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GENERALIZATION OF PELL SEQUENCE AND PELL-LUCAS SEQUENCE, NEW RESULTS AND CIRCULANT MATRICES ASPECTS

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Abstract. Some generalizations of Pell sequence and Pell-Lucas sequence, namely, (k,h)-Pell sequence and (k,h)-Pell-Lucas sequence are considered in this paper. We obtain generating functions, some identities and formulas for the sums of a finite number of terms and consecutive terms, sums of squares of consecutive terms and alternating sums of consecutive terms of these sequences. Then, we obtain the eigenvalues and determinants of particular circulant matrices involving (k,h)-Pell sequence and (k,h)-Pell-Lucas sequence. Finally, we obtain some upper and lower bounds for the spectral norms of these circulant matrices.
Keywords: Lucas sequence, Pell sequence, circulant matrices.

1. Introduction

Circulant matrices have excited many researchers ever since their first occurrence in a paper by Catalan and have been widely used in the analysis of time series . Also, being a special type of Toeplitz matrix, it has many applications in solutions to differential and integral equations, spline functions and various problems in physics, mathematics, statistics and signal processing [14].

Because of interesting and exciting applications of well-known Fibonacci sequences and some generalizations of this sequence in the area of numerical analysis,

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combinatory theory, matrix theory and computer sciences, mathematicians have been fascinated by these sequences.

For instance, Bueno [1] considered a particular right circulant matrix involving Fibonacci sequence and obtained the formulae for the Euclidean norm and eigenvalues of this matrix.

Bozkurt [4] computed the spectral norms of some matrices connected integer sequences such as Fibonacci, Lucas, Pell and Perrin numbers. Cerda Morales [5] considered a particular Jacobsthal sequence, namely, q-Jacobsthal sequence $\{J_{q,n}\}$ and obtained a generating matrix for the terms of sequence $\{J_{q,kn}\}$ for a positive integer k. Then, by the aid of this matrix, new identities for this sequence were established.

Nali and Sen [11] obtained norms of circulant matrices involving a generalization of Fibonacci numbers. The authors in [12] studied (k,h)-Pell sequence and (k,h)-Pell-Lucas sequence and found a formulae of n^{th} term and sum of the first n terms of these sequences.

The first objective of this paper is to investigate new results about (k,h)-Pell sequence and (k,h)-Pell Lucas sequence. The second objective is to study the norm properties of circulant matrices associated these recursive sequences. The paper is organized as follows:

In section 2, we give the generating functions of (k,h)-Pell sequence and (k,h)-Pell-Lucas sequence. We obtain some new identities and give formulas for the sums of a finite number of terms and squares of finite terms of (k,h)-Pell sequence and (k,h)-Pell-Lucas sequence .

In section 3, inspired by the work of Cerin [6], firstly, we demonstrate a theorem that gives formulas for the sums of a finite number of consecutive terms of (k,h)-Pell sequence and (k,h)-Pell-Lucas sequence. By using this theorem, we give some examples that state formulas for the sums of squares of consecutive terms, product of consecutive terms alternating sums of consecutive terms and alternating sums of product of consecutive terms of these sequences.

In section 4, we consider particular circulant matrices involving (k,h)-Pell sequence and (k,h)-Pell-Lucas sequence. We obtain some formulas for computing the eigenvalues and determinants of these circulant matrices. Then, we obtain upper and lower bounds for their spectral norms.

