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# ON GENERALIZED STATISTICAL CONVERGENCE OF ORDER $\alpha$ OF FUNCTIONS

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**Abstract.** In this paper, we introduce the concept of generalized statistical convergence of measurable functions of order $\alpha$ for $0 < \alpha \leq 1$ at $\infty$ and at a point $c \in \mathbb{R}$. In addition to this, we defined generalized strongly $p$-Cesàro summability $(0 < p < \infty)$ of a locally integrable function at $\infty$ and at a point $c \in \mathbb{R}$. Using these definitions, we present some basic results.
**Keywords:** statistical convergence, measurable function, $p$-Cesàro summability.

## 1. Introduction

In 1951, Steinhaus [22] and Fast [11] introduced the notion of statistical convergence and later in 1959, Schoenberg [20] reintroduced it independently. Bilalov and Sadigova [3], Caserta et al. [4], Çolak [6], Connor [7], Et et al. [10], Fridy [12], Isik

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and Akbas ([1],[15]), Salat [19], Sadigova et al. ([17], [18]), Şengül et al. ([2],[21]) and many others investigated some arguments related to this notion.

Recently, different approaches to statistical convergence have been made by some authors, namely: Çolak [6] defined the $\alpha$-density of a subset $K$ of $\mathbb{N}$ as follows:

$$\delta_\alpha(K) = \lim_{n\to\infty} \frac{1}{n^\alpha} \left|\{ k \leq n : k \in K\}\right|$$

provided that the limit exists, $\delta_\alpha(K)$ is said to be the $\alpha$-density of a subset $K$, where $\alpha$ is a real number such that $0 < \alpha \leqslant 1$. Also, statistical convergence of order $\alpha$ and strong $p$-Cesàro summability of order $\alpha$ were studied by Çolak [6].

Móricz [16] defined the statistical limit of measurable function at $\infty$ as follows:

$$\lim_{b\to\infty} \frac{1}{b-a} \left|\{a < x < b : |f(x) - \ell| > \varepsilon\}\right| = 0.$$

Gökhan et al. ([13],[14]) introduced the definition of pointwise and uniform statistical convergence of sequences of real valued functions and Duman and Orhan [8] studied independently. Then, Çınar et al. [5] defined pointwise and uniform statistical convergence of order $\alpha$ for sequences of functions and pointwise $\lambda$ and lacunary statistical convergence of order $\alpha$ for sequences of functions were introduced by Et et al. [9].

## 2. Main Results

Let's begin our work by introducing some new definitions.

A closed interval in $\mathbb{R}^m$ is given by $I(a,b) = \{x = (x_1, x_2, ..., x_m) : a_i \leqslant x_i \leqslant b_i\}$ where $a = (a_1, ..., a_m)$ and $b = (b_1, ..., b_m)$. Let $(I^m(a,\infty), \mathcal{B}(\mathbb{R}^m))$ be a measurable space where $\mathcal{B}$ is Borel $\sigma$-algebra where $I^m(a,\infty) = \{(x_1, x_2, ..., x_m) : a_i \leq x_i, i = 1, 2, ..., m\}$ and $\mu : I(a,\infty) \to [0,\infty]$ be a measure function.

**Definition 2.1.** Let $0 < \alpha \leqslant 1$ and $K \subset I\,(a,\infty)$ be a measurable function. We can define the generalized density of $K$ on $I\,(a,\infty)$ of order $\alpha$ at infinity as follows:

$$\delta^\alpha(K) = \lim_{\min b_i\to\infty} \frac{\mu(\{I(a,b) \cap K\})}{\left(\mu(I(a,b))\right)^\alpha}$$

where $\mu(I(a,b)) = \prod_{i=1}^{m} (b_i - a_i)$.

The generalized density of order $\alpha$ is well defined for $0 < \alpha \leqslant 1$. If $\alpha > 1$, then we can see that the generalized density of every subset of $I$ is zero.

Let $K$ be an arbitrary subset of $I(a,\infty)$ such that $K=\prod_{i=1}^{m} K_i$ where $K_i$ is a subset of $(a_i,\infty)$, $i=1,2,3,...,m$. For any $b=(b_1,b_2,...,b_m)\subset\mathbb{R}^m$, it is clear that

$$K\cap I(a,b)=\prod_{i=1}^{m} K_i\cap(a_i,b_i)$$

holds. This implies that

$$\mu\left(K\cap I(a,b)\right)=\prod_{i=1}^{m}\mu\left(K_i\cap(a_i,b_i)\right). \tag{2.1}$$

So, we can give the following fact:

**Lemma 2.1.** *Let $K\subset I(a,\infty)$ such that $K=\prod_{i=1}^{m} K_i$. If $\mu_\alpha-$density of $K_i$, exists for all $i=1,2,3,...,m$, then*

$$\delta^\alpha(K)=\prod_{i=1}^{m}\delta^\alpha(K_i).$$

*Proof.* From (2.1), we have

$$\frac{\mu\left(K\cap I(a,b)\right)}{\mu\left(I(a,b)\right)^\alpha}=\frac{\prod_{i=1}^{m}\mu\left(K_i\cap(a_i,b_i)\right)}{\mu\left(I(a,b)\right)^\alpha}$$

$$=\frac{\mu\left(K_1\cap(a_1,b_1)\right)}{\mu\left((a_1,b_1)\right)^\alpha}\cdots\frac{\mu\left(K_m\cap(a_m,b_m)\right)}{\mu\left((a_m,b_m)\right)^\alpha}.$$

