theta(X), Y\rangle$ which satisfies $V_0 = radK_0$. Consider $\{f_1, f_2, f_3, ..., f_r\}$ be an orthonormal basis of $V = V_0 + V_1 + ... + V_r$ and θ be an endomorphism $\theta(f_i) = \lambda_i f_i$ for $i = 1, 2, ..., r$. Suppose that $X = X_0 + \sum_{i=1}^r x_i f_i, Y = Y_0 + \sum_{i=1}^r y_i f_i, \quad X_0, Y_0 \in V_0, \quad x_i, y_i \in \mathbb{R}$. Using Theorem 3.1, $Y \in \mathfrak{g}$ is a geodesic vector if and only if equation 3.1 equals to zero.

Hence, let us consider

$$\begin{aligned}
&[(m+1)\langle X, Y_{\mathfrak{m}}\rangle\langle Y_{\mathfrak{m}}, [Y, Z]_{\mathfrak{m}}\rangle - m\langle Y_{\mathfrak{m}}, Y_{\mathfrak{m}}\rangle\langle X, [Y, Z]_{\mathfrak{m}}\rangle] \\
&= [(m+1)\langle X_0 + \sum_{i=1}^r x_i f_i, Y_0 + \sum_{i=1}^r y_i f_i\rangle\langle Y_0 + \sum_{i=1}^r y_i f_i, [Y, Z]_{\mathfrak{m}}\rangle \\
&\quad - m\langle Y_0 + \sum_{i=1}^r y_i f_i, Y_0 + \sum_{i=1}^r y_i f_i\rangle\langle X_0 + \sum_{i=1}^r x_i f_i, [Y, Z]_{\mathfrak{m}}\rangle] \\
&= [(m+1)\langle X_0 + \sum_{i=1}^r x_i f_i, Y_0 + \sum_{i=1}^r y_i f_i\rangle\langle Y_0, [Y, Z]_{\mathfrak{m}}\rangle \\
&\quad - m\langle Y_0 + \sum_{i=1}^r y_i f_i, Y_0 + \sum_{i=1}^r y_i f_i\rangle\langle X_0, [Y, Z]_{\mathfrak{m}}\rangle] \\
&\quad + (m+1)\langle X_0 + \sum_{i=1}^r x_i f_i, Y_0 + \sum_{i=1}^r y_i f_i\rangle K\left([Y, Z]_{\mathfrak{m}}, \sum_{i=1}^r y_i \frac{f_i}{\lambda_i}\right) \\
&\quad - m\langle Y_0 + \sum_{i=1}^r y_i f_i, Y_0 + \sum_{i=1}^r y_i f_i K\left([Y, Z]_{\mathfrak{m}}, \sum_{i=1}^r x_i f_i\right)\rangle
\end{aligned} \tag{4.1}$$

(4.2)
$$= (m+1)\left[\langle X_0 + \sum_{i=1}^{r} x_i f_i, Y_0 + \sum_{i=1}^{r} y_i f_i\rangle\right]\left[\langle Y_0, [Y,Z]_{\mathfrak{m}}\rangle + K\left([Y,Z]_{\mathfrak{m}}, \sum_{i=1}^{r} y_i \frac{f_i}{\lambda_i}\right)\right]$$
$$- m\left[\langle Y_0 + \sum_{i=1}^{r} y_i f_i, Y_0 + \sum_{i=1}^{r} y_i f_i\rangle][\langle X_0, [Y,Z]_{\mathfrak{m}}\rangle + K\left([Y,Z]_{\mathfrak{m}}, \sum_{i=1}^{r} x_i \frac{f_i}{\lambda_i}\right)\right]$$
$$= (m+1)[\langle X_0 + \sum_{i=1}^{r} x_i f_i, Y_0 + \sum_{i=1}^{r} y_i f_i\rangle]K(Z, [Y,Y]_{\mathfrak{m}})$$
$$- m\langle Y_0 + \sum_{i=1}^{r} y_i f_i, Y_0 + \sum_{i=1}^{r} y_i f_i\rangle K(Z, X_0 + \sum_{i=1}^{r} x_i\lambda_i f_i, Y_0 + \sum_{i=1}^{r} y_i\lambda_i f_i).$$

