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AN L^p-L^q-VERSION OF MORGAN'S THEOREM FOR THE GENERALIZED DUNKL TRANSFORM

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Abstract. In this article, we prove An L^p-L^q-version of Morgan's theorem for the generalized Dunkl transform.
Keywords: Morgan's theorem, generalized Dunkl transform, Heisenberg inequality, Dunkl transform.

1. Introduction

Heisenberg in [8] proved the uncertainty principle. It states that the more precisely the position of some particle is determined, the less precisely its momentum can be known, and vice versa. This physical idea is illustrated by Heisenberg inequality as mathematical formulations

$$\int_{\mathbb{R}} x^2|f(x)|^2dx \int_{\mathbb{R}} \xi^2|\widehat{f}(\xi)|^2d\xi \geq \frac{1}{4}\int_{\mathbb{R}}|f(x)|^2dx.$$

As a description of this, one has Hardy's theorem [1], Morgan's theorem [2]. These theorems have been generalized to many other situations (see, for example, [5]). An L^p-L^q-version of Morgan's theorem has been proved by Ben Farah and Mokni [7], they proved that for $1 \leq p, q \leq \infty$, $a > 0$, $b > 0$, $\gamma > 2$ and $\eta = \frac{\gamma}{\gamma-1}$, then for all measurable function f on $\mathbb{R}$, the conditions

$$e^{a|x|^\gamma} f \in L^p(\mathbb{R}) \ \ and \ \ e^{b|\lambda|^\eta}\widehat{f}(\lambda) \in L^q(\mathbb{R})$$

$$imply \ \ f = 0 \ \ if \ \ (a\gamma)^{\frac{1}{\gamma}}(b\eta)^{\frac{1}{\eta}} > \left(\sin\left(\frac{\pi}{2}(\eta-1)\right)\right)^{\frac{1}{\eta}}.$$

Some applications of approximation of functions in L^p-spaces ($p \geq 1$) can be seen in [9, 10]. In this paper we establish an analogous of L^p-L^q-version of Morgan's theorem for the Generalized Dunkl transform $\mathcal{F}_{\alpha,n}$ associated with the Generalized Dunkl operator $\Lambda_{\alpha,n}$ [4] which generalized the Dunkl operator, we refer the reader

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to [11, 12] for more interesting work. We prove that for $1 \leq p, q \leq \infty$, $a > 0$, $b > 0$, $\gamma > 2$ and $\eta = \frac{\gamma}{\gamma-1}$, then for all measurable function f on $\mathbb{R}$, the conditions

$$e^{a|x|^{\gamma}} f \in L^p_{\alpha,n}(\mathbb{R}) \ \ and \ \ e^{b|\lambda|^{\eta}} \mathcal{F}_{\alpha,n}(f)(\lambda) \in L^q_{\alpha,n}(\mathbb{R})$$

imply

$$f = 0 \ \ if \ \ (a\gamma)^{\frac{1}{\gamma}} (b\eta)^{\frac{1}{\eta}} > \left(\sin\left(\frac{\pi}{2}(\eta - 1)\right)\right)^{\frac{1}{\eta}}.$$

Throughout this paper, we assume that $\alpha > \frac{-1}{2}$, and we denote by

- $\mathcal{M}$ the map defined by $\mathcal{M}f(x) = x^{2n} f(x)$.
- $E(\mathbb{R})$ the space of functions $\mathbb{C}^{\infty}$ on $\mathbb{R}$, provided with the topology of compact convergence for all derivatives. That is the topology defined by semi-norms

$$P_{a,m}(f) = sup_{x\in[-a,a]} \sum_{k=0}^{m} | \frac{d^k}{dx^k} f(x) |, \quad a > 0, \quad m = 0, 1, ...$$