Hence, by taking limit we get

$$\delta^\alpha(K)=\prod_{i=1}^{m}\delta^\alpha(K_i).$$

□

**Corollary 2.1.** *If $\delta^\alpha(K_i)=0$ for any $i=1,2,3,...,m$, then $\delta^\alpha(K)=0$ holds.*

**Corollary 2.2.** *If $\delta^\alpha(K)=0$, then $\delta^\alpha(K_i)$ exists for all $i=1,2,3,...,m$. Furthermore $\exists i\in\{1,2,3,...,m\}$, $\delta^\alpha(K_i)=0$.*

**Remark 2.1.** If $\delta^\alpha(K_i)$ does not exist for any $i\in\{1,2,3,...,m\}$ and $\delta^\alpha(K_j)\neq 0$ for all $j\in\{1,2,3,...,m\}\setminus\{i\}$, then $\delta^\alpha(K)$ does not exist.

*Proof.* Without loss of generality let $K_1$ and $K_2$ such that $\delta^\alpha(K_1)$ does not exist, $\delta^\alpha(K_2)$ exists but is not zero. Since $0 \leqslant \frac{\mu(K_1 \cap I(a,b))}{\mu(I(a,b))} \leqslant 1$, then

$$\limsup_{b_1\to\infty} \frac{\mu\left(K_1 \cap (a_1,b_1)\right)}{\mu\left((a_1,b_1)\right)} \neq \liminf_{b_1\to\infty} \frac{\mu\left(K_1 \cap (a_1,b_1)\right)}{\mu\left((a_1,b_1)\right)}.$$

Therefore,

$$\limsup_{\min b_i\to\infty} \frac{\mu\left(K \cap I(a,b)\right)}{\mu\left(I(a_1,b_1)\right)} \neq \liminf_{\min b_i\to\infty} \frac{\mu\left(K \cap I(a,b)\right)}{\mu\left(I(a,b)\right)}$$

holds. This completes the proof. $\square$

**Definition 2.2.** Let $0 < \alpha \leqslant 1$ and $f : I(a,\infty) \to \mathbb{R}$ be a measurable function. $f$ is said to be generalized statistically convergent to $\ell$ with order $\alpha$ at $\infty$, if for every $\varepsilon > 0$, the following limit

$$\lim_{\min\left\{b_i : i=\overline{1,2,3,\ldots m}\right\}\to\infty} \frac{1}{\left(\mu(I(a,b))\right)^\alpha} \mu\left(\{x \in I(a,b) : |f(x) - \ell| > \varepsilon\}\right) = 0,$$

exists. It is denoted by $st^\alpha - \lim_{x\to\infty} f(x) = \ell$. The set of all generalized statistically convergent function of order $\alpha$ at $\infty$ is denoted by $S^\alpha(I)$.

Throughout the paper we shall assume that $\mu(I(a,b)) = \prod_{i=1}^{m} (b_i - a_i)$ and $f : I(a,\infty) \to \mathbb{R}$ be a measurable function (in Lebesgue's sense).

**Remark 2.2.** If $\lim_{x\to\infty} f(x) = \ell$ then $st^\alpha - \lim_{x\to\infty} f(x) = \ell$ holds. So, $S^\alpha(I) \neq \emptyset$.

**Remark 2.3.** There exists a function $f \in S^\alpha(I)$ such that $f$ has no classical limit.

**Example 2.1.** Let $f : I(a,\infty) \to \mathbb{R}$ be a function and $b = (b_1, b_2, \ldots, b_m) \subset I(a,\infty)$ be an arbitrary points. Define the following function

$$f(x) = \begin{cases} 1, & \text{if } b_i - \sqrt{b_i} < x_i < b_i,\ i = \overline{1,2,3,\ldots m}, \\ 0, & \text{otherwise} \end{cases}.$$

Hence,

$$\begin{aligned} &\lim_{\min b_i\to\infty} \frac{\mu\left(\{x \in I(a,b) : |f(x) - 0| > \varepsilon\}\right)}{\left(\mu(I(a,b))\right)^\alpha} \\ = &\lim_{\min b_i\to\infty} \frac{\sqrt{b_1}\sqrt{b_2}\ldots\sqrt{b_m}}{\prod_{i=1}^{m} (b_i - a_i)^\alpha} = 0 \end{aligned}$$

holds for every $\varepsilon > 0$ and $\alpha \in (\frac{1}{2}, 1]$. So, $f$ is generalized statistically convergent to zero with of order $\alpha$, but not a convergent function.

