FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 38, No 4 (2023), 793–803
https://doi.org/10.22190/FUMI230611051C
Original Scientific Paper

ON RICCI SEMI-SYMMETRIC SUPER QUASI-EINSTEIN HERMITIAN MANIFOLD

Braj Bhushan Chaturvedi[1] and Brijesh Kumar Gupta[2]

[1]**Department of Mathematics, Mahatma Gandhi Post Graduate College Gorakhpur Gorakhpur (Uttar Pradesh), Pin-273001, India**
[2]**Department of Pure and Applied Mathematics**
Guru Ghasidas Vishwavidyalaya, Bilaspur (Chhattisgarh), Pin-495009, India

Abstract. The object of the present paper is to study the Bochner Ricci semi-symmetric super quasi-Einstein Hermitian manifold and a holomorphically projective Ricci-semi symmetric super quasi Einstein Hermitian manifold.
Keywords: Super quasi-Einstein manifold, pseudo quasi-Einstein manifold, Bochner curvature tensor and holomorphically projective curvature tensor.

1. Introduction

An even dimensional differentiable manifold M^n is said to be a Hermitian manifold if a complex structure J of type (1, 1) and a Riemannian metric g of the manifold satisfy

$$J^2 = -I, \tag{1.1}$$

and

$$g(JX, JY) = g(X, Y), \tag{1.2}$$

where $X, Y \in \chi(M)$ and $\chi(M)$ is Lie algebra of vector fields on the manifold.
An Einstein manifold is a Riemannian or pseudo-Riemannian manifold $(M^n, g)(n \geq 2)$ in which the Ricci tensor is a scalar multiple of the Riemannian metric i.e.

$$S(X, Y) = \alpha g(X, Y), \tag{1.3}$$

Received June 11, 2023, accepted: October 05, 2023
Communicated by Uday Chand De
Corresponding Author: Brijesh Kumar Gupta (brijeshggv75@gmail.com)
2010 *Mathematics Subject Classification.* Primary 53C25; Secondary 53B35

where S denotes the Ricci tensor of the manifold $(M^n, g)(n \geq 2)$ and α is a non-zero scalar.
From the equation (1.3), we get

$$r = n\alpha. \tag{1.4}$$

The notion of a quasi-Einstein manifold arose during the study of exact solutions of the Einstein field equations as well as during considerations of quasi-umbilical hypersurfaces. The same notion of quasi-Einstein manifolds is also studied by M. C. Chaki and R.K. Maity [10].
A semi-Riemannian manifold $(M^n, g)(n \geq 2)$ is said to be a quasi-Einstein manifold [2, 15] if its Ricci tensor S of type (0, 2) of the manifold satisfies

$$S(X, Y) = \alpha g(X, Y) + \beta A(X)A(Y), \tag{1.5}$$

where α, β are scalars such that $\beta \neq 0$ and A are a non-zero 1-form associated with a unit vector field ρ defined by $g(X, \rho) = A(X)$, for every vector field X. ρ denotes the unit vector called the generator of the manifold. An n-dimensional quasi-Einstein manifold is denoted by $(QE)_n$.
Contraction of the equation (1.5), gives

$$r = \alpha\, n + \beta. \tag{1.6}$$

From the equations (1.2) and (1.5), we can easily write

$$\begin{aligned} S(X, \rho) = (\alpha + \beta)A(X), \quad S(\rho,\, \rho) = (\alpha + \beta), \\ g(J\rho,\, \rho) = 0 \quad \textit{and} \quad S(J\rho,\, \rho) = 0. \end{aligned} \tag{1.7}$$