The first term in last resultant of above equation 4.2 vanishes, which on plugging into equation 3.1, we get

$$m\frac{\langle Y,Y\rangle^{m+1}}{\langle X,Y\rangle^{2m+1}}\left[K(Z, [x_0 + \sum_{i=1}^{r} x_i\lambda_i y_i], y_0 + \sum_{i=1}^{r} y_i\lambda_i f_i)\right]. \tag{4.3}$$

Above equation vanishes, whenever we have a solution in the form $(Y_0, y_1, ..., y_r, t)$. It is obvious to check that $\{Y_0 = X_0, y_1 = t_0 x_1, ..., y_r = t_0 x_r, t = t_0\}$ is a solution to satisfy above equation. This completes the proof. □

In fact, in [11] author has showed existence of two homogeneous geodesics in any arbitrary homogeneous Finsler spaces. Hence, in particular, above proposition can be extended to say that there exists two homogeneous geodesics in this space. With this motivation in the next section, we construct an $(n+1)$-dimensional and 4-dimensional example and find homogeneous geodesics.

5. Examples of some homogeneous geodesic vectors

In this section, we visualize the homogeneous geodesics in an $(n+1)$-dimensional space and a 4-dimensional space. Let us consider a Lie algebra $\mathfrak{n}$ with orthonormal basis $\mathfrak{B} = \{e_1, e_2, ..., e_{n+1}\}$ generated by Lie brackets as follows:

$$[e_i, e_j] = 0, \quad \forall i, j \leq n$$
$$[e_{n+1}, e_i] = a_i e_i + e_{i+1}, \quad \forall i < n$$
$$[e_{n+1}, e_n] = a_n e_n$$

for arbitrary non-zero parameters $a_1, a_2, ..., a_n \in \mathbb{R}$. The family of Lie algebras $(\mathfrak{n}, \langle,\rangle)$ generates an (n-parameter) solvable Lie groups $\mathcal{N}$ with a set of invariant Riemannian metrics. In [18], authors showed that for generic choices of $\{a_i\}_{i=1}^{n}$ the corresponding group $\mathcal{N}$ acting by left translations is the maximal group of isometries. In [18] authors have assumed that $\mathcal{N}$ is diffeomorphic to $(n+1)$-dimensional

Euclidean space. We use a similar approach as in [10] to solve our further result. For the sake of simplicity, we shall consider metric F generated by the vector $X = ke_1, 0 < k < 1$. which are suitable for our purpose.

Example 5.1. Let (G, F) be an $(n+1)$-dimensional homogeneous generalized m-Kropina space, such that the parameters constructed above satisfies $min\{a_i\} > n$ and left-invariant metric F is determined by $X = ke_1$ and also $ka_1 < 1$. Then (G, F) admits exactly two geodesics whose initial vectors are $\tau_1 = c_1 e_{n+1} + \frac{m}{m+1} kF(Y_\mathfrak{m})e_1$, and $\tau_2 = -c_1 e_{n+1} + \frac{m}{m+1} kF(Y_\mathfrak{m})e_1$.

An arbitrary vector $Y \in \mathfrak{g}$ can be expressed with respect to the basis $\mathfrak{B} = \{e_1, e_2, ..., e_{n+1}\}$ as $Y = y_1e_1 + y_2e_2 + ... + y_{n+1}e_{n+1}$. The Lie brackets can be calculated as follows:

$$[Y, e_i] = y_{n+1}(a_i e_i + e_{i+1}), \quad 1 \leq i < n,$$

$$[Y, e_n] = y_{n+1} a_n e_n,$$

$$[Y, e_{n+1}] = -y_1 a_1 e_1 - \sum_{i=2}^{n} (y_{i-1} + y_i a_i) e_i.$$

Next, we plug the vector $Z \in \mathfrak{m}$ in equation 3.1 step by step for all elements of orthonormal basis $\mathfrak{B}$. Using Theorem 3.1 we get the $Y \in \mathfrak{g}$ is geodesic vector, if it satisfies the following homogeneous system of equations:

$$(m+1)[y_{n+1}(a_1 y_1 + y_2) - mF(Y_\mathfrak{m})ka_1] = 0,$$
$$(m+1)[y_{n+1}(a_i y_i + y_{i+1})] = 0, \quad 1 < i < n$$
$$(m+1) y_{n+1} a_n y_n = 0,$$
$$(m+1)[-y_1^2 a_1 - \sum_{i=2}^{n} (y_{i-1} + y_i a_i) y_i] - mF(Y_\mathfrak{m})ky_1a_1 = 0.$$

In order to solve system of equations, first let us consider the case if $y_{n+1} \neq 0$. Due to homogeneity of equations, without loss of generality we may assume $y_{n+1} = \pm\, c$. Consequently, from all equations for $i = 1, ..., n$ we immediately get $y_n = y_{n-1} = ... = y_2 = 0$ and $y_1 = \left(\frac{m}{m+1}\right) kF(Y_\mathfrak{m})$. Hence, we obtain just two geodesics solutions for above system of equations.
Next, let us consider second case that $y_{n+1} = 0$, first n equations are satisfied immediately. For the last equation, we solve for polynomial $p(y_i) = 0$, where

$$p(y_i) = (m+1)y_1^2 a_1 + (m+1)\sum_{i=2}^{n} y_i y_{i-1} + \sum_{i=2}^{n} y_i^2 a_i + \sum_{i=2}^{n} y_i^2 a_i + mka_1 y_1 F(Y_\mathfrak{m}).$$

On using the estimates $|y_i y_{i+1}| < 1$ and $\min a_i > n$, we get that $p(y_i) > 0$, which implies above system of equation doesn't have any other non trivial solution. This completes the proof.

Example 5.2. Consider a 4-dimensional (R^4, F) equipped with m-Kropina metric, which can be written as homogeneous space G/H where G is the 5-dimensional group of equiaffine transformations of a Euclidean space and H is group of rotations around origin. Also $\mathfrak{g}$

has reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, an orthonormal basis (e_1, e_2, e_3, e_4) of $\mathfrak{m}$ and generarator Λ of $\mathfrak{h}$. Using the multiplication table from [16], we have

$$[e_1, e_2] = 0, [e_1, e_3] = -e_1, [e_1, e_4] = e_1,$$

$$[e_2, e_3] = e_2, [e_2, e_4] = e_1, [e_3, e_4] = -2\Lambda,$$

$$[\Lambda, e_1] = -e_2, [\Lambda, e_2] = e_1, [\Lambda, e_3] = 2e_4, [\Lambda, e_4] = -2e_3.$$

Also, we have $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. Suppose $y \in \mathfrak{g}$ be geodesic vector,

$$y = y_1e_1 + y_2e_2 + y_3e_3 + y_4e_4 + q\Lambda$$

Using equation 3.1, we get the following set of equations:

$$\begin{aligned}(m+1)(x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4)(y_1(y_3 - y_4) - y_2q)\\ - m(y_1^2 + y_2^2 + y_3^2 + y_4^2)(x_1(y_3 - y_4) - x_2q) = 0,\end{aligned} \tag{5.1}$$

$$\begin{aligned}(m+1)(x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4)(y_1(y_3 - y_4) - y_2q)\\ - m(y_1^2 + y_2^2 + y_3^2 + y_4^2)(x_1(q - y_4) - x_2y_3) = 0,\end{aligned} \tag{5.2}$$

$$\begin{aligned}(m+1)(x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4)(-y_1^2 + y_2^2 + 2qy_4)\\ - m(y_1^2 + y_2^2 + y_3^2 + y_4^2)(-x_1y_1 + x_2y_2 + 2qx_4) = 0,\end{aligned} \tag{5.3}$$

$$\begin{aligned}(m+1)(x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4)(y_1(y_1 + y_2) - 2qy_3)\\ - m(y_1^2 + y_2^2 + y_3^2 + y_4^2)(x_1(y_1 + y_2) - 2qx_3) = 0.\end{aligned} \tag{5.4}$$

