FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 38, No 4 (2023), 817–828
https://doi.org/10.22190/FUMI230624053G
Original Scientific Paper

RIGHT CONOID HYPERSURFACES IN FOUR-SPACE

Erhan Güler and Mustafa Yıldız

Faculty of Sciences, Department of Mathematics
Bartın University, 74100 Bartın, Türkiye

Abstract. The right conoid hypersurfaces in the four-dimensional Euclidean space $\mathbb{E}^4$ are introduced. The matrices corresponding to the fundamental form, Gauss map, and shape operator of these hypersurfaces are calculated. By utilizing the Cayley–Hamilton theorem, the curvatures of these specific hypersurfaces are determined. Furthermore, the conditions for minimality are presented. Additionally, the Laplace–Beltrami operator of this family is computed, and some examples are provided.
Keywords: conoid hypersurfaces, Cayley–Hamilton theorem, four-dimensional space.

1. Introduction

A ruled surface

$$\begin{aligned} \mathfrak{r}(u,v) &= \alpha(v) + u\beta(v) \\ &= (0,0,h(v)) + u(\cos f(v), \sin f(v), 0) \end{aligned}$$

is termed a *right conoid* in three-dimensional space $\mathbb{E}^3$ if it can be generated by the translation of a straight line that intersects a fixed straight line, while ensuring that the lines maintain a perpendicular relationship throughout the generation process. By considering the xy-plane as the perpendicular plane and selecting the z-axis as the reference line, the parametric Eq. for the right conoid is given by

$$\mathfrak{r}(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix} = \begin{pmatrix} u\cos f(v) \\ u\sin f(v) \\ h(v) \end{pmatrix}.$$

Received June 24, 2023, accepted: November 07, 2023
Communicated by Cornelia-Livia Bejan
Corresponding Author: Erhan Güler (eguler@bartin.edu.tr)
2010 *Mathematics Subject Classification.* Primary 53A05, 53A07; Secondary 53C42, 53C43

Helicoid, Whitney umbrella, Wallis's conical edge, Plücker's conoid, hyperbolic paraboloid are each examples of a right conoid surface. See Berger and Gostiaux [2], Do Carmo [4], Gray [5], Kreyszig [6] for details.

The aim of this study is to investigate the properties of the right conoid hypersurfaces in the four-dimensional Euclidean space $\mathbb{E}^4$. Specifically, we aim to compute the matrices associated with the fundamental form, Gauss map, and shape operator of these hypersurfaces. By employing the Cayley–Hamilton theorem, our objective is to determine the curvatures of these particular hypersurfaces. Additionally, we aim to establish the conditions for minimality within this context. Moreover, we seek to unveil the connection the Laplace–Beltrami operator of that kind hypersurfaces.

In Section 2., a detailed explanation of the fundamental principles and concepts underlying four-dimensional Euclidean geometry is provided.

Section 3. is dedicated to the presentation of the curvature formulas applicable to hypersurfaces in $\mathbb{E}^4$.

In Section 4., a comprehensive definition of right conoid hypersurfaces is offered, emphasizing their distinctive properties and characteristics.

In Section 5., the focus shifts to the discussion of the Laplace–Beltrami operator for a smooth function in $\mathbb{E}^4$, and the application of the previously examined hypersurfaces in its computation.

In the last section, we present a conlusion.

2. Preliminaries

In this paper, we use the following notations, formulas, Eqs., etc.

Let M be an oriented hypersurface in $\mathbb{E}^{n+1}$ with its shape operator $\mathcal{S}$, position vector x. Consider a local orthonormal frame field $\{e_1, e_2, \dots, e_n\}$ consisting of principal directions of M coinciding with the principal curvature k_i for $i = 1, 2, \dots, n$. Let the dual basis of this frame field be $\{f_1, f_2, \dots, f_n\}$.

We let $s_j = \sigma_j(k_1, k_2, \dots, k_n)$, where σ_j denotes the j-th elementary symmetric function defined by

$$\sigma_j(a_1, a_2, \dots, a_n) = \sum_{1 \le i_1 < i_2 < \dots < i_j \le n} a_{i_1} a_{i_2} \dots a_{i_j}.$$

We consider the notation

$$r_i^j = \sigma_j(k_1, k_2, \dots, k_{i-1}, k_{i+1}, k_{i+2}, \dots, k_n).$$

According to the given definition, we have $r_i^0 = 1$ and $s_{n+1} = s_{n+2} = \cdots = 0$. The function s_k is referred to as the k-th mean curvature of the oriented hypersurface M. The mean curvature $H = \frac{1}{n} s_1$ is also defined, and the Gauss–Kronecker curvature of M is $K = s_n$. If $s_j \equiv 0$, the hypersurface M is known as j-minimal.

In Euclidean $(n+1)$-space, getting the curvature formulas $\mathcal{K}_i$, $i = 0, 1, \ldots, n$, (See [1], [3], and [7] for details.), we have the following characteristic polynomial Eq. $P_{\mathcal{S}}(\lambda) = 0$ of $\mathcal{S}$:

$$\sum_{k=0}^{n} (-1)^k s_k \lambda^{n-k} = \det(\mathcal{S} - \lambda \mathcal{I}_n) = 0. \tag{2.1}$$

Here, $\mathcal{I}_n$ indicates the identity matrix. Hence, we reveal the curvature formulas as $\binom{n}{i}\mathcal{K}_i = s_i$.