In 2004 U. C. De and G. C. Ghosh [22] introduced the notion of generalised quasi-Einstein manifolds. A Riemannian manifold $(M^n, g)(n \geq 2)$ is said to be a generalised quasi-Einstein manifold if a non-zero Ricci tensor of type (0, 2) satisfies the condition

$$S(X, Y) = \alpha g(X, Y) + \beta A(X)A(Y) + \gamma C(X)C(Y), \tag{1.8}$$

where α, β and γ are scalars such that $\beta \neq 0$, $\gamma \neq 0$ and A, C are non-vanishing 1-forms associated with two orthogonal unit vectors ρ and μ by

$$\begin{aligned} g(X, \rho) = A(X),\, g(X, \mu) = C(X), \\ g(\rho,\, \rho) = g(\mu,\, \mu) = 1, \end{aligned} \tag{1.9}$$

An n-dimensional generalised quasi-Einstein manifold is denoted by $G(QE)_n$.
After contraction of the equation (1.8), we get

$$r = \alpha\, n + \beta + \gamma. \tag{1.10}$$

From the equations (1.2), (1.8) and (1.9), we can easily write

(1.11)

$$\begin{aligned} &S(X,\, \rho) = (\alpha + \beta)A(X),\, S(X,\, \mu) = (\alpha + \gamma)C(X), S(\mu,\, \mu) = \alpha + \gamma, \\ &S(\rho,\, \rho) = \alpha + \beta,\; g(J\rho,\, \rho) = g(J\mu, \mu) = 0, \; \textit{and} \quad S(J\mu,\, \mu) = S(J\rho,\, \rho) = 0. \end{aligned}$$

Some classes of generalised quasi-Eiinstein manifold studied by A. A. Shaikh and S. K. Hui [3]. Also in 2004, M. C. Chaki [11] introduced the notion of super quasi-Einstein manifolds. A Riemannian manifold $(M^n, g)(n \geq 2)$ is said to be a super quasi-Einstein manifold if a non-zero Ricci tensor of type (0, 2) satisfies

(1.12)
$$S(X,Y) = \alpha g(X,Y) + \beta A(X)A(Y) + \gamma[A(X)C(Y) + C(X)A(Y)] + \delta D(X,Y),$$

where α, β, γ and δ are non-zero scalars, A, C are non-vanishing 1-forms defined as (1.9) and ρ, μ are orthogonal unit vector fields, D is symmetric tensor of (0, 2) with a zero trace which satisfies the condition

(1.13)
$$D(X, \rho) = 0, \forall X.$$

An n-dimensional super quasi-Einstein manifold is denoted by $S(QE)_n$. From the equations (1.2), (1.9), (1.12) and (1.13), we can easily write

(1.14)
$$S(X,\, \rho) = (\alpha + \beta)A(X) + \gamma C(X),\, S(X,\, \mu) = \alpha C(X) + \gamma A(X),$$
$$S(\mu,\, \mu) = \alpha + \delta D(\mu, \mu),\, S(\rho,\, \rho) = \alpha + \beta + \delta D(\rho, \rho),\;\, g(J\rho,\, \rho) = g(J\mu, \mu) = 0,$$
$$S(J\mu,\, \mu) = \gamma A(J\mu) + \delta D(J\mu, \mu),\;\, S(J\rho,\, \rho) = \gamma C(J\rho) + \delta D(J\rho, \rho).$$

Super quasi-Einstein manifold studied P. Debnath and A. Konar [13] and S. K. Hui and R. S. Lemence [19]. In 2009, A. A. Shaikh [1] introduced the notion of a pseudo quasi-Einstein manifold. A semi-Riemannian manifold $(M^n, g)(n \geq 2)$ is said to be a pseudo quasi-Einstein manifold if a non-zero Ricci tensor of type (0, 2) satisfies the condition

(1.15)
$$S(X,Y) = \alpha g(X,Y) + \beta A(X)A(Y) + \delta D(X,Y),$$

where α, β, and δ are non-zero scalars and A is a non-zero 1-form defined by $g(X, \rho) = A(X)$. ρ denotes the unit vector called the generator of the manifold and D is symmetric tensor of type (0, 2) with a zero trace defined as (1.13). An n-dimensional pseudo quasi-Einstein manifold is denoted by $P(QE)_n$.
From the equations (1.2), (1.9) (1.13) and (1.15), we can easily write