Using above equations, we also get

$$\begin{aligned}(m+1)(x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4)(y_1 + y_2)(y_3 - q)\\ - m(y_1^2 + y_2^2 + y_3^2 + y_4^2)(x_1 + x_2)(y_3 - q) = 0,\end{aligned} \tag{5.5}$$

$$\begin{aligned}(m+1)(x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4)(y_2(y_1 + y_2) + 2q(y_4 - y_3))\\ - m(y_1^2 + y_2^2 + y_3^2 + y_4^2)(y_2(x_1 + x_2) + 2q(x_4 - x_3)) = 0.\end{aligned} \tag{5.6}$$

We consider some assumptions to see geodesic vectors explicitly:
A: $X = x_3e_3$
B: $X = x_4e_4$
C: $X = x_3(e_3 + e_4)$
D: $X = x_1(e_1 - e_2)$
A : For $X = x_3e_3$, equation 5.5 gives

$$(m+1)(x_3y_3)(y_1 + y_2)(y_3 - q) = 0.$$

(1) Let us suppose $y_1 = -y_2$, $y_3 \neq 0$ and $y_3 \neq q$. In this case equation 5.3, implies $2(m+1)(x_3y_3)qy_4 = 0$, which again gives two cases (a) and (b)	(a) $q = 0$ and $y_4 \neq 0$, this implies $y_1(y_3 - y_4) = 0$. If $y_1 = 0$, we also have $y_2 = 0$, which shows $y = y_3e_3 + y_4e_4$. If $y_3 - y_4 = 0$, implies $\boldsymbol{y = y_1(e_1 - e_2) + y_3(e_3 + e_4)}$.	(b) If $q \neq 0$ and $y_4 = 0$, then using equations 5.1 and 5.2, we get $y_1(y_1 + q) = 0$. And again here, if $y_1 = 0$, $y_3 \neq -q$ we have $y = y_3e_3 + q\Lambda$, otherwise for $y_1 \neq 0$ and $y_3 = -q$, we have $\boldsymbol{y = y_1(e_1 - e_2) + y_3(e_3 - \Lambda)}$.

(2) Next, we assume $y_1 \neq -y_2, y_3 \neq 0$, and $y_3 = q$. On plugging these into equation 5.3, we get $y_4 = \frac{y_1^2 - y_2^2}{2y_3}$, which gives geodesic vector $\boldsymbol{y = y_1e_1 + y_2e_2 + y_3e_3 + \frac{y_1^2 - y_2^2}{2y_3}e_4 + y_3\Lambda}$.

(3) At last we suppose $y_1 = -y_2, y_3 \neq 0$ and $y_3 = q$. On plugging into (5.3), we get $2(m+1)x_3q^2y_4 = 0$. This takes us to two cases:	(a) If $q = 0$ and $y_4 \neq 0$, this implies $\boldsymbol{y = y_4e_4}$ or $\boldsymbol{y = y_1(e_1 - e_2)}$.	(b) On taking $q \neq 0$, $y_4 = 0$ in (5.1), we get $2qy_1 = 0$ which vanishes y_1. So the geodesic vector is $\boldsymbol{y = y_3(e_3 + \Lambda)}$.

(4) Next we suppose, $y_1 \neq -y_2, y_3 \neq q, y_3 = 0$, from equation 5.4, we have $2mqx_3(y_1^2 + y_2^2 + y_4^2) = 0$. This gives that q vanishes and we get the geodesic vector as $\boldsymbol{y = y_1e_1 + y_2e_2 + y_4e_4}$.