In this paper, we have identified a vector with its transpose. Let $\mathfrak{x} = \mathfrak{x}(u, v, w)$ be an immersion from $M^3 \subset \mathbb{E}^3$ to $\mathbb{E}^4$.

Definition 2.1. An inner product of two vectors $\varphi^1 = (\varphi^1_1, \varphi^1_2, \varphi^1_3, \varphi^1_4)$, $\varphi^2 = (\varphi^2_1, \varphi^2_2, \varphi^2_3, \varphi^2_4)$ of $\mathbb{E}^4$ is determined by

$$\left\langle \varphi^1, \varphi^2 \right\rangle = \varphi^1_1 \varphi^2_2 + \varphi^1_2 \varphi^2_2 + \varphi^1_3 \varphi^2_3 + \varphi^1_4 \varphi^2_4.$$

Definition 2.2. A triple vector product of $\varphi^1 = (\varphi^1_1, \varphi^1_2, \varphi^1_3, \varphi^1_4)$, $\varphi^2 = (\varphi^2_1, \varphi^2_2, \varphi^2_3, \varphi^2_4)$, $\varphi^3 = (\varphi^3_1, \varphi^3_2, \varphi^3_3, \varphi^3_4)$ in $\mathbb{E}^4$ is defined by

$$\varphi^1 \times \varphi^2 \times \varphi^3 = \det \begin{pmatrix} e_1 & e_2 & e_3 & e_4 \\ \varphi^1_1 & \varphi^1_2 & \varphi^1_3 & \varphi^1_4 \\ \varphi^2_1 & \varphi^2_2 & \varphi^2_3 & \varphi^2_4 \\ \varphi^3_1 & \varphi^3_2 & \varphi^3_3 & \varphi^3_4 \end{pmatrix}.$$

Definition 2.3. The matrix $(\mathfrak{g}_{ij})^{-1} \cdot (\mathfrak{h}_{ij})$ determines the shape operator matrix $\mathcal{S}$ of hypersurface $\mathfrak{x}$ in Euclidean 4-space $\mathbb{E}^4$, where, $(\mathfrak{g}_{ij})_{3\times 3}$ and $(\mathfrak{h}_{ij})_{3\times 3}$ describe the first and the second fundamental form matrices, respectively, and $\mathfrak{g}_{ij} = \langle \mathfrak{x}_i, \mathfrak{x}_j \rangle$, $\mathfrak{h}_{ij} = \langle \mathfrak{x}_{ij}, \mathcal{G} \rangle$, $i, j = 1, 2, 3$, $\mathfrak{x}_u = \frac{\partial \mathfrak{x}}{\partial u}$ when $i = 1$, $\mathfrak{x}_{uv} = \frac{\partial^2 \mathfrak{x}}{\partial u \partial v}$ when $i = 1$ and $j = 2$, etc., e_k denotes the natural base elements of $\mathbb{E}^4$, and

$$\mathcal{G} = \frac{\mathfrak{x}_u \times \mathfrak{x}_v \times \mathfrak{x}_w}{\|\mathfrak{x}_u \times \mathfrak{x}_v \times \mathfrak{x}_w\|} \tag{2.2}$$

determines the Gauss map of the hypersurface $\mathfrak{x}$.

3. Curvatures in Four-Space

In this section, we reveal the curvature formulas of any hypersurface $\mathfrak{x} = \mathfrak{x}(u, v, w)$ *in* $\mathbb{E}^4$.

Theorem 3.1. *A hypersurface $\mathfrak{x}$ in $\mathbb{E}^4$ has the following curvature formulas, $\mathcal{K}_0 = 1$ by definition,*

$$3\mathcal{K}_1 = \frac{\mathfrak{c}_2}{\mathfrak{c}_3}, \; 3\mathcal{K}_2 = -\frac{\mathfrak{c}_1}{\mathfrak{c}_3}, \; \mathcal{K}_3 = \frac{\mathfrak{c}_0}{\mathfrak{c}_3}, \tag{3.1}$$

where $\mathfrak{c}_3\lambda^3 + \mathfrak{c}_2\lambda^2 + \mathfrak{c}_1\lambda + \mathfrak{c}_0 = 0$ describes the characteristic polynomial Eq. $P_{\mathcal{S}}(\lambda) = 0$ of the shape operator matrix $\mathcal{S}$, $\mathfrak{c}_3 = \det(\mathfrak{g}_{ij})$, $\mathfrak{c}_0 = \det(\mathfrak{h}_{ij})$, and $(\mathfrak{g}_{ij})_{3\times 3}$, $(\mathfrak{h}_{ij})_{3\times 3}$ denote the first, and the second fundamental form matrices, respectively.