- $D_a(\mathbb{R})$, the space of $\mathbb{C}^{\infty}$ function on $\mathbb{R}$, which are supported in $[-a, a]$, equipped with the topology induced by $E(\mathbb{R})$.
- $D(\mathbb{R}) = \bigcup_{a>0} D_a(\mathbb{R})$, endowed with inductive limit topology.
- $E_n(\mathbb{R})$ (resp $D_n(\mathbb{R})$) stand for the subspace of $E(\mathbb{R})$ (resp $D_n(\mathbb{R})$) consisting of functions f such that

$$f(0) = ... = f^{(2n-1)}(0).$$

- L^p_{α} the class of measurable functions f on $\mathbb{R}$ for which $||f||_{p,\alpha} < \infty$, where

$$||f||_{p,\alpha} = \left(\int_{\mathbb{R}} |f(x)|^p |x|^{2\alpha+1} dx\right)^{\frac{1}{p}}, \quad if p < \infty,$$

  and $||f||_{\infty,\alpha} = ||f||_{\infty} = esssup_{x\in\mathbb{R}} |f(x)|$.
- $L^p_{\alpha,n}$ the class of measurable functions f on $\mathbb{R}$ for which

$$||f||_{p,\alpha,n} = ||\mathcal{M}^{-1} f||_{p,\alpha+2n} < \infty.$$

- $D^p_{\alpha,n}(\mathbb{R}) = D_n(\mathbb{R}) \bigcap L^p_{\alpha,n}(\mathbb{R})$.

2. Dunkl transform

In this section, we recall some facts about harmonic analysis that are related to Dunkl operator Λ_α associated with reflection group $\mathbb{Z}_2$ on $\mathbb{R}$. We cite here, as briefly as possible, only some properties. For more details we refer the reader to [3].

The Dunkl operator Λ_α is defined as follow:

$$\Lambda_\alpha f(x) = f'(x) + \left(\alpha + \frac{1}{2}\right)\frac{f(x) - f(-x)}{x}. \tag{2.1}$$

The Dunkl kernel e_α is defined by

$$e_\alpha(z) = j_\alpha(iz) + \frac{z}{2(\alpha+1)} j_{\alpha+1}(z) \quad (z \in \mathbb{C}) \tag{2.2}$$

where

$$j_\alpha(z) = \Gamma(\alpha+1)\Sigma_{n=0}^{\infty}\frac{(-1)^n(\frac{z}{2})^{2n}}{n!\,\Gamma(n+\alpha+1)} \quad (z \in \mathbb{C}).$$

is the normalized spherical Bessel function of index α.

The one-dimensional Dunkl transform of a function $f \in D(\mathbb{R})$ is defined by

$$\mathcal{F}_\alpha(f)(\lambda) = \int_{\mathbb{R}} f(x) e_\alpha(-i\lambda x)|x|^{2\alpha+1}dx, \quad \lambda \in \mathbb{C}. \tag{2.3}$$

Theorem 2.1. **(i)** *The Dunkl transform $\mathcal{F}_\alpha$ is a topological automorphism from $D(\mathbb{R})$ onto $\mathbb{H}$. More precisely $f \in D_a(\mathbb{R})$ if, and only if, $\mathcal{F}_\alpha(f) \in \mathbb{H}_a$*

(ii) *For every $f \in D(\mathbb{R})$,*

$$f(x) = \int_{\mathbb{R}} \mathcal{F}_\alpha(f)(\lambda) e_\alpha(i\lambda x)|\lambda|^{2\alpha+1}d\lambda,$$

$$\int_{\mathbb{R}} |f(x)|^2|x|^{2\alpha+1}dx = m_\alpha \int_{\mathbb{R}} |\mathcal{F}_\alpha(f)(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda,$$

where

$$m_\alpha = \frac{1}{2^{2(\alpha+1)}(\Gamma(\alpha+1))^2}. \tag{2.4}$$

3. Harmonic analysis associated with $\Lambda_{\alpha,n}$

In this section, we collect relevant material from the harmonic analysis associated with $\Lambda_{\alpha,n}$, which was developed recently in [4].

Let $\alpha > \frac{1}{2}$ and n, and let $E_n(\mathbb{R})$(resp.$D_n(\mathbb{R})$) stand for the subspace of $E(\mathbb{R})$(resp. $D(\mathbb{R})$) consisting of functions f such that

$$f(0) = \cdots = f^{(2n-1)}(0) = 0.$$

For $a > 0$, put

$$D_{a,n}(\mathbb{R}) = D_a(\mathbb{R}) \cap E_n(\mathbb{R}).$$

The first-order singular differential-difference operator on $\mathbb{R}$ is defined as follow

$$\Lambda_{\alpha,n} f(x) = f'(x) + (\alpha + \frac{1}{2})\frac{f(x) - f(-x)}{x} - 2n\frac{f(-x)}{x}. \tag{3.1}$$