(1.16)
$$S(X,\, \rho) = (\alpha + \beta)A(X),\, S(\rho,\, \rho) = \alpha + \beta,$$
$$g(J\rho,\, \rho) = 0 \; and \; S(J\rho,\, \rho) = \delta D(J\rho,\, \rho).$$

2. Semi-symmetric and Ricci semi-symmetric manifold

Let (M^n, g) be a Riemannian manifold and ∇ be the Levi-Civita connection on (M^n, g) then, a Riemannian manifold is said to be locally symmetric if $\nabla R = 0$, where R is the Riemannian curvature tensor of (M^n, g). The locally symmetric manifold has been studied by different geometers through different approaches and different notions have been developed, e.g., a semi-symmetric manifold by *Szabò* [23], recurrent manifold by Walker [4], conformally recurrent manifold by Adati

and Miyazawa [20] Ricci recurrent manifolds by E. M. Patterson [8], Concircular recurrent manifold by T. Miyazawa [17, 21] and weakly symmetric manifolds by T. $Tamássy$ and T. Q. Binh[25].
According to Z. I. $Szabò$[23], if the manifold M satisfies the condition

$$(R(X,Y).R)(U,V)W = 0, \quad X,Y,U,V,W \in \chi(M) \tag{2.1}$$

for all vector fields X and Y, then the manifold is called a semi-symmetric manifold. For a (0, k)- tensor field T on M, $k \geq 1$ and a symmetric (0, 2)-tensor field A on M, the $(0, k+2)$-tensor fields R.T and Q(A, T) are defined by

$$\begin{aligned}(R.T)(X_1,.....X_k;X,Y) &= -T(R(X,Y)X_1,X_2,.......X_k)\\ &\quad - .... - T(X_1,..........X_{k-1},R(X,Y)X_k),\end{aligned} \tag{2.2}$$

and

$$\begin{aligned}Q(A,T)(X_1,.....X_k;X,Y) &= -T((X\wedge_A Y)X_1,X_2,.......X_k)\\ &\quad - .... - T(X_1,..........X_{k-1},(X\wedge_A Y)X_k),\end{aligned} \tag{2.3}$$

where $X \wedge_A Y$ is the endomorphism given by

$$(X\wedge_A Y)Z = A(Y,Z)X - A(X,Z)Y. \tag{2.4}$$

Definition 2.1. ([14]) A semi-Riemannian manifold is said to be Ricci semi-symmetric if

$$(R(X,\,Y).S)(Z,W) = -S(R(X,Y)Z,W) - S(Z,R(X,Y)W) = 0, \tag{2.5}$$

for all $X,Y,Z,W \in \chi(M^n)$.
The above developments allow several authors to generalize the notion of quasi Einstein manifolds. In this process, generalized quasi-Einstein manifolds are studied by Prakasha and Venkatesha [7] and N(k)-quasi Einstein manifolds are studied by [5, 12]. In 2012, S. K. Hui and R. S. Lemence [18] discussed generalised quasi-Einstein manifold admitting a W_2- curvature tensor and they proved that if a W_2-curvature tensor satisfies $W_2.S = 0$, then either the associated scalars β and γ are equal or the curvature tensor R satisfies a definite condition. D. G. Prakasha and H. Venkatesha [7] studied some results on generalised quasi-Einstein manifolds and they proved that in generalised quasi-Einstein manifold if a conharmonic curvature tensor satisfies $L.S = 0$, then either M is a nearly quasi-Einstein manifold $N(QE)_n$ or the curvature tensor R satisfies a definite condition. Recently, B. B. Chatuurvedi and B. K. Gupta [6] have studied Ricci pseudo-symmetric mixed generalized quasi-Einstein hermitian manifolds. We have studied the above developments in quasi-Einstein manifold $(QE)_n$, generalised quasi- Einstein manifold $G(QE)_n$, a super quasi-Einstein manifold and decided to study Bochner Ricci semi-symmetric and holomorphically projective Ricci semi-symmetric super quasi-Einstein Hermitian manifold .