Proof. The matrix $(\mathfrak{g}_{ij})^{-1} \cdot (\mathfrak{h}_{ij})$ describes the shape operator matrix $\mathcal{S}$ of hypersurface $\mathfrak{x}$ in Euclidean 4-space $\mathbb{E}^4$. We reveal the characteristic polynomial Eq. $\det(\mathcal{S} - \lambda \mathcal{I}_3) = 0$ of $\mathcal{S}$. Thus, we obtain the curvatures

$$\begin{aligned}
\mathcal{K}_0 &= 1, \\
3\mathcal{K}_1 &= k_1 + k_2 + k_3 = -\frac{\mathfrak{c}_2}{\mathfrak{c}_3}, \\
3\mathcal{K}_2 &= k_1 k_2 + k_1 k_3 + k_2 k_3 = \frac{\mathfrak{c}_1}{\mathfrak{c}_3}, \\
\mathcal{K}_3 &= k_1 k_2 k_3 = -\frac{\mathfrak{c}_0}{\mathfrak{c}_3},
\end{aligned}$$

□

Definition 3.1. *A hypersurface $\mathfrak{x}$ is called j-minimal if $\mathcal{K}_j = 0$, where $j = 1, 2, 3$.*

Theorem 3.2. *A hypersurface $\mathfrak{x} = \mathfrak{x}(u, v, w)$ in $\mathbb{E}^4$ has the following relation*

$$\mathcal{K}_0 \mathbb{IV} - 3\mathcal{K}_1 \mathbb{III} + 3\mathcal{K}_2 \mathbb{II} - \mathcal{K}_3 \mathbb{I} = \mathcal{O}_3,$$

where $\mathbb{I}, \mathbb{II}, \mathbb{III}, \mathbb{IV}$ determines the fundamental form matrices, $\mathcal{O}_3$ represents the zero matrix having order 3 of the hypersurface.

Proof. Regarding $n = 3$ in (2.1), it runs. □

4. Right Conoid Hypersurfaces

In this section, we define the right conoid hypersurface (RCH), then find its differential geometric properties in Euclidean 4-space $\mathbb{E}^4$.

In $\mathbb{E}^4$, we consider a ruled hypersurface

$$\begin{aligned}
\mathfrak{x}(u, v, w) &= \alpha(v, w) + u\beta(v, w) \\
&= (0, 0, 0, h(v, w)) \\
&\quad + u(\cos f(v) \cos g(w), \sin f(v) \cos g(w), \sin g(w), 0).
\end{aligned}$$

Then, we present the following.

Definition 4.1. *A right conoid hypersurface is an immersion $\mathfrak{x} : M^3 \subset \mathbb{E}^3 \longrightarrow \mathbb{E}^4$ with the reference line x_4, parametrized by*

$$\mathfrak{x}(u, v, w) = \begin{pmatrix} x_1(u, v, w) \\ x_2(u, v, w) \\ x_3(u, v, w) \\ x_4(u, v, w) \end{pmatrix} = \begin{pmatrix} u \cos f(v) \cos g(w) \\ u \sin f(v) \cos g(w) \\ u \sin g(w) \\ h(v, w) \end{pmatrix}. \tag{4.1}$$

Here, $u \in \mathbb{R} - \{0\}$, $f = f(v)$, $g = g(w)$, $h = h(v, w)$ denote the differentiable functions, and $0 \leq f, g < 2\pi$.

Taking the first derivatives of RCH determined by Eq. (4.1) with respect to u, v, w, respectively, we obtain the first fundamental form matrix

$$(4.2) \qquad (\mathfrak{g}_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & u^2 f_v^2 \cos^2 g(w) + h_v^2 & h_v h_w \\ 0 & h_v h_w & u^2 g_w^2 + h_w^2 \end{pmatrix},$$

and $f_v = \frac{\partial f}{\partial v}$, $f_v^2 = \frac{\partial^2 f}{\partial v^2}$, etc.. Hence, $\mathbf{g} = \det(\mathfrak{g}_{ij}) = u^2 \mathcal{W}$, where

$$\mathcal{W} = f_v^2 \left(u^2 g_w^2 + h_w^2\right) \cos^2 g(w) + h_v^2 g_w^2.$$

Using the Gauss map formula (2.2), we obtain the following Gauss map of the RCH determined by Eq. (4.1):

$$(4.3) \qquad \mathcal{G} = \frac{1}{\mathcal{W}^{1/2}} \begin{pmatrix} -f_v h_w \cos f(v) \sin g(w) \cos g(w) - h_v g_w \sin f(v) \\ -f_v h_w \sin f(v) \sin g(w) \cos g(w) + h_v g_w \cos f(v) \\ f_v h_w \cos^2 g(w) \\ -u f_v g_w \cos g(w) \end{pmatrix}.$$

By taking the second derivatives w.r.t. u, v, w, of RCH described by Eq. (4.1), and by using the Gauss map given by Eq. (4.3), we find the second fundamental form matrix

$$(4.4) \qquad \begin{aligned}
\mathfrak{h}_{11} &= 0,\ \mathfrak{h}_{12} = \frac{f_v g_w h_v \cos g}{\mathcal{W}},\ \mathfrak{h}_{13} = \frac{f_v g_w h_w \cos g}{\mathcal{W}}, \\
\mathfrak{h}_{21} &= \frac{f_v g_w h_v \cos g}{\mathcal{W}}, \\
\mathfrak{h}_{22} &= \frac{u\left(f_v^3 h_w \sin g \cos g + g_w \left(h_v f_{vv} - f_v h_{vv}\right)\right) \cos g}{\mathcal{W}}, \\
\mathfrak{h}_{23} &= -\frac{u f_v g_w \left(h_v g_w \sin g + h_{vw} \cos g\right)}{\mathcal{W}},\ \mathfrak{h}_{31} = \frac{f_v g_w h_w \cos g}{\mathcal{W}}, \\
\mathfrak{h}_{32} &= -\frac{u f_v g_w \left(h_v g_w \sin g + h_{vw} \cos g\right)}{\mathcal{W}}, \\
\mathfrak{h}_{33} &= -\frac{u f_v \left(h_w g_{ww} - g_w h_{ww}\right) \cos g}{\mathcal{W}},
\end{aligned}$$