Lemma 3.1. **(i)** *The map*

$$M_n(f)(x) = x^{2n} f(x)$$

is a topological isomorphism

- *from $E(\mathbb{R})$ onto $E_n(\mathbb{R})$;*
- *from $D(\mathbb{R})$ onto $D_n(\mathbb{R})$.*

(ii) *For all $f \in E(\mathbb{R})$,*

$$\Lambda_{\alpha,n} \circ M_n(f) = M_n \circ \Lambda_{\alpha+2n}(f),$$

where $\Lambda_{\alpha+2n}$ is the Dunkl operator of order $\alpha + 2n$ given by (2.1).

(iii) *Let $f \in E_n(\mathbb{R})$ and $g \in D_n(\mathbb{R})$. Then*

$$\int_{\mathbb{R}} \Lambda_{\alpha,n} f(x) g(x) |x|^{2\alpha+1} dx = -\int_{\mathbb{R}} f(x) \Lambda_{\alpha,n} g(x) |x|^{2\alpha+1} dx.$$

3.1. Generalized Fourier Transform

For $\lambda \in \mathbb{C}$ and $x \in \mathbb{R}$ put

$$\Psi_{\lambda,\alpha,n}(x) = x^{2n} e_{\alpha+2n}(i\lambda x), \tag{3.2}$$

where $e_{\alpha+2n}$ is the Dunkl kernel of index $\alpha + 2n$ given by (2.2).

Proposition 3.1. **(i)** *$\Psi_{\lambda,\alpha,n}$ satisfies the differential-difference equation*

$$\Lambda_{\alpha,n} \Psi_{\lambda,\alpha,n} = i\lambda \Psi_{\lambda,\alpha,n}.$$

(ii) $\Psi_{\lambda,\alpha,n}$ *possesses the Laplace integral representation*

$$\Psi_{\lambda,\alpha,n}(x) = a_{\alpha+2n}x^{2n}\int_{-1}^{1}(1-t^2)^{\alpha+2n-\frac{1}{2}}(1+t)e^{i\lambda xt}dt,$$

where

$$a_{\alpha+2n} = \frac{\Gamma(\alpha+2n+1)}{\sqrt{\pi}\Gamma(\alpha+2n+\frac{1}{2}}).$$

(iii) *For all* $m = 0, 1, ...$

$$\left|\frac{\partial^m}{\partial\lambda^m}\Psi_{\lambda,\alpha,n}(x)\right| \leq |x|^{2n+m}e^{|Im\ \lambda||x|}. \tag{3.3}$$

In particular

$$|\ \Psi_{\lambda,\alpha,n}(x)\ |\leq |x|^{2n}e^{|Im\ \lambda||x|}. \tag{3.4}$$

Definition 3.1. *The generalized Fourier transform of a function* $f \in D_n(\mathbb{R})$ *is defined by*

$$\mathcal{F}_{\alpha,n}(f)(\lambda) = \int_{\mathbb{R}} f(x)\Psi_{-\lambda,\alpha,n}(x)|x|^{2\alpha+1}dx, \quad \lambda \in \mathbb{C}. \tag{3.5}$$

Theorem 3.1. **(i)** *For all* $f \in D_n(\mathbb{R})$, *we have the inversion formula*

$$f(x) = m_{\alpha+2n}\int_{\mathbb{R}}\mathcal{F}_{\alpha,n}(f)(\lambda)\Psi_{\lambda,\alpha,n}(x)|\lambda|^{2\alpha+4n+1}d\lambda,$$

where

$$m_{\alpha+2n} = \frac{1}{2^{2(\alpha+2n+1)}(\Gamma(\alpha+2n+1))^2}.$$

(ii) *For every* $f \in D_n(\mathbb{R})$, *we have the Plancherel formula*

$$\int_{\mathbb{R}}|f(x)|^2|x|^{2\alpha+1}dx = m_{\alpha+2n}\int_{\mathbb{R}}|\mathcal{F}_{\alpha,n}(f)(\lambda)|^2|\lambda|^{2\alpha+4n+1}d\lambda.$$

(iii) *The generalized Fourier transform* $\mathcal{F}_{\alpha,n}$ *extends uniquely to an isometric isomorphism from* $L^2(\mathbb{R}, |x|^{2\alpha+1}dx)$ *onto* $L^2(\mathbb{R}, m_{\alpha+2n}|\lambda|^{2\alpha+4n+1}d\lambda)$.