**Theorem 2.1.** *Let $0 < \alpha \leqslant 1$ and $f$ be a measurable function and $c \in \mathbb{R}$. If $st^{\alpha} - \lim f(x) = \ell_1$ and $st^{\alpha} - \lim g(x) = \ell_2$, then*

*(i) $st^{\alpha} - \lim_{x\to\infty} cf(x) = c\ell_1$,*

*(ii) $st^{\alpha} - \lim_{x\to\infty}(f(x) + g(x)) = \ell_1 + \ell_2$.*

*Proof.* (i) The proof is evident when $c = 0$. Assume that $c \neq 0$, then we have the proof (i) follows from

$$\frac{1}{(\mu(I(a,b)))^{\alpha}}\mu\left(\{x \in I(a,b) : |cf(x) - \ell_1| > \varepsilon\}\right)$$

$$= \frac{1}{(\mu(I(a,b)))^{\alpha}}\mu\left(\left\{x \in I(a,b) : |f(x) - \ell_1| > \frac{\varepsilon}{|c|}\right\}\right).$$

(ii) For all $b_i > a_i$ and $\varepsilon > 0$, we have

$$\frac{1}{(\mu(I(a,b)))^{\alpha}}\mu\left(\{x \in I(a,b) : |f(x) + g(x) - (\ell_1 + \ell_2)| > \varepsilon\}\right)$$

$$\leqslant \frac{1}{(\mu(I(a,b)))^{\alpha}}\mu\left(\left\{x \in I(a,b) : |f(x) - \ell_1| > \frac{\varepsilon}{2}\right\}\right)$$

$$+\frac{1}{(\mu(I(a,b)))^{\alpha}}\mu\left(\left\{x \in I(a,b) : |g(x) - \ell_2| > \frac{\varepsilon}{2}\right\}\right).$$

This proves the proof. □

**Theorem 2.2.** *Let $0 < \alpha \leqslant 1$ and $f$ be a measurable function. If for each $x \in I(a,b)$, $st^{\alpha} - \lim_{x\to\infty} f(x) = \ell_1$ and $st^{\alpha} - \lim_{x\to\infty} f(x) = \ell_2$, then $\ell_1 = \ell_2$.*

*Proof.* Omitted. □

**Corollary 2.3.** *$S^{\alpha}(I)$ is a real vector space for all $0 < \alpha \leqslant 1$.*

**Theorem 2.3.** *Let $0 < \alpha \leqslant \beta \leqslant 1$ and $f$ be a measurable function. Then $S^{\alpha}(I) \subseteq S^{\beta}(I)$ and this inclusion is strict for some $\alpha$ and $\beta$ such that $\alpha < \beta$.*

*Proof.* Omitted. □

To show that the inclusion is strict, consider the sequence in Example 2.1.

**Definition 2.3.** Let $0 < \alpha \leqslant 1$ and $f$ be a measurable function. $f$ is said to be generalized statistically Cauchy function of order $\alpha$ in a neighborhood of $\infty$, if there exists an element $s = (s_1, s_2, ...s_m) \in I(a, \infty)$ and $\min s_i > \min a_i$ such that

$$\lim_{\min b_i \to \infty} \frac{1}{(\mu(I(a,b)))^\alpha} \mu\left(\{x \in I(a,b) : |f(x) - f(s)| > \varepsilon\}\right) = 0, \tag{2.2}$$

holds for every $\varepsilon > 0$.

**Theorem 2.4.** *Let $0 < \alpha \leqslant 1$ and $f$ be a measurable function. Then the following statements are equivalent:*

*(i) $f$ is generalized statistically convergent with order $\alpha$.*

*(ii) $f$ is a generalized statistically Cauchy with order $\alpha$.*

*(iii) $f$ can be represented as the sum of two measurable functions $g$ and $h$, such that*

$$\lim_{x \to \infty} g(x) = st^\alpha - \lim_{x \to \infty} f(x) \tag{2.3}$$

*and*

$$\lim_{\min b_i \to \infty} \frac{1}{(\mu(I(a,b)))^\alpha} \mu\left(\{x \in I(a,b) : h(x) \neq 0\}\right) = 0, \quad i = 1, 2, 3, ...m. \tag{2.4}$$

*Furthermore, in case $f$ is bounded, then both $g$ and $h$ are also bounded.*

*Proof.* "$(i) \Rightarrow (ii)$" : Assume that the function $f$ is generalized statistically convergent to $\ell$ with $\alpha$. Let $\varepsilon > 0$, by the definition of generalized statistically convergent we have

$$|f(x) - \ell| < \frac{\varepsilon}{2} \tag{2.5}$$

for almost all $x \in I(a, b)$. Let us choose one of $x \in I(a, b)$ satisfying (2.5) and denoted it by $s$. Then, for every $a_i < x_i \leqslant b_i$,$i = \overline{1, 2, 3, ...m}$, following inequality

$$|f(x) - f(s)| \leqslant |f(x) - \ell| + |\ell - f(s)| \leqslant |f(x) - \ell| + \frac{\varepsilon}{2}$$

holds for almost all $x \in I(a, b)$. Then, following inclusion

$$\{x \in I(a,b) : |f(x) - f(s)| > \varepsilon\} \subseteq \left\{x \in I(a,b) : |f(x) - \ell| > \frac{\varepsilon}{2}\right\}$$

holds and implies that $f$ is a generalized statistically Cauchy with order $\alpha$.