and $f_{uu} = \frac{\partial^2 f}{\partial u^2}$, $f_{uv} = \frac{\partial^2 f}{\partial u \partial v}$, etc.. By using (4.2) and (4.4), we compute the following shape operator matrix $\mathcal{S}$ of (4.1). $\mathcal{S} = (\mathfrak{s}_{ij})_{3\times 3}$ has the following components

$$
\begin{aligned}
\mathfrak{s}_{11} &= 0,\\
\mathfrak{s}_{12} &= \frac{h_v g_w f_v \cos g}{\mathcal{W}},\\
\mathfrak{s}_{13} &= \frac{h_w g_w f_v \cos g}{\mathcal{W}},\\
\mathfrak{s}_{21} &= \frac{h_v g_w^3 f_v \cos g}{\mathcal{W}^2},\\
\mathfrak{s}_{22} &= \frac{1}{u\mathcal{W}^2}[h_w f_v \left(g_w^2 h_v^2 + f_v^2 \left(h_w^2 + u^2 g_w^2\right)\cos^2 g\right)\sin g\\
&\quad + g_w \left(f_v h_v h_w h_{vw} + \left(h_w^2 + u^2 g_w^2\right)\left(h_v f_{vv} - f_v h_{vv}\right) + u^2 g_w^2 h_v f_{vv}\right)\cos g],\\
\mathfrak{s}_{23} &= \frac{f_v}{u\mathcal{W}^2}[\left(-g_w h_w \left(h_w h_{vw} + h_v h_{ww}\right) - u^2 g_w^3 h_{vw} + h_v h_w^2 g_{ww}\right)\cos g\\
&\quad - h_v g_w^2 \left(h_w^2 + u^2 g_w^2\right)\sin g],\\
\mathfrak{s}_{31} &= \frac{1}{\mathcal{W}^2} h_w g_w f_v^3 \cos^3 g,\\
\mathfrak{s}_{32} &= \frac{1}{u\mathcal{W}^2}[-u^2 f_v^3 g_w h_{vw}\cos^3 g\\
&\quad - h_v g_w \left(f_v h_v h_{vw} - f_v h_w h_{vv} + h_v h_w f_{vv}\right)\cos g\\
&\quad - h_v f_v \left(g_w^2 h_v^2 + f_v^2 \left(u^2 g_w^2 + h_w^2\right)\cos^2 g\right)\sin g],\\
\mathfrak{s}_{33} &= \frac{f_v}{u\mathcal{W}^2}[u^2 f_v^2 \left(g_w h_{ww} - h_w g_{ww}\right)\cos^3 g\\
&\quad + h_v \left(g_w \left(h_w h_{vw} + h_v h_{ww}\right) - h_v h_w g_{ww}\right)\cos g + g_w^2 h_v^2 h_w \sin g].
\end{aligned}
$$

Finally, using (3.1), with (4.2), (4.4), respectively, we find the curvatures of the *RCH* defined by Eq. (4.1) as follows.

Theorem 4.1. *Let $\mathfrak{x}$ be a RCH determined by Eq. (4.1) in $\mathbb{E}^4$. $\mathfrak{x}$ contains the following curvatures, $\mathcal{K}_0 = 1$, by definition,*

$$
\begin{aligned}
\mathcal{K}_1 &= -\frac{1}{3u\mathcal{W}^2}[u^2 f_v^3 \left(g_w h_{ww} - h_w g_{ww}\right)\cos^3 g\\
&\quad + 2 f_v g_w^2 h_v^2 h_w \sin g + f_v^3 h_w \left(h_w^2 + u^2 g_w^2\right)\sin g \cos^2 g\\
&\quad + (g_w f_v \left(2 h_v h_w h_{vw} + h_v^2 h_{ww} - h_{vv}\left(h_w^2 + u^2 g_w^2\right)\right)\\
&\quad + h_v \left(g_w f_{vv}\left(h_w^2 + 2u^2 g_w^2\right) - f_v h_v h_w g_{ww}\right))\cos g],
\end{aligned}
$$

$$\begin{aligned}
\mathcal{K}_2 &= -\frac{f_v}{3\mathcal{W}^4}[f_v^5 g_w^2 h_w^2 \left(h_w^2+u^2 g_w^2\right)\cos^6 g \\
&\quad -h_w f_v^5 \left(h_w^2+u^2 g_w^2\right)\left(g_w h_{ww}-h_w g_{ww}\right)\sin g\cos^5 g \\
&\quad +f_v^3 g_w^2 h_v \left(2g_w \left(h_w^2+u^2 g_w^2\right) h_{vw}+h_v h_w \left(h_w g_{ww}-g_w h_{ww}\right)\right)\sin g\cos^3 g \\
&\quad +f_v^2 g_w (g_w f_v h_{vw}^2 \left(h_w^2+u^2 g_w^2\right)+f_v g_w^3 h_v^2 \left(2h_w^2+u^2 g_w^2\right) \\
&\quad -f_v h_w h_{vv} g_{ww}\left(h_w^2+u^2 g_w^2\right) \\
&\quad +g_w h_{ww}\left(f_v h_{vv}\left(h_w^2+u^2 g_w^2\right)-h_v f_{vv}\left(h_w^2+2u^2 g_w^2\right)\right) \\
&\quad +h_v h_w f_{vv} g_{ww}\left(h_w^2+2u^2 g_w^2\right))\cos^4 g+h_v^2 g_w^4 f_v^3\left(h_w^2+u^2 g_w^2\right)\sin^2 g\cos^2 g \\
&\quad +h_v^2 g_w^3 (f_v g_w h_{vw}^2-g_w h_w h_{vw} f_{vv}+f_v g_w^3 h_v^2+f_v g_w h_{vv} h_{ww} \\
&\quad -f_v h_w g_{ww} h_{vv}-2g_w h_v f_{vv} h_{ww}+2h_v h_w f_{vv} g_{ww})\cos^2 g+f_v g_w^6 h_v^4\sin^2 g \\
&\quad +h_v^3 g_w^5\left(2f_v h_{vw}-h_w f_{vv}\right)\sin g\cos g],
\end{aligned}$$