4. An L^p-L^q-version of Morgan's theorem for $\mathcal{F}_{\alpha,n}$

We start by getting the following lemma of Phragmen-Lindlöf type using the same technique as in [6, 7]. We need this lemma to prove the main result of this paper.

Lemma 4.1. *Suppose that $\rho \in]1,2[$, $q \in [1,\infty]$, $\sigma > 0$ and $B > \sigma \sin\left(\frac{\pi}{2}(\rho-1)\right)$. If g is an entire function on $\mathbb{C}$ verifying:*

$$|g(x+iy)| \leq C.e^{\sigma|y|^{\rho}} \tag{4.1}$$

and

$$e^{B|x|^{\rho}} g/_{\mathbb{R}} \in L^{q}_{\alpha,n}(\mathbb{R}) \tag{4.2}$$

for all $x, y \in \mathbb{R}$ then $g = 0$.

Theorem 4.1. *Let $1 \leq p, q \leq \infty$, $a > 0$, $b > 0$, $\gamma > 2$ and $\eta = \frac{\gamma}{\gamma-1}$, then for all measurable function f on $\mathbb{R}$, the conditions*

$$e^{a|x|^{\gamma}} f \in L^{p}_{\alpha,n}(\mathbb{R}) \tag{4.3}$$

and

$$e^{b|\lambda|^{\eta}} \mathcal{F}_{\alpha,n}(f)(\lambda) \in L^{q}_{\alpha,n}(\mathbb{R}) \tag{4.4}$$

imply $f = 0$ if

$$(a\gamma)^{\frac{1}{\gamma}} (b\eta)^{\frac{1}{\eta}} > \left(\sin\left(\frac{\pi}{2}(\eta-1)\right)\right)^{\frac{1}{\eta}}.$$

Proof. The function

$$\mathcal{F}_{\alpha,n}(f)(\lambda) = \int_{\mathbb{R}} f(x)\Psi_{\lambda,\alpha,n}(x)x^{2\alpha+1}dx, \quad \lambda \geq 0.$$

is well defined, entirely on $\mathbb{C}$, and satisfies the condition

$$\begin{aligned} |\mathcal{F}_{\alpha,n}(f)(\lambda)| &= |\int_{\mathbb{R}} f(x)\Psi_{\lambda,\alpha,n}(x)d\mu_{\alpha}(x)|, \\ &\leq \int_{\mathbb{R}} |f(x)||x|^{2n} e^{|x||\zeta|} d\mu_{\alpha}(x), \\ &= \int_{\mathbb{R}} |\mathcal{M}_n^{-1} f(x)| e^{|x||\zeta|} d\mu_{\alpha+2n}(x), \quad \forall \lambda = \xi + i\zeta \in \mathbb{C} \end{aligned}$$

Applying Hölder inequality, we get

$$\begin{aligned} |\mathcal{F}_{\alpha,n}(f)(\lambda)| &\leq \left| \left(\int_{\mathbb{R}} \left(\mathcal{M}_n^{-1} f(x)| e^{a|x|^{\gamma}}\right)^{p} d\mu_{\alpha+2n}(x)\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}} \left(|e^{-a|x|^{\gamma}} e^{|x||\zeta|}\right)^{p'} d\mu_{\alpha+2n}(x)\right)^{\frac{1}{p'}} \right|, \\ &\leq C\left(\int_{\mathbb{R}} \left(|e^{-a|x|^{\gamma}} e^{|x||\zeta|}\right)^{p'} d\mu_{\alpha+2n}(x)\right)^{\frac{1}{p'}}. \end{aligned}$$

where p' is the conjugate exponent of p.