"$(ii) \Rightarrow (iii)$" : If $K$ and $J$ are two intervals such that each of them contains the function value $f(x)$ for almost all $x \in I(a, b)$, then so does their intersection $K \cap J$. Now we apply (2.1) with $\varepsilon_1 = 1/2$. It can be concluded that the interval $K = [f(s) - 1/2, f(s) + 1/2]$ contains the function value $f(x)$ for almost all $x \in$

$I(a,b)$. Then, we apply (2.1) with $\varepsilon_2 = 1/4$ to obtain some $t > a$( means $t_i > a_i$ for each $i = 1, 2, ..., m$) such that the interval $J := [f(t) - 1/4, f(t) + 1/4]$ contains the function value $f(x)$ for almost all $x \in I(a,b)$. Considering the above observation, the interval $K_1 := K \cap J$ also contains the function value $f(x)$ for almost all $x$. Plainly, $K_1$ is a $K_1 \subset \mathbb{R}$ closed interval whose length $|K_1| \leqslant 1/2$.

Next, we apply (2.1) with $\varepsilon_3 = 1/8$ to obtain some $r > a$ such that the interval $J_1 = [f(r) - 1/8, f(r) + 1/8]$ contains the function value $f(x)$ for almost all $x \in I(a,b)$. Hence we have $K_2 = K_1 \cap J_1$ contains the function value $f(x)$ for almost all $x \in I(a,b)$. Also, $K_2$ is a closed interval whose length $|K_2| \leqslant 1/4$.

By induction, we construct a sequence $\{K_n : n = 1, 2, ...\}$ of closed intervals such that $K_n \supseteq K_{n+1}$, $|K_n| \leqslant 2^{-n}$ and $K_m$ contains the function value $f(x)$ for almost all $x \in I(a,b)$. Using the Nested Intervals Theorem, there is a unique number $\ell$ such that $\ell \in K_n$ for each $n \geqslant 1$.

Moreover, we can select numbers $(a_i <)c_1 < c_2 < ...$ such that $c_n \to \infty$ as $n \to \infty$ and

$$\frac{1}{(\mu(I(a,b)))^{\alpha}} \mu\left(\{x \in I(a,b) : f(x) \notin K_n\}\right) < 1/n, \tag{2.6}$$

if $\quad b_i > c_n \quad i = 1, 2, 3, ...m.$

This makes the definition of the functions $g$ and $h$ plausible. For $a_i < x_i \leqslant c_1$, we set

$$g(x) = f(x) \quad \text{and} \quad h(x) := 0.$$

If $x_i > c_1$, then $c_n < x_i \leqslant c_{n+1}$ for some $n \geqslant 1$, and we set

$$g(x) = \begin{cases} f(x), & \text{if } f(x) \in K_n \\ \ell, & \text{otherwise} \end{cases}$$

and $h = f - g$.

It is clear that $g$ and $h$ are measurable functions and $f = g + h$. We want to show that the ordinary limit of $g(x)$ exists as $x \to \infty$ and equals $\ell$. For this, let an arbitrary $\varepsilon > 0$ such that $\varepsilon < 1$, and choose a positive integer $t$ such that $2^{-t} < \varepsilon$. If $x_i > c_t$, then $c_n < x_i \leqslant c_{n+1}$ for some $n \geqslant t$. If $f(x) \in K_n$, then $g(x) = f(x)$ and by definition,

$$|g(x) - \ell| \leqslant |K_n| \leqslant 2^{-n} \leqslant 2^{-t} < \varepsilon \tag{2.7}$$

while if $x \notin K_n$, then $g(x) := \ell$. Then, (2.7) holds for all $x_i > c_t$. Thus we have (2.3).

By definition $h(x) \neq 0$ if and only if $g(x) \neq f(x)$. It means that if $c_n < b_i \leqslant c_{n+1}$ for some $n \geqslant 1$, then by definition,

$$\{x \in I(a,b) : h(x) \neq 0\} = \bigcup_{t=1}^{n-1} \{c_t < x_i \leqslant c_{t+1} : f(x) \notin K_t\}$$

$$\cup\{c_n < x_i \leqslant b_i : f(x) \notin K_n\} \subseteq \{a < x \leqslant b : f(x) \notin K_n\}.$$

By (2.6), we have

$$\frac{1}{(\mu(I(a,b)))^{\alpha}}\mu\left(\{x \in I(a,b) : h(x) \neq 0\}\right) < 1/n, \quad \text{for} \quad c_n < b_i \leqslant c_{n+1} \quad n = 1, 2, 3, ....$$

Since $n \to \infty$ as $b \to \infty$, this proves (2.4).

If there exists $C > 0$ such that $|f(x)| \leqslant C$ for all $x$, then $f$ is bounded. Then by definitions of $g$ and $h$, we have

$$|g(x)| \leqslant \max\{C, \ell\} \quad \text{and} \quad |h(x)| \leqslant C + \ell,$$

for all $x \in I(a,b)$ with $x_i \in (a_i, \infty)$, $i = \overline{1, 2, 3, ...m}$.