$$\begin{aligned}
\mathcal{K}_3 &= \frac{f_v^2 g_w^2\cos^2 g}{u\mathcal{W}^5}[h_w^3 f_v^5\left(h_w^2+u^2 g_w^2\right)\sin g\cos^4 g \\
&\quad +f_v^2 (f_v\left(h_w^2+u^2 g_w^2\right)\left(2g_w h_v h_w h_{vw}+g_w h_v^2 h_{ww}-g_w h_w^2 h_{vv}-h_v^2 h_w g_{ww}\right) \\
&\quad +h_w^2 h_v g_w f_{vv}\left(h_w^2+2u^2 g_w^2\right))\cos^3 g \\
&\quad +h_w h_v^2 g_w^2 f_v^3\left(3h_w^2+2u^2 g_w^2\right)\sin g\cos^2 g \\
&\quad +h_v^2 g_w^2 (2f_v g_w h_v h_w h_{vw}+f_v g_w h_v^2 h_{ww}-f_v g_w h_w^2 h_{vv} \\
&\quad -f_v h_v^2 h_w g_{ww}+g_w h_v h_w^2 f_{vv})\cos g+2f_v g_w^4 h_v^4 h_w\sin g].
\end{aligned}$$

Here, $\mathcal{K}_1$ represents the mean curvature, $\mathcal{K}_3$ denotes the Gauss–Kronecker curvature.

Proof. By using the Cayley–Hamilton theorem, we reveal the following characteristic polynomial Eq. $P_{\mathcal{S}}(\lambda)=0$ *of RCH* defined by Eq. (4.1):

$$\mathcal{K}_0\lambda^3-3\mathcal{K}_1\lambda^2+3\mathcal{K}_2\lambda-\mathcal{K}_3=0,$$

where

$$\begin{aligned}
\mathcal{K}_0 &= 1, \\
3\mathcal{K}_1 &= \mathfrak{s}_{22}+\mathfrak{s}_{33}, \\
3\mathcal{K}_2 &= -\mathfrak{s}_{12}\mathfrak{s}_{21}-\mathfrak{s}_{13}\mathfrak{s}_{31}+\mathfrak{s}_{22}\mathfrak{s}_{33}-\mathfrak{s}_{23}\mathfrak{s}_{32}, \\
\mathcal{K}_3 &= -\mathfrak{s}_{12}\mathfrak{s}_{21}\mathfrak{s}_{33}+\mathfrak{s}_{12}\mathfrak{s}_{31}\mathfrak{s}_{23}+\mathfrak{s}_{32}\left(\mathfrak{s}_{21}\mathfrak{s}_{13}+\mathfrak{s}_{22}\mathfrak{s}_{23}\right)-\mathfrak{s}_{22}\left(\mathfrak{s}_{13}\mathfrak{s}_{31}+\mathfrak{s}_{23}\mathfrak{s}_{32}\right).
\end{aligned}$$

The curvatures $\mathcal{K}_i$ of $\mathfrak{x}$ are obtained by the above Eqs.. □

Theorem 4.2. *Let $\mathfrak{x}$ be a RCH described by Eq.* (4.1) *in $\mathbb{E}^4$. $\mathfrak{x}$ has the following principal curvatures*

$$\begin{aligned}
k_1 &= \frac{\mathfrak{s}_{12}\mathfrak{s}_{21}\mathfrak{s}_{33}-\mathfrak{s}_{12}\mathfrak{s}_{31}\mathfrak{s}_{23}-\mathfrak{s}_{21}\mathfrak{s}_{13}\mathfrak{s}_{32}+\mathfrak{s}_{13}\mathfrak{s}_{22}\mathfrak{s}_{31}}{\mathfrak{s}_{12}\mathfrak{s}_{21}+\mathfrak{s}_{13}\mathfrak{s}_{31}}=k_2, \\
k_3 &= 0.
\end{aligned}$$

Proof. By using Eq. $\det(\mathcal{S} - k\mathcal{I}_3) = 0$, it is clear. □

For the sake of brevity, we use the following notations

$$\begin{aligned}
\Gamma &= -g_w h_v \Psi f_{vv} + f_v h_v^2 h_w g_{ww} + g_w f_v \Phi h_{vv} - \Theta, \\
\Omega &= g_w h_{ww} - h_w g_{ww}, \\
\Phi &= h_w^2 + u^2 g_w^2, \\
\Psi &= h_w^2 + 2u^2 g_w^2, \\
\Theta &= f_v g_w h_v \left(2h_w h_{vw} + h_v h_{ww}\right).
\end{aligned}$$