3. Bochner Ricci semi-symmetric super quasi-Einstein Hermitian manifold

The notion of Bochner curvature tensor was introduced by S. Bochner [16]. The Bochner curvature tensor B is defined by

(3.1)
$$\begin{aligned}B(Y,Z,U,V) =& R(Y,Z,U,V) - \frac{1}{2(n+2)}\Big\{S(Y,V)g(Z,U) - S(Y,U)g(Z,V)\\&+ g(Y,V)S(Z,U) - g(Y,U)S(Z,V) + S(JY,V)g(JZ,U)\\&- S(JY,U)g(JZ,V) + S(JZ,U)g(JY,V) - g(JY,U)S(JZ,V)\\&- 2S(JY,Z)g(JU,V) - 2g(JY,Z)S(JU,V)\Big\}\\&+ \frac{r}{(2n+2)(2n+4)}\Big\{g(Z,U)g(Y,V) - g(Y,U)g(Z,V)\\&+ g(JZ,U)g(JY,V) - g(JY,U)g(JZ,V) - 2g(JY,Z)g(JU,V)\Big\},\end{aligned}$$

where r is a scalar curvature of the manifold.
In a Hermitian manifold a Bochner curvature tensor satisfies the condition

$$B(X,Y,U,V) = -B(X,Y,V,U). \tag{3.2}$$

Now we introduce the following:

Definition 3.1. A Hermitian manifold is said to be a super quasi-Einstein Hermitian manifold if it satisfies the equation (1.12). Throughout this paper, we denote the super quasi-Einstein Hermitian manifold by $S(QEH)_n$.

Definition 3.2. An even dimensional Hermitian manifold (M^n, g) is said to be a Bochner Ricci semi-symmetric super quasi-Einstein Hermitian manifold if the Bochner curvature tensor of the manifold satisfies $B.S = 0$, i.e.

$$(B(X,\,Y).S)(Z,W) = -S(B(X,Y)Z,W) - S(Z,(B(X,Y)W) = 0, \tag{3.3}$$

for all $X, Y, Z, W \in \chi(M^n)$.
If we take a Bochner Ricci semi-symmetric super quasi-Einstein Hermitian manifold, then from the equations (1.12) and (3.3), we have

$$\begin{aligned}&\alpha[B(X,Y,Z,U) + B(X,Y,U,Z)]\\&+ \beta[A(B(X,Y)Z)A(U) + A(Z)A(B(X,Y)U)]\\&+ \gamma[A(B(X,Y)Z)C(U) + C(B(X,Y)Z)A(U)\\&+ A(Z)C(B(X,Y)U) + C(Z)A(B(X,Y)U)]\\&+ \delta[D(B(X,Y)Z,U) + D(Z,B(X,Y)U)] = 0,\end{aligned} \tag{3.4}$$

where $g(B(X,Y)U,Z) = B(X,Y,U,Z)$.
Now from the equations (3.2) and (3.4), we have

$$\begin{aligned}
&\beta[A(B(X,Y)Z)A(U) + A(Z)A(B(X,Y)U)] \\
&+ \gamma[A(B(X,Y)Z)C(U) + C(B(X,Y)Z)A(U) \\
&+ A(Z)C(B(X,Y)U) + C(Z)A(B(X,Y)U)] \\
&+ \delta[D(B(X,Y)Z,U) + D(Z,B(X,Y)U)] = 0,
\end{aligned} \tag{3.5}$$

putting $Z = U = \rho$ in equation (3.5), we get

$$\gamma B(X,Y,\rho,\mu) = 0, \tag{3.6}$$

this implies either $\gamma = 0$ or $B(X,Y,\rho,\mu) = 0$.
From equation (3.6) if $\gamma = 0$ then from equation (1.12), we have

$$S(X,Y) = \alpha g(X,Y) + \beta A(X)A(Y) + \delta D(X,Y), \tag{3.7}$$

this is the condition for pseudo quasi-Einstein manifold.
Thus we come to the conclusion:

Theorem 3.1. *A Bochner Ricci semi-symmetric super quasi-Einstein Hermitian manifold is either a Bochner Ricci semi-symmetric pseudo quasi-Einstein Hermitian manifold or*

$$B(X,Y,\rho,\mu) = 0.$$

If we take a Bochner flat curvature tensor then from equation (3.1), we have

$$\begin{aligned}
&R(Y,Z,U,V) = \frac{1}{2(n+2)}\Big\{S(Y,V)g(Z,U) - S(Y,U)g(Z,V) \\
&+ g(Y,V)S(Z,U) - g(Y,U)S(Z,V) + S(JY,V)g(JZ,U) \\
&- S(JY,U)g(JZ,V) + S(JZ,U)g(JY,V) - g(JY,U)S(JZ,V) \\
&- 2S(JY,Z)g(JU,V) - 2g(JY,Z)S(JU,V)\Big\} \\
&- \frac{r}{(2n+2)(2n+4)}\Big\{g(Z,U)g(Y,V) - g(Y,U)g(Z,V) \\
&+ g(JZ,U)g(JY,V) - g(JY,U)g(JZ,V) - 2g(JY,Z)g(JU,V)\Big\},
\end{aligned} \tag{3.8}$$

From equation (2.5) and (3.8), we infer

$$\begin{aligned}(3.9)\quad &\Big\{S(QY,V)g(Z,U) - g(Y,U)S(QZ,V) + S(QJY,V)g(JZ,U)\\ &- g(JY,U)S(JQZ,V) - 2g(JY,Z)S(JQU,V)\\ &+ S(QY,U)g(Z,V) - g(Y,V)S(QZ,U) + S(QJY,U)g(JZ,V)\\ &- g(JY,V)S(JQZ,U) - 2g(JY,Z)S(JQV,U)\Big\}\\ &- \frac{r}{(2n+2)}\Big\{g(Z,U)S(Y,V) - g(Y,U)S(Z,V) + g(JZ,U)S(JY,V)\\ &- g(JY,U)S(JZ,V) + g(Z,V)S(Y,U) - g(Y,V)S(Z,U)\\ &+ g(JZ,V)S(JY,U) - g(JY,V)S(JZ,U)\Big\} = 0,\end{aligned}$$

If we take λ be an eigen value of Q and JQ corresponding to eigen vectors X and JX respectively then $QX = \lambda X$ and $QJX = \lambda JX$ i.e. $S(X,U) = \lambda g(X,U)$ (where the manifold is not Einstein) and hence

$$(3.10)\qquad S(QX,U) = \lambda S(X,U) \quad \textit{and} \quad S(QJX,U) = \lambda S(JX,U).$$

Using equation (3.10) in equation (3.9), we have

$$\begin{aligned}(3.11)\quad &\left(\lambda - \frac{r}{(2n+2)}\right)\Big\{S(Y,V)g(Z,U) - g(Y,U)S(Z,V)\\ &+ S(Y,U)g(Z,V) - g(Y,V)S(Z,U) + S(JY,V)g(JZ,U)\\ &- S(JZ,V)g(JY,U) + S(JY,U)g(JZ,V) - g(JY,V)S(JZ,U)\Big\} = 0,\end{aligned}$$

If we take $\lambda \neq \frac{r}{(2n+2)}$, then from equation (3.11), we obtain

$$\begin{aligned}(3.12)\quad &S(Y,V)g(Z,U) - g(Y,U)S(Z,V)\\ &+ S(Y,U)g(Z,V) - g(Y,V)S(Z,U) + S(JY,V)g(JZ,U)\\ &- S(JZ,V)g(JY,U) + S(JY,U)g(JZ,V) - g(JY,V)S(JZ,U) = 0.\end{aligned}$$