Let

$$C \in I =](b\eta)^{\frac{-1}{\eta}} \sin\left(\frac{\pi}{2}(\eta-1)\right)^{\frac{1}{\eta}}, (a\gamma)^{\frac{1}{\gamma}}[.$$

Applying the convex inequality

$$|ty| \leq (\frac{1}{\gamma})|t|^{\gamma} + (\frac{1}{\eta})|y|^{\eta}$$

to the positive numbers $C|x|$ and $\frac{|\zeta|}{C}$, we obtain

$$|x||\zeta| \leq (\frac{C^{\gamma}}{\gamma})|x|^{\gamma} + (\frac{1}{\eta C^{\eta}})|\zeta|^{\eta}$$

and the following relation holds

$$\int_{\mathbb{R}} e^{-ap'|x|^{\gamma}} e^{p'|x||\zeta|} d\mu_{\alpha+2n}(x) \leq e^{\frac{p'|\zeta|^{\eta}}{\eta C^{\eta}}} \int_{\mathbb{R}} e^{-p'(a-\frac{C^{\gamma}}{\gamma})|x|^{\gamma}} d\mu_{\alpha+2n}(x).$$

Since $C \in I$, then $a > \frac{C^{\gamma}}{\gamma}$, and thus the integral

$$\int_{\mathbb{R}} e^{-p'(a-\frac{C^{\gamma}}{\gamma})|x|^{\gamma}} d\mu_{\alpha+2n}(x)$$

is finite. Moreover

$$|\mathcal{F}_{\alpha,n}(f)(\lambda)| \leq \text{ Const.} e^{\frac{p'|\zeta|^{\eta}}{\eta C^{\eta}}}, \quad \text{for all } \lambda \in \mathbb{C}. \tag{4.5}$$

By virtue of relations (4.3), (4.4), (4.5) and Lemma 4.1, we obtain that $\mathcal{F}_{\alpha,n} f = 0$. Then $f = 0$ by Theorem 3.1. □

REFERENCES

[1] G.H. Hardy, "A theorem concerning Fourier transforms," Journal of the London Mathematical Society, vol. 8, pp. 227–231, 1933.

[2] G.W. Morgan, "A note on Fourier transforms," Journal of the London Mathematical Society, vol. 9, pp. 187-192, 1934.

[3] M.A. Mourou and K. Trimeche, Transmutation operators and Paley-Wiener theorem associated with a singular Differential-Difference operator on the real line, Analysis and Applications, Vol. 1 (2003), 43–70.

[4] Al Sadhan, S.A., Al Subaie, R.F. and Mourou, M.A., Harmonic Analysis Associated with A First-Order Singular Differential-Difference Operator on the Real Line. Current Advances in Mathematics Research, 1 (2014), 23–34.

[5] S.C. Bagchi and S. K. Ray, "Uncertainty principles like Hardy's theorem on some Lie groups," Journal of the Australian Mathematical Society. Series A, 65(3) (1998), 289–302,

[6] S. Ayadi, An L^p-L^q-version of Morgan's theorem for the dunkl transform, Math. Sci. Res. J. 8(11) (2004) 318-327.

[7] S. Ben Farah and K. Mokni, Uncertainty principle and the L^p-L^q-version of Morgan's theorem on some groups, Russ. J. Math. Phys. 10(3) (2003) 1–16.

[8] W. Heisenberg, Uber den anschaulichen inhalt der quantentheoretischen kinematik und machanik, Z. Phys. 43 (1927) 172–198.

[9] L.N. Mishra, V.N. Mishra, K. Khatri, Deepmala, On The Trigonometric approximation of signals belonging to generalized weighted Lipschitz $W(L^r, \xi(t))(r \geq 1)-$ class by matrix $(C^1.N_p)$ Operator of conjugate series of its Fourier series, Applied Mathematics and Computation, Vol. 237 (2014) 252-263.

[10] V.N. Mishra, Kejal Khatri and L. N. Mishra, Product Summability Transform of Conjugate Series of Fourier series, International Journal of Mathematics and Mathematical Sciences, Vol. 2012 (2012), Article ID 298923, 13 pages, DOI: 10.1155/2012/298923 (Hindawi Publishing Corporation, USA).

[11] Deepmala, L.N. Mishra, Differential operators over modules and rings as a path to the generalized differential geometry, FACTA UNIVERSITATIS (NIŠ), Ser. Math. Inform., (2015), in press.

[12] L.N. Mishra, M. Sharma, V.N. Mishra, Lakshmi - Manoj generalized Yang-Fourier transforms to heat-conduction in a semi-infinite fractal bar, Pure and Applied Mathematics Journal 4(2) (2015), 57–61.

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