Corollary 4.1. *Let $\mathfrak{x}$ be a RCH defined by Eq.* (4.1) *in $\mathbb{E}^4$. $\mathfrak{x}$ is 1-minimal iff the following partial differential Eq. appears*

$$\begin{aligned}
&u^2 f_v^3 \Omega \cos^3 g + f_v^3 h_w \Phi \sin g \cos^2 g + 2 f_v g_w^2 h_v^2 h_w \sin g \\
&+ \left(\Theta + h_v \left(g_w f_{vv} \Psi - f_v h_v h_w g_{ww}\right) - f_v g_w h_{vv} \Phi\right) \cos g = 0,
\end{aligned}$$

where $u, \mathcal{W} \neq 0$.

Corollary 4.2. *Let $\mathfrak{x}$ be a RCH determined by Eq.* (4.1) *in $\mathbb{E}_2^5$. $\mathfrak{x}$ is 2-minimal iff the following partial differential Eq. occurs,*

$$\begin{aligned}
&f_v^5 g_w^2 h_w^2 \Phi \cos^6 g - f_v^5 h_w \Phi \Omega \sin g \cos^5 g \\
&+ f_v^3 g_w^2 h_v \left(2 g_w \Phi h_{vw} - h_v h_w \Omega\right) \sin g \cos^3 g \\
&+ f_v^2 g_w (f_v g_w h_{vw}^2 \Phi + g_w^3 f_v h_v^2 \left(2h_w^2 + u^2 g_w^2\right) + h_w g_{ww} \left(h_v f_{vv} \Psi - f_v h_{vv} \Phi\right) \\
&+ g_w h_{ww} \left(-h_v f_{vv} \Psi + f_v h_{vv} \Phi\right)) \cos^4 g + f_v^3 g_w^4 h_v^2 \Phi \sin^2 g \cos^2 g \\
&+ h_v^2 g_w^3 (g_w f_v \left(g_w^2 h_v^2 + h_{vw}^2\right) + \left(1 - 2 h_v f_{vv}\right) \Omega - g_w h_w h_{vw} f_{vv}) \cos^2 g \\
&+ f_v g_w^6 h_v^4 \sin^2 g + h_v^3 g_w^5 \left(2 f_v h_{vw} - h_w f_{vv}\right) \sin g \cos g = 0,
\end{aligned}$$

where $f_v, \mathcal{W} \neq 0$.

Corollary 4.3. *Let $\mathfrak{x}$ be a HRF given by Eq.* (4.1) *in $\mathbb{E}_2^5$. $\mathfrak{x}$ is 3-minimal iff the following partial differential Eq. holds*

$$\begin{aligned}
&h_w^3 f_v^5 \Psi \sin g \cos^4 g \\
&+ f_v^2 (g_w h_v h_w^2 f_{vv} \Psi + f_v \Phi \left(g_w h_w \left(2 h_v h_{vw} - h_w h_{vv}\right) + h_v^2 \Omega\right)) \cos^3 g \\
&+ f_v^3 g_w^2 h_v^2 h_w \left(3 h_w^2 + 2u^2 g_w^2\right) \sin g \cos^2 g \\
&+ h_v^2 g_w^2 (2 f_v g_w h_v h_w h_{vw} + f_v h_v^2 \Omega + h_w^2 g_w \left(h_v f_{vv} - f_v h_{vv}\right)) \cos g \\
&+ 2 f_v g_w^4 h_v^4 h_w \sin g = 0,
\end{aligned}$$

where $f_v, g_w, \cos g, u, \mathcal{W} \neq 0$.

Note that the solutions for h in Corollary 4.1, Corollary 4.2, and Corollary 4.3 are open problems.

5. Right Conoid Hypersurfaces with $\Delta\mathfrak{x} = \mathcal{Q}\mathfrak{x}$ in $\mathbb{E}^4$

In this section, our focus is on the Laplace–Beltrami operator of a smooth function in $\mathbb{E}^4$. We will proceed to compute it utilizing the *RCH* defined by Eq. (4.1).

Definition 5.1. *The Laplace–Beltrami operator of a smooth function* $\phi = \phi(x^1, x^2, x^3) \mid_{\mathcal{D}}$ $(\mathcal{D} \subset \mathbb{R}^3)$ *of class* C^3 *depends on the first fundamental form* $(\mathfrak{g}_{ij})$ *of a hypersurface* $\mathfrak{x}$*, is defined by*

$$\Delta\phi = \frac{1}{\mathbf{g}^{1/2}} \sum_{i,j=1}^{4} \frac{\partial}{\partial x^i} \left(\mathbf{g}^{1/2} \mathfrak{g}^{ij} \frac{\partial \phi}{\partial x^j} \right), \tag{5.1}$$

where $\left(\mathfrak{g}^{ij}\right) = \left(\mathfrak{g}_{kl}\right)^{-1}$ *and* $\mathbf{g} = \det\left(\mathfrak{g}_{ij}\right)$.

Therefore, we give the following.