Now putting $V = \rho$ and $U = \mu$, we get

(3.13)

$$\begin{aligned}&[S(Y,\rho)g(Z,\mu) - g(Y,\mu)S(Z,\rho) + S(Y,\mu)g(Z,\rho) - g(Y,\rho)S(Z,\mu)\\ &+ S(JY,\rho)g(JZ,\mu) - S(JZ,\rho)g(JY,\mu) + S(JY,\mu)g(JZ,\rho) - S(JZ,\mu)g(JY,\rho)] = 0.\end{aligned}$$

Now using equations (1.9) and (1.14) in equation (3.13), we get

$$(3.14)\qquad \gamma[A(Y)C(Z) - A(Z)C(Y) + A(JY)C(JZ) - A(JZ)C(JY)] = 0,$$

this implies either $\gamma = 0$ or

$$(3.15)\qquad A(Y)C(Z) - A(Z)C(Y) = A(JZ)C(JY) - A(JY)C(JZ).$$

If we take $\lambda \neq \frac{r}{(2n+2)}$ and $\gamma = 0$, then equations (1.2), (1.9) and (3.15) imply $g(Y,\rho)\,g(Z,\mu) - g(Z,\rho)\,g(Y,\mu) = g(Z,J\rho)\,g(Y,J\mu) - g(Y,J\rho)\,g(Z,J\mu)$, i.e. $g(Y,\rho)\,g(Z,\mu) = g(Z,\rho)\,g(Y,\mu)$ if and only if $g(Z,J\rho)\,g(Y,J\mu) = g(Y,J\rho)\,g(Z,J\mu)$, therefore we can say that if $\lambda \neq \frac{r}{(2n+2)}$ and $\gamma = 0$ the vector fields ρ and μ corresponding to 1-forms A and C respectively are codirectional if and only if the vector fields $\overline{\rho}$ and $\overline{\mu}$ corresponding to 1-forms A and C respectively are codirectional.
Thus we conclude:

Theorem 3.2. *In a Bochner flat Ricci semi-symmetric super quasi-Einstein Hermitian manifold if $\frac{r}{(2n+2)}$ is not an eigen value of the Ricci operator Q and JQ and $\gamma \neq 0$ then the vector fields ρ and μ corresponding to 1-forms A and C respectively are codirectional if and only if the vector fields $\overline{\rho}$ and $\overline{\mu}$ corresponding to 1-forms A and C respectively are codirectional.*

4. Holomorphically projective Ricci semi-symmetric super quasi-Einstein Hermitian manifold

The holomorphically projective curvature tensor is defined by [24]

$$\begin{aligned} P(X,Y,Z,W) = R(X,Y,Z,W) - \frac{1}{n-2}[&S(Y,Z)g(X,W) - S(X,Z)g(Y,W) \\ &+ S(JX,Z)g(JY,W) - S(JY,Z)g(JX,W)]. \end{aligned} \tag{4.1}$$

This tensor has the following properties

$$P(X,Y,Z,W) = -P(Y,X,Z,W), \quad P(JX,JY,Z,W) = P(X,Y,Z,W). \tag{4.2}$$

Now we introduce the following:

Definition 4.1. An even dimensional Hermitian manifold (M^n, g) is said to be a holomorphically projective Ricci semi-symmetric super quasi-Einstein Hermitian manifold if the holomorphically projective curvature tensor of the manifold satisfies $P.S = 0$, i.e.