"$(iii) \Rightarrow (i)$" : This implication is valid under even the weaker assumptions that $f = g + h$,

$$st^{\alpha} - \lim_{x \to \infty} g(x) = \ell \quad \text{and} \quad st^{\alpha} - \lim_{x \to \infty} h(x) = 0,$$

because of the additivity property of generalized statistical limit. □

## 3. Strong Cesàro summability at $\infty$

In this section, we are going to define generalized strongly $p-$Cesaro summable of order $\alpha$ at $\infty$ of multi variable measurable functions. Denote $dx = dx_1 dx_2 ... dx_m$ and $\int\limits_{I(a,b)} := \int_{a_1}^{b_1} \int_{a_2}^{b_2} ... \int_{a_m}^{b_m}$.

**Definition 3.1.** Let $f$ be a measurable (in Lebesgue's sense) function on some interval $(a_i, \infty)$, where $a_i > 0$ for $i = 1, 2, ..m$ and $0 < \alpha \leq 1$. Then it is said that $f$ is generalized strongly $p-$Cesaro summable of order $\alpha$ at $\infty$ if there exists $\ell \in \mathbb{C}$ such that

$$\lim_{\min b_i \to \infty} \frac{1}{(\mu(I(a,b)))^{\alpha}} \int\limits_{I(a,b)} |f(x) - \ell|^p \, dx = 0$$

where $|f(x) - \ell|^p$ is locally integrable (in Lebesgue's sense) over the interval $(a_i, \infty)$.

**Theorem 3.1.** *(i) If $f$ is generalized strongly p-Cesàro summable with order $\alpha$ at $\infty$ to $\ell \in \mathbb{C}$ for some $0 < p < \infty$, then the generalized statistical limit with order $\alpha$ of $f$ at $\infty$ exists and equals the same $\ell$.*

*(ii) If the generalized statistical limit with order $\alpha$ of $f$ at $\infty$ exists and equals $\ell \in \mathbb{C}$, and $f$ is bounded, then $f$ is generalized strongly p-Cesàro summable at $\infty$ to the same $\ell$ for every $0 < p < \infty$.*

*Proof.* $(i)$ Let $f$ be a generalized strongly $p-$Cesàro summable function to $\ell$ with order $\alpha$. For a given $\varepsilon > 0$, we can easily get

$$\int\limits_{I(a,b)} |f(x) - \ell|^p \, dx \geq \varepsilon^p \mu(\{x \in I(a,b) : |f(x) - \ell| > \varepsilon\})$$

from Markov's inequality for all $0 < p < \infty$. Dividing both sides of the above inequality by $(\mu(I(a,b)))^{\alpha}$ and taking the limit as $\min b_i \to \infty$, we obtain

$$\begin{aligned} &\lim_{\min b_i \to \infty} \frac{1}{(\mu(I(a,b)))^{\alpha}} \mu(\{x \in I(a,b) : |f(x) - \ell| > \varepsilon\}) \\ \leq\; & \frac{1}{\varepsilon^p} \lim_{\min b_i \to \infty} \frac{1}{(\mu(I(a,b)))^{\alpha}} \int\limits_{I(a,b)} |f(x) - \ell|^p \, dx = 0. \end{aligned}$$

Thus, $f$ is generalized statistically convergent function of order $\alpha$ at $\infty$ to $\ell \in \mathbb{C}$.

$(ii)$ Assume $|f(x)| \leq B$ for all $x$ and $f$ is a generalized statistically convergent function of order $\alpha$ at $\infty$ to $\ell \in \mathbb{C}$. We can write

$$\lim_{\min b_i \to \infty} \frac{1}{(\mu(I(a,b)))^{\alpha}} \mu(\{x \in I(a,b) : |f(x) - \ell| > \varepsilon\}) = 0.$$

Then

$$\begin{aligned} &\int\limits_{I(a,b)} |f(x) - \ell|^p \, dx \\ =\; & \left\{ \begin{array}{c} \left[\int_{\{a_1 \leq x_1 \leq b_1 : |f(x)-\ell| \leq \varepsilon\}} + \int_{\{a_1 \leq x_1 \leq b_1 : |f(x)-\ell| > \varepsilon\}}\right] \\ \left[\int_{\{a_2 \leq x_2 \leq b_2 : |f(x)-\ell| \leq \varepsilon\}} + \int_{\{a_2 \leq x_2 \leq b_2 : |f(x)-\ell| > \varepsilon\}}\right] \\ \ldots \left[\int_{\{a_m \leq x_m \leq b_m : |f(x)-\ell| \leq \varepsilon\}} + \int_{\{a_m \leq x_m \leq b_m : |f(x)-\ell| > \varepsilon\}}\right] \end{array} \right\} \\ & |f(x) - \ell|^p \, dx \\ \leq\; & \prod_{i=1}^{m} (b_i - a_i) \, \varepsilon^p + (B + |\ell|)^p \, \mu(\{x \in I(a,b) : |f(x) - \ell| > \varepsilon\}). \end{aligned}$$

We conclude that

$$\lim_{\min b_i \to \infty} \frac{1}{\mu(I(a,b))} \int\limits_{I(a,b)} |f(x) - \ell|^p \, dx \leq \varepsilon^p.$$