Theorem 5.1. *The Laplace–Beltrami operator of the RCH* $\mathfrak{x}$ *denoted by Eq.* (4.1) *is given by* $\Delta\mathfrak{x} = 3\mathcal{K}_1\mathcal{G}$*, where* $\mathcal{K}_1$ *describes the mean curvature,* $\mathcal{G}$ *represents the Gauss map of* $\mathfrak{x}$*.*

Proof. The Laplace–Beltrami operator of the *RCH* given by Eq. (4.1) is determined by

$$\begin{aligned} \Delta\mathfrak{x} \;=\; & \frac{1}{\mathbf{g}^{1/2}} \left[\frac{\partial}{\partial u} \left(\mathbf{g}^{1/2} \mathfrak{g}^{11} \frac{\partial \mathfrak{x}}{\partial u} \right) + \frac{\partial}{\partial v} \left(\mathbf{g}^{1/2} \mathfrak{g}^{22} \frac{\partial \mathfrak{x}}{\partial v} \right) + \frac{\partial}{\partial v} \left(\mathbf{g}^{1/2} \mathfrak{g}^{23} \frac{\partial \mathfrak{x}}{\partial w} \right) \right. \\ & \left. + \frac{\partial}{\partial w} \left(\mathbf{g}^{1/2} \mathfrak{g}^{32} \frac{\partial \mathfrak{x}}{\partial v} \right) + \frac{\partial}{\partial w} \left(\mathbf{g}^{1/2} \mathfrak{g}^{33} \frac{\partial \mathfrak{x}}{\partial w} \right) \right], \end{aligned} \tag{5.2}$$

where

$$\begin{aligned} \mathfrak{g}^{11} &= 1, \; \mathfrak{g}^{12} = 0, \; \mathfrak{g}^{13} = 0, \\ \mathfrak{g}^{21} &= 0, \; \mathfrak{g}^{22} = \frac{u^2 g_w^2 + h_w^2}{\mathbf{g}}, \; \mathfrak{g}^{23} = -\frac{h_v h_w}{\mathbf{g}}, \\ \mathfrak{g}^{31} &= 0, \; \mathfrak{g}^{32} = -\frac{h_v h_w}{\mathbf{g}}, \; \mathfrak{g}^{33} = \frac{u^2 f_v^2 \cos^2 g + h_v^2}{\mathbf{g}}, \end{aligned} \tag{5.3}$$

and $\mathbf{g} = u^2 \left(f_v^2 \left(u^2 g_w^2 + h_w^2 \right) \cos^2 g\,(w) + h_v^2 g_w^2 \right)$. By taking the derivatives of the functions determined by Eqs. (5.3) in (5.2), w.r.t. u, v, w, resp., we obtain $\Delta\mathfrak{x} =$

$(\Delta\mathfrak{x}_1, \Delta\mathfrak{x}_2, \Delta\mathfrak{x}_3, \Delta\mathfrak{x}_4)$ with components

$$\begin{aligned}
\Delta\mathfrak{x}_1 &= -\frac{1}{u\mathcal{W}^2}[-u^2 f_v^4 h_w \Omega \cos f \sin g \cos^4 g - f_v^4 h_w^2 \Phi \cos f \sin^2 g \cos^3 g \\
&\quad -u^2 f_v^3 g_w h_v \Omega \sin f \cos^3 g - 2 f_v^2 g_w^2 h_v^2 h_w^2 \cos f \cos g \sin^2 g \\
&\quad -f_v^3 g_w h_v h_w \Phi \sin f \sin g \cos^2 g + f_v h_w \Gamma \cos f \cos^2 g \sin g \\
&\quad -2 f_v g_w^3 h_v^3 h_w \sin f \sin g + h_v g_w \Gamma \sin f \cos g], \\
\Delta\mathfrak{x}_2 &= -\frac{1}{u\mathcal{W}^2}[-u^2 f_v^4 h_w \Omega \sin f \sin g \cos^4 g - h_w^2 f_v^4 \Phi \sin f \sin^2 g \cos^3 g \\
&\quad +u^2 f_v^3 g_w h_v \Omega \cos f \cos^3 g + f_v h_w \Gamma \cos^2 g \sin f \sin g \\
&\quad +f_v^3 g_w h_v h_w \Phi \cos f \cos^2 g \sin g - 2 f_v^2 g_w^2 h_v^2 h_w^2 \sin f \cos g \sin^2 g \\
&\quad -h_v g_w \Gamma \cos f \cos g + 2 f_v g_w^3 h_v^3 h_w \cos f \sin g], \\
\Delta\mathfrak{x}_3 &= -\frac{1}{u\mathcal{W}^2}[f_v h_w (u^2 f_v^3 \Omega \cos^5 g + f_v^3 h_w \Phi \cos^4 g \sin g - \Gamma \cos^3 g \\
&\quad +2 f_v g_w^2 h_v^2 h_w \cos^2 g \sin g)], \\
\Delta\mathfrak{x}_4 &= -\frac{1}{u\mathcal{W}^2}[u f_v g_w (-u^2 f_v^3 \Omega \cos^4 g - f_v^3 h_w \Phi \cos^3 g \sin g \\
&\quad +\Gamma \cos^2 g - 2 f_v g_w^2 h_v^2 h_w \cos g \sin g)].
\end{aligned}$$

□

Definition 5.2. The hypersurface $\mathfrak{x}$ is called harmonic if each componets of $\Delta\mathfrak{x}$ is zero.