Case (B) can be seen similar to the case(A). And it also coincides with the homogeneous geodesic in 4-dimensional Randers space example [12].

Case (C): On considering $X = x_3(e_3 + e_4)$, again from 5.5, we get

$$(m+1)x_3(y_3 + y_4)(y_1 + y_3)(y_3 - q) = 0.$$

This leads to different possiblities: (1) Let us Suppose $y_1 = -y_2, y_3 \neq q, y_3 \neq -y_4$,

(1) also from equation 5.6, we have $2q(m+1)x_3(y_3^2 - y_4^2) = 0$ which implies two cases, i.e., either $q = 0$ or $y_3 = y_4$	(a) If $q = 0$, and $y_3 \neq y_4$, from equation (5.1), we get $y_1(y_3 - y_4) = 0$, implies $y_1 = y_2 = 0$, which gives geodesic vector $\boldsymbol{y = y_3e_3 + y_4e_4}$.	If $q \neq 0$, $y_3 = y_4$, equation(5.1), gives $2qy_2(m+1)x_3(y_3+y_4) = 0$, which vanishes $y_3 = y_4 = 0$. Hence the geodesic vector takes the form $\boldsymbol{y = y_3(e_3 + e_4) + q\Lambda}$.

(2) In this case assume $y_1 \neq y_2, y_3 = q, y_3 \neq y_4$ using equation 5.6, we have

$$x_3(y_3 + y_4)[y_2(y_1 + y_2) + 2y_3y_4 - 2y_3^2] = 0.$$

Since, in this case $y_3 + y_4$ can't vanish. Hence, we get $2y_3^2 - 2y_3y_4 + y_2(y_3 + y_4) = 0$, which is quadratic in y_3. So the roots are $y_3 = \frac{y_4 \pm \sqrt{y_4^2 + 2(y_1 + y_2)y_2}}{2}$. So the geodesic vector y is written as

$$\boldsymbol{y_1 e_1 + y_2 e_2 + \frac{y_4 \pm \sqrt{y_4^2 + 2(y_1 + y_2) y_2}}{2}(e_3 + \Lambda) + y_4 e_4.}$$

(3) In third case, we assume $y_1 = -y_2, y_3 = q, y_3 \neq -y_4$, from using equation 5.6, we get $2x_3 y_3(y_3 + y_4)(y_3 - y_4) = 0$, which leads to two cases:

(a) If $y_3 = 0$, the geodesic vector y takes form $\boldsymbol{y = y_1(e_1 - e_2) + y_4 e_4}$.

(b) If $y_3 = y_4$, then $\boldsymbol{y = y_1(e_1 + e_2) + y_3(e_3 + e_4 + \Lambda)}$.

(4) In this, let us assume $y_1 \neq y_2$, $y_3 \neq q$, $y_3 = y_4$, using the above assumptions in equation (5.4), we have $-4my_3^2 q x_3 = 0$, which leads to two cases, i.e., either $y_3 = 0$ or $q = 0$	(a) $y_3 = 0$ implies geodesic vector takes the form $\boldsymbol{y = y_1 e_1 + y_2 e_2 + q\Lambda}$.	(b) If $q = 0$, we get $y = y_1 e_1 + y_2 e_2 + y_3(e_3 - e_4)$. If both $q = y_3 = 0$, y reduces to $\boldsymbol{y_1 e_1 + y_2 e_2}$.

(5) For this case, let us suppose $y_1 = -y_2, y_3 = -y_4, y_3 \neq q$. On plugging into equation 5.3, we get $-4mqy_3^2 x_4 = 0$, which is similar to the case (4).

(6) For the last case, we take $y_1 \neq y_2, y_3 = q, y_3 = -y_4$. From equation 5.4, we have $-4my_3^3 x_3 = 0$, which implies $y_3 = 0$ and this gives geodesic vector is $\boldsymbol{y = y_1 e_1 + y_2 e_2}$. For the last assumption $X = x_3(e_3 - e_4)$, we can retrace the steps of above to get the homogeneous geodesics.

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