□

## 4. Statistical limit inferior and superior at $\infty$

In this Section, we consider the real-valued measurable function $f$ on some interval $(a_i, \infty)$, where $a_i \geq 0$. $A(f)$ will denote the set of those $u \in \mathbb{R}$ such that for $0 < \alpha \leq 1$

$$\lim_{\min b_i \to \infty} \frac{1}{(\mu(I(a,b)))^\alpha} \mu(\{x \in I(a,b) : f(x) < u\}) \neq 0$$

holds, by which we mean that either this limit does not exist at all, or it does exist but positive. Now, the generalized statistical limit inferior of order $\alpha$ of the function $f$ at $\infty$ is defined by

$$st^\alpha - \liminf_{x \to \infty} f(x) = \inf A(f),$$

provided $A(f)$ is not empty; otherwise we set $st^\alpha - \liminf_{x\to\infty} f(x) = \infty$.

Similarly, the statistical limit superior of order $\alpha$ of the function $f$ at $\infty$ defined by $st^\alpha$-$\limsup_{x\to\infty} f(x) = \sup B(f)$. $B(f)$ will denote the set of those $v \in \mathbb{R}$ for which

$$\lim_{\min b_i \to \infty} \frac{1}{(\mu(I(a,b)))^\alpha} \mu(\{x \in I(a,b) : f(x) > v\}) \neq 0.$$

Here are some basic features of these concepts.

$(i)$ $\liminf_{x\to\infty} f(x) \leq st^\alpha - \liminf_{x\to\infty} f(x) \leq st^\alpha - \limsup_{x\to\infty} f(x) \leq \limsup_{x\to\infty} f(x)$

$(ii)$ If $u = st^\alpha - \liminf_{x\to\infty} f(x)$ is finite, then for every $\varepsilon > 0$,

$$\lim_{\min b_i \to \infty} \frac{1}{(\mu(I(a,b)))^\alpha} \mu(\{x \in I(a,b) : f(x) < u + \varepsilon\}) \neq 0$$

and

$$\lim_{\min b_i \to \infty} \frac{1}{(\mu(I(a,b)))^\alpha} \mu(\{x \in I(a,b) : f(x) < u - \varepsilon\}) = 0.$$

Conversely, if the two relations mentioned above hold for every $\varepsilon > 0$, then $u = st^\alpha - \liminf_{x\to\infty} f(x)$.

$(iii)$ A function $f$ is said to be statistically bounded if for some $C \in \mathbb{R}$,

$$\lim_{\min b_i \to \infty} \frac{1}{(\mu(I(a,b)))^\alpha} \mu(\{x \in I(a,b) : |f(x)| > C\}) = 0$$

holds. If $f$ is statistically bounded, then $st^{\alpha}-\lim_{x\to\infty} f(x)=\ell$ exists if and only if

$$st^{\alpha}-\lim\inf_{x\to\infty} f(x)=st^{\alpha}-\limsup_{x\to\infty} f(x)=\ell.$$

$(iv)$The relation $st^{\alpha}-\liminf_{x\to\infty} f(x)=\infty$ is equivalent to the following one: for every $C\in\mathbb{R}$

$$\lim_{\min b_i\to\infty}\frac{1}{(\mu(I(a,b)))^{\alpha}}\mu(\{x\in I(a,b):f(x)<C\})=0,$$

holds.

We denote it by $st^{\alpha}-\lim_{x\to\infty} f(x)=\infty$ . The symmetric counterpart $st^{\alpha}-\lim_{x\to\infty} f(x)=-\infty$ is meant analogously.

$(v)$ $st^{\alpha}-\limsup_{x\to\infty} f(x)=-st^{\alpha}-\liminf_{x\to\infty}(-f(x))$ .

$(vi)$ A function $f$ is said to be statistically positive if

$$\lim_{\min b_i\to\infty}\frac{1}{(\mu(I(a,b)))^{\alpha}}\mu(\{x\in I(a,b):f(x)\le 0\})=0.$$

If $f$ is statistically positive, then

$$st^{\alpha}-\limsup_{x\to\infty} f(x)=1/st^{\alpha}-\liminf_{x\to\infty}(1/f(x))$$

with the agreements that $1/0=\infty$ and $1/\infty=0$.

$(vii)$ $st^{\alpha}-\liminf_{x\to\infty}(f_1(x)+f_2(x))\ge st^{\alpha}-\liminf_{x\to\infty} f_1(x)+st^{\alpha}-\liminf_{x\to\infty} f_2(x)$.