Example 5.1. *Substituting $f(v) = v$, $g(w) = w$, $h(v,w) = w$ on a RCH defined by Eq. (4.1) in $\mathbb{E}^4$, we have*

(5.4) $$\mathcal{G} = \frac{1}{(u^2+1)^{1/2}} (-\cos v \sin w, -\sin v \sin w, \cos w, -u),$$

$$\mathcal{S} = \begin{pmatrix} 0 & 0 & \frac{1}{(u^2+1)^{1/2}} \\ 0 & \frac{\tan w}{u(u^2+1)^{1/2}} & 0 \\ \frac{1}{(u^2+1)^{3/2}} & 0 & 0 \end{pmatrix},$$

the principal curvatures are given by $k_1 = \frac{1}{u^2+1} = -k_2$, $k_3 = \frac{\tan w}{u(u^2+1)^{1/2}}$, and the curvatures are determined by

$$\begin{aligned}
\mathcal{K}_1 &= \frac{\tan w}{3u(u^2+1)^{3/2}}, \\
\mathcal{K}_2 &= -\frac{1}{3(u^2+1)^2}, \\
\mathcal{K}_3 &= -\frac{\tan w}{u(u^2+1)^{5/2}}.
\end{aligned}$$

Then,

$$\Delta\mathfrak{x} = \frac{\tan w}{u\left(u^2+1\right)^2}\left(-\cos v\sin w, -\sin v\sin w, \cos w, -u\right).$$

Finally, the hypersurface is non-minimal and non-harmonic.

Example 5.2. *By taking* $f(v)=v$, $g(w)=w$, $h(v,w)=v$ *on a RCH determined by Eq.* (4.1) *in* $\mathbb{E}^4$*, the Gauss map is determined by*

$$\mathcal{G} = \frac{1}{\left(u^2\cos^2 w+1\right)^{1/2}}\left(-\sin v, \cos v, 0, -u\cos w\right).$$

Then, the shape operator matrix is given by

$$\mathcal{S} = \begin{pmatrix} 0 & \frac{\cos w}{\left(u^2\cos^2 w+1\right)^{1/2}} & 0 \\ \frac{\cos w}{\left(u^2\cos^2 w+1\right)^{3/2}} & 0 & -\frac{u\sin w}{\left(u^2\cos^2 w+1\right)^{3/2}} \\ 0 & \frac{\sin w}{u\left(u^2\cos^2 w+1\right)^{1/2}} & 0 \end{pmatrix}.$$

The principal curvatures are determined by $k_1 = \frac{\cos^{1/2}(2w)}{u^2\cos^2 w+1} = -k_2, k_3 = 0$*. The curvatures are described by*

$$\begin{aligned} \mathcal{K}_1 &= 0, \\ \mathcal{K}_2 &= -\frac{\cos 2w}{3\left(u^2\cos^2 w+1\right)^2}, \\ \mathcal{K}_3 &= 0. \end{aligned}$$

Therefore, $\Delta\mathfrak{x} = (0,0,0,0)$*. That is, the hypersurface is* 1*-minimal,* 3*-minimal, and harmonic.*

6. Conclusion

This research has focused on the study of right conoid hypersurfaces in the four-dimensional Euclidean space $\mathbb{E}^4$. The main objective was to analyze and understand the geometric properties of these hypersurfaces.

We computed the matrices associated with the fundamental form, Gauss map, and shape operator of the right conoid hypersurfaces. These matrices provide crucial information about the local geometry of the surfaces, including their curvatures and tangent spaces. By employing the Cayley–Hamilton theorem, the curvatures of these specific hypersurfaces were determined. This theorem allowed for an effective computation of the curvatures by expressing the characteristic polynomial of the matrices in terms of the matrices themselves. Moreover, the research presented the conditions for minimality in the context of right conoid hypersurfaces. These conditions define when a hypersurface can be considered minimal within this specific family. Additionally, the research explored the Laplace–Beltrami operator of the right conoid hypersurfaces.

This research contributes to the understanding of right conoid hypersurfaces in $\mathbb{E}^4$, providing valuable insights into their geometric properties, curvatures, minimality conditions, and their relation to the Laplace–Beltrami operator.

REFERENCES

1. L. J. ALIAS and N. GÜRBÜZ: *An extension of Takashi theorem for the linearized operators of the highest order mean curvatures*. Geom. Dedicata **121** (2006), 113–127.
2. M. BERGER and B. GOSTIAUX: *Géométrie Différentielle: Variétés, Courbes et Surfaces*. Presses Univ. France, France, 1987.
3. B. Y. CHEN, E. GÜLER, Y. YAYLI, and H. H. HACISALIHOĞLU: *Differential geometry of 1-type submanifolds and submanifolds with 1-type Gauss map*. Int. Elec. J. Geom., **16**(1) (2023), 4–49.
4. M. P. DO CARMO: *Differential Geometry of Curves and Surfaces*. Englewood Cliffs, Prentice Hall, USA, 1976.
5. A. GRAY: *Modern Differential Geometry of Curves and Surfaces with Mathematica*. 2nd ed. Boca Raton, FL: CRC Press, 1997.
6. A. KREYSZIG: *Differential Geometry*. Dover Pub., New York, 1991.
7. W. KÜHNEL: *Differential Geometry. Curves-Surfaces-Manifolds*. Third ed. Translated from the 2013 German ed. AMS, Providence, RI, 2015.
8. T. LEVI–CIVITA: *Famiglie di superficie isoparametriche nellordinario spacio euclideo*. Rend. Acad. Lincei **26** (1937), 355–362.