$$(P(X,\,Y).S)(Z,W) = -S(P(X,Y)Z,W) - S(Z,(P(X,Y)W) = 0, \tag{4.3}$$

for all $X,Y,Z,W \in \chi(M^n)$.
If we take a holomorphically projective Ricci semi-symmetric super quasi-Einstein Hermitian manifold, then from the equations (1.12) and (4.1), we have

$$\begin{aligned} &\alpha[P(X,Y,Z,W) + P(X,Y,W,Z)] \\ &+ \beta[A(P(X,Y)Z)A(W) + A(Z)A(P(X,Y)W)] \\ &+ \gamma[A(P(X,Y)Z)C(W) + A(W)C(P(X,Y)Z) \\ &+ A(Z)C(P(X,Y)W) + C(Z)A(P(X,Y)W)] \\ &+ \delta[D(P(X,Y)Z,W) + D(Z,P(X,Y)W)] = 0. \end{aligned} \tag{4.4}$$

Now putting $Z = W = \rho$ in equation (4.4) and using equation (1.14), we have

$$(\alpha+\beta)P(X,Y,\rho,\rho)+\gamma P(X,Y,\rho,\mu)=0. \tag{4.5}$$

Using $Z = W = \rho$ in equation (4.1), we have

$$\begin{aligned} P(X,Y,\rho,\rho) = -\frac{\gamma}{n-2}[&C(Y)A(X)-A(Y)C(X)\\ &+C(JX)A(JY)-C(JY)A(JX)]. \end{aligned} \tag{4.6}$$

Similarly putting $Z = \rho$ and $W = \mu$ in equation (4.1), we get

$$\begin{aligned} P(X,Y,\rho,\mu) = R(X,Y,\rho,\mu) - \frac{(\alpha+\beta)}{n-2}[&A(Y)C(X)-C(Y)A(X)\\ &+C(JX)A(JY)-C(JY)A(JX)]. \end{aligned} \tag{4.7}$$

Using equations (4.6) and (4.7) in (4.5), we get

$$\gamma R(X,Y,\rho,\mu)=0, \tag{4.8}$$

this implies that either $\gamma = 0$ or $R(X,Y,\rho,\mu) = 0$. If $\gamma = 0$ then from equation (1.12), we get the condition of a pseudo quasi-Einstein manifolds.
Thus we can conclude:

Theorem 4.1. *A holomorphically projectively Ricci semi-symmetric super quasi-Einstein Hermitian manifold is either a holomorphically projective Ricci semi-symmetric pseudo quasi-Einstein Hermitian manifold or*

$$R(X,Y,\rho,\mu)=0.$$

Putting $Z = \rho$ and $W = \mu$ in equation (4.4), we get

$$\alpha[P(X,Y,\rho,\mu)+P(X,Y,\mu,\rho)]+\beta P(X,Y,\mu,\rho)+\gamma[P(X,Y,\rho,\rho)+P(X,Y,\mu,\mu)]=0. \tag{4.9}$$

Putting $Z = U = \mu$ in (4.1), we get

$$\begin{aligned} &P(X,Y,\mu,\mu)\\ &= -\frac{\gamma}{n-2}[A(Y)C(X)-C(Y)A(X)+C(JY)A(JX)-C(JX)A(JY)]. \end{aligned} \tag{4.10}$$

Adding equations (4.6) and (4.10), we get

$$P(X,Y,\mu,\mu)+P(X,Y,\rho,\rho)=0, \tag{4.11}$$

from equations (4.9) and (4.11), we have

$$\alpha[P(X,Y,\rho,\mu)+P(X,Y,\mu,\rho)]+\beta P(X,Y,\mu,\rho)=0. \tag{4.12}$$

From equations (4.1), (4.7) and (4.12), we have

$$\beta R(X,Y,\mu,\rho)=0. \tag{4.13}$$

This implies either $\beta = 0$ or $R(X,Y,\mu,\rho) = 0$.