$(viii)$ However, if the statistical limit $st^{\alpha}-\lim_{x\to\infty} f_1(x)$ exists, then

$$st^{\alpha}-\liminf_{x\to\infty}(f_1(x)+f_2(x))=st^{\alpha}-\lim_{x\to\infty} f_1(x)+st^{\alpha}-\liminf_{x\to\infty} f_2(x).$$

Similarly, statements are valid for "$st^{\alpha}-\limsup$" in place of "$st^{\alpha}-\liminf$". We formulate only the counterpart of $(ii)$ as follows:

$(ii')$ If $v=st^{\alpha}-\limsup_{x\to\infty} f(x)$ is finite, then for every $\varepsilon>0$,

$$\lim_{\min b_i\to\infty}\frac{1}{(\mu(I(a,b)))^{\alpha}}\mu(\{x\in I(a,b):f(x)>v-\varepsilon\})\ne 0$$

and

$$\lim_{\min b_i\to\infty}\frac{1}{(\mu(I(a,b)))^{\alpha}}\mu(\{x\in I(a,b):f(x)>v+\varepsilon\})=0.$$

Conversely, if the two relations mentioned above hold for every $\varepsilon>0$, then $v=st^{\alpha}-\limsup_{x\to\infty} f(x)$.

**Theorem 4.1.** $st^{\alpha}-\lim f(x)=\ell \iff st^{\alpha}-\liminf f(x)=st^{\alpha}-\limsup f(x)$.

*Proof.* Assume that $st^{\alpha}-\lim f(x)=\ell$ holds. So $\alpha-$statistical density of

$$A(\varepsilon)=\{x\in I(a,\infty):|f(x)-\ell|\geq\varepsilon\} \tag{4.1}$$

is zero for every $\varepsilon>0$. We have

$$\{x\in I(a,\infty):|f(x)-\ell|\geq\varepsilon\}=\{x\in I(a,\infty):f(x)\geq\varepsilon+\ell\}\cup\{x\in I(a,\infty):f(x)<\ell-\varepsilon\}.$$

Then, from (4.1) we have

$$\delta^{\alpha}(\{x\in I(a,\infty):f(x)\geq\varepsilon+\ell\})=0 \tag{4.2}$$

and

$$\delta^{\alpha}(\{x\in I(a,\infty):f(x)<\ell-\varepsilon\})=0 \tag{4.3}$$

hold. Equation (4.1) also implies that

$$\delta^{\alpha}(\{x\in I(a,\infty):|f(x)-\ell|<\varepsilon\})=\delta^{\alpha}(I(a,\infty)) \tag{4.4}$$

and we have

$$\delta^{\alpha}(\{x\in I(a,\infty):f(x)<\varepsilon+\ell\})=\delta^{\alpha}(I(a,\infty)) \tag{4.5}$$

and

$$\delta^{\alpha}(\{x\in I(a,\infty):f(x)\geq\ell-\varepsilon\})=\delta^{\alpha}(I(a,\infty)). \tag{4.6}$$

So, (4.2)-(4.6) implies that $\varepsilon+\ell\in A(f)$, $\ell-\varepsilon\in B(f)$ for all $\varepsilon>0$. Then, $st^{\alpha}-\liminf f(x)=\ell$ and $st^{\alpha}-\limsup f(x)=\ell$.

Now assume that $st^{\alpha}-\liminf f(x)=st^{\alpha}-\limsup f(x)=\ell$ holds. From this assumption, we have

$$\inf A(f)=\sup B(f)=\ell.$$

So, for every $\varepsilon>0$, there exists $\ell'\in A(f)$ and $\ell''\in B(f)$ such that $\ell-\varepsilon<\ell''$, $\ell'<\varepsilon+\ell$ satisfied. Then,

$$\delta^{\alpha}(\{x\in I(a,\infty):f(x)\geq\varepsilon+\ell\})\leq\delta^{\alpha}(\{x\in I(a,\infty):f(x)>\ell'\})=0 \tag{4.7}$$

and

$$\delta^{\alpha}(\{x\in I(a,\infty):f(x)<\ell-\varepsilon\})\leq\delta^{\alpha}(\{x\in I(a,\infty):f(x)<\ell''\})=0. \tag{4.8}$$

From (4.7) and (4.8), the following equality

$$\delta^{\alpha}(\{x\in I(a,\infty):|f(x)-\ell|\geq\varepsilon\})=\delta^{\alpha}(\{x\in I(a,\infty):f(x)\geq\varepsilon+\ell\})$$
$$+\delta^{\alpha}(\{x\in I(a,\infty):f(x)<\ell-\varepsilon\})$$

is satisfied. So, this fact completes the proof of theorem. □

## 5. Statistical Limit at $c \in \mathbb{R}^n$

Let $A \subset \mathbb{R}^m$ for $m \geq 1$ and $A$ be a Lebesgue measurable subset of $\mathbb{R}^m$. Let us define the approximate density of $A$ in the $r$ neighborhood of an $x$ point in $\mathbb{R}^m$ as follows

$$d_r(x) = \frac{\mu(A \cap B_r(x))}{\mu(B_r(x))}$$

where $B_r(x)$ denotes the closed ball of radius $r$ centered at $x$.

According to Lebesgue's density theorem, the density of almost every point $x$ of $A$ is

$$d(x) = \lim_{r \to 0} d_r(x)$$

exists and is equal to 0 or 1.