Theorem 4.2. *In a holomorphically projective Ricci semi-symmetric super quasi-Einstein Hermitian manifold either $\beta \neq 0$ or $R(X,Y,\mu,\rho) = 0$.*

REFERENCES

1. A. A. Shaikh: *On pseudo quasi Einstein manifold.* Period. Math. Hungar. **59**(2009),119-146.
2. A. A. Shaikh, D. W. Yoon and S. K. Hui: *On quasi-Einstein space times.* Tsukuba J. of Math. **33(2)** (2009), 305-326.
3. A. A. Shaikh and S. K. Hui: *On some classes of generalised quasi-Einstein manifolds.* Commun. Korean Math. Soc. **24(3)** (2009), 415-424.
4. A. G. Walker: *On Ruse's spaces of recurrent curvature.* Proc. London Math. Soc. **52**(1950), 36-64.
5. C. Özgür and S. Sular: *On N(k)-quasi-Einstein manifolds satisfying certain conditions.* Balkan J. Geom. Appl. **13**(2008), 74-79.
6. B. B. Chaturvedi and B. K. Gupta: *On Ricci pseudo-Symmetric mixed generalized quasi-Einstein hermitian manifolds.* Proceedings of the National Academy of Sciences, India Section A: Physical Sciences **91(1)** 2021.
7. D. G. Prakasha and H. Venkatesha: *Some results on generalised quasi-Einstein manifolds.* Chinese Journal of Mathematics (Hindawi Publishing Corporation), 2014.
8. E. M. Patterson: *Some theorems on Ricci recurrent spaces.* J. London. Math. Soc. **27**(1952), 287-295.
9. F. Defever: *Ricci-semisymmetric hypersurfaces.* Balkan Journal of Geometry and its appl. **5**(2000), 81-91.
10. M. C. Chaki and R. K. Maity: *On quasi-Einstein manifolds.* Publ. Math. Debrecen **57**(2000), 297-306.
11. M. C. Chaki: *On super quasi-Einstein manifold.* Publ. Math. Debrecen **64**(2004), 481-488.
12. M. M. Tripathi and J. S Kim: *On N(k)-quasi-Einstein manifolds.* Commun. Korean Math. Soc. **22(3)**2007, 411-417.
13. P. Debnath and A. Konar: *On super quasi-Einstein manifold.* Institut Math. Publ. Nouvelle Serie **89** (103) (2011), 95-104.
14. R. Deszcz: *On pseudo symmetric spaces.* Bull. Soc. Math. Belg. Ser. A **44** (1992)1-34.
15. R. Deszcz, M. Glogowska, M. Hotlos and Z. Senturk: *On certain quasi-Einstein semi-symmetric hypersurfaces.* Annales Univ. Sci. Budapest, Eotovos sect. Math **41**(1998), 151-164
16. S. Bochner: *Curvature and Betti numbers II.* Ann. of Math. **50**(1949), 77-93.
17. K. Yano: *Concircular geometry I.* Proc. Imp. Acad. Tokyo **16** (1940), 195-200.
18. S. K. Hui and R. S. Lemence: *On generalized quasi Einstein manifold admitting W_2-curvature tensor.* Int. Journal of Math. Analysis, **6**, 2012, no. 23, 1115 - 1121.
19. S. K. Hui and R. S. Lemence: *Some results of super quasi-Einstein manifolds.* ISRN Geometry, 2012, Article ID 217132.
20. T. Adati and T. Miyazawa: *On a Riemannian space with recurrent conformal curvature.* Tensor N. S. **18**(1967), 348-354.
21. T. Miyazawa: *On Riemannian space admitting some recurrent tensor.* TRU Math. J. **2** (1996), 1118

22. U. C. De and G. C. Ghosh: *On generalized quasi Einstein manifolds*. Kyungpook Math.J. **44**(2004), 607-615

23. Z. I. Szabó: *Structure theorems on Riemannian spaces satisfying R(X,Y).R=0*. J. Diff. Geom. **17**(1982), 531-582.

24. K. Yano: *Differential geometry of complex and almost complex spaces*. Pergamon Press, New York, 1965.

25. L. Tamássy and T. Q. Binh: *On weakly symmetric and weakly projective symmetric Riemannian manifolds*. Colloq. Math. Soc. J´anos Bolyai **56** (1989), 663-670.