**Definition 5.1.** The Lebesgue measurable function $f$ is called statistically convergent of order $\alpha$ ($\alpha \in (0, 1]$) at $c$ if there exists some $\ell \in \mathbb{C}$ such that for every $\varepsilon > 0$

$$\lim_{\substack{\mu(I(a,b)) \to 0 \\ I(a,b) \ni c}} \frac{1}{(\mu(I(a,b)))^\alpha} \mu(\{x \in I(a,b) \subset \mathbb{R}^m : |f(x) - \ell| \geq \varepsilon\}) = 0.$$

It is denoted by $st^\alpha - \lim_{x \to c} f(x) = \ell$. Clearly $\ell$ is unique.

**Definition 5.2.** The Lebesgue measurable function $f$ is called statistically Cauchy of order $\alpha$ ($\alpha \in (0, 1]$) at $c$ if there exists some $\ell \in \mathbb{C}$ such that for every $\varepsilon > 0$

$$\lim_{\substack{\mu(I(a,b)) \to 0 \\ I(a,b) \ni c}} \frac{1}{(\mu(I(a,b)))^\alpha} \mu(\{x \in I(a,b) \subset \mathbb{R}^m : |f(x) - f(x_0)| \geq \varepsilon\}) = 0.$$

**Theorem 5.1.** *The following statements are equivalent:*

*i) $f$ has a statistical limit of order $\alpha$ at $c \in \mathbb{R}$,*

*ii) $f$ is Cauchy of order $\alpha$ in a neighborhood of $c$,*

*iii) $f$ can be represented in a neighborhood of $c$ as the sum of two measurable functions $g$ an $h$ in the same neighborhood of $c$ such that $\lim_{x \to c} g(x) = st^\alpha - \lim_{x \to c} f(x)$ and $\lim_{\mu(I(a,b)) \to 0} {}_{I(a,b) \ni c} \frac{1}{(\mu(I(a,b)))^\alpha} \mu(\{x \in I(a,b) \subset \mathbb{R}^m : h(x) \neq 0\}) = 0.$*

*Furthermore, in case $f$ is bounded in a neighborhood of $c$, then both $g$ and $h$ are also bounded there.*

*Proof.* The proof is similar to that of Theorem 2.4. □

**Definition 5.3.** Let $0 < p < \infty$. $f$ is said to be strongly $p$- Cesàro summable of order $\alpha$ at $c \in \mathbb{R}^m$ if there exists a number $\ell \in \mathbb{C}$ such that

$$\lim_{\substack{\mu(I(a,b))\to 0\\ I(a,b)\ni c}} \frac{1}{(\mu(I(a,b)))^\alpha} \int_{I(a,b)} |f(x)-\ell|^p dx = 0.$$

We accept that the function $|f(x)-\ell|^p$ is integrable in the neighborhood of c, although it is not written here.

**Theorem 5.2.** *Let $0 < p < \infty$. If $f$ at $c \in \mathbb{R}^m$ to $\ell \in \mathbb{C}$ for some $p$ is strongly $p$-Cesàro summable of order $\alpha$, then the statistical limit of $f$ at $c$ exists and equals the same $\ell$.*

*Proof.* The proof is similar to that of the Theorem 3.2. □

## 6. Statistical limit inferior and superior at $c \in \mathbb{R}^m$

In this section, we will consider functions that can be measured in the neighborhood of point $c \in \mathbb{R}$. This time, we indicate the set of those $u \in \mathbb{R}$ for which $A_x(f)$

$$\lim_{\substack{\mu(I(a,b))\to 0\\ I(a,b)\ni c}} \frac{1}{(\mu(I(a,b)))^\alpha} \mu(\{x \in I(a,b) \subset \mathbb{R}^m : f(x) < u\}) \neq 0.$$

If $A_x(f)$ is not empty, $st^\alpha - \liminf_{x\to c} f(x) = \inf A_x(f)$ is the statistical limit inferior of $f$ at $c$ otherwise we set $st^\alpha - \liminf_{x\to c} f(x) := \infty$.

Let $B_x(f)$ denote the set

$$B_x(f) = \{v \in \mathbb{R} : \delta_\mu(x \in I : f(x) > v) \neq 0\}.$$

The statistical limit superior of order $\alpha$ of $f$ at point $c$ is given by

$$st^\alpha - \limsup f = \begin{cases} \sup B_x(f), & \text{if } B_x(f) \neq \emptyset \\ -\infty, & \text{if } B_x(f) = \emptyset \end{cases}$$

denoted by $st^\alpha - \liminf_{x\to c} f(x)$.

If $u = st^\alpha - \liminf_{x\to c} f(x)$ is finite then for every $\varepsilon > 0$,

$$\lim_{\substack{\mu(I(a,b))\to 0\\ I(a,b)\ni c}} \frac{1}{(\mu(I(a,b)))^\alpha} \mu(\{x \in I(a,b) \subset \mathbb{R}^m : f(x) < u+\varepsilon\}) \neq 0 \tag{6.1}$$

and

$$\lim_{\substack{\mu(I(a,b))\to 0\\ I(a,b)\ni c}} \frac{1}{(\mu(I(a,b)))^\alpha} \mu(\{x \in I(a,b) \subset \mathbb{R}^m : f(x) < u-\varepsilon\}) = 0. \tag{6.2}$$

Conversely, if (6.1) and (6.2) hold for every $\varepsilon > 0$, then $u = st^\alpha - \liminf_{x\to c} f(x)$.

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