In section 5, we give numerical examples related to eigenvalues, determinants and Euclidean norm of particular circulant matrices connected (k,h)-Pell sequence and (k,h)-Pell-Lucas sequence for $k=1$ and $h=1$.
The well-known Pell sequence $\{P_n\}$ has the recursive relation

$$P_n = 2P_{n-1} + P_{n-2}, \tag{1.1}$$

[6], where $P_0 = 0, P_1 = 1$. Now, we consider a generalization of this sequence [12] which is called (k,h)-Pell sequence and is denoted by Γ_n. This sequence has the recursive relation

$$\Gamma_n = 2k\Gamma_{n-1} + h\Gamma_{n-2}, \tag{1.2}$$

where $\Gamma_0 = 0$, $\Gamma_1 = 2k$ and $k, h \in Z$, $k^2 + h > 0$. Also, we consider generalization of Pell-Lucas sequence which is called (k, h)-Pell-Lucas sequence and is denoted by $\{\Theta_n\}$. This sequence has the recursive relation

$$\Theta_n = 2k\Theta_{n-1} + h\Theta_{n-2}, \tag{1.3}$$

where, $\Theta_0 = 2$ and $\Theta_1 = 2k$ (see [12]).

More information about Pell sequence, Pell -Lucas sequence, Fibonacci sequence and some generalizations and applications of these sequences can be found in [3], [7]- [10] and [13]- [18].

It is known that

$$\sum_{k=0}^{n-1} x^k = 1 + x + x^2 + \cdots + x^{n-1} = \frac{x^n - 1}{x - 1}, \tag{1.4}$$

Let $A = (a_{ij})$ is an $n \times n$ matrix, then the maximum column length norm $c_1(.)$ and maximum row length norm $r_1(.)$ of matrix A are defined respectively by

$$c_1(A) = \max_j \sqrt{\sum_i |a_{ij}|^2} \quad , \quad r_1(A) = \max_i \sqrt{\sum_j |a_{ij}|^2}. \tag{1.5}$$

The ℓ_p norm of A is defined by

$$\|A\|_p = (\sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}|^p)^{\frac{1}{p}} \quad . \tag{1.6}$$

For $p = 2$, this norm is called " Frobenius" or "Euclidean" norm and showed by $\|A\|_E$.
Let $A = (a_{ij})$ and $B = (b_{ij})$ are $m \times n$ matrices. Then, the Hadamard product of A and B is defined by $A \circ B = (a_{ij}b_{ij})$.

The spectral norm of A is defined by

$$\|A\|_2 = \sqrt{\max_{1\le i\le n} \lambda_i}, \tag{1.7}$$

where, λ_i is the eigenvalue of matrix AA^H and A^H is conjugate transpose of matrix A. There is a relation between Frobenius and spectral norm, that is

$$\frac{1}{\sqrt{n}}\|A\|_E \le \|A\|_2 \le \|A\|_E. \tag{1.8}$$

Let A, B and C be $m \times n$ matrices and $A = B \circ C$, (Hadamard product of B and C) then we have

$$\|A\|_2 \le r_1(B)c_1(C). \tag{1.9}$$

2. New identities and summation formulas

In this section, we give the generating functions of (k,h)-Pell sequence and (k,h)-Pell-Lucas sequence. Then, using the Binet like formulas of (k,h)-Pell sequence and (k,h)-Pell-Lucas sequence, we represent new identities related to these sequences. Also, we obtain some formulas for the sums of a finite number of terms, sums of squares of finite terms of (k,h)-Pell sequence and (k,h)-Pell-Lucas sequence.
We start this section with a lemma, which is similar to the well-known Binet formula [12].

Lemma 2.1. *(Binet-Like formula)Let Γ_n be a sequence as1.2 in and Θ_n be a sequence as in 1.3. Then, we have*

$$\Gamma_n = \frac{2k}{p}\left(\alpha^n - \beta^n\right), \qquad \Theta_n = \alpha^n + \beta^n. \tag{2.1}$$

where, $\alpha = k + \sqrt{k^2+h}, \beta = k - \sqrt{k^2+h}$ and $p = \alpha - \beta$.

As a result of lemma 2.1, we have the following theorem.

Theorem 2.1. *Let Γ_n be a sequence as in 1.2 and Θ_n be a sequence as in 1.3. Then, the following identities are valid.*

$$\begin{aligned}
&\Gamma_{n+1}\Gamma_{n-1} - \Gamma_n^2 = 4k^2(-1)^n h^{n-1},\\
&\Theta_{n+1}\Theta_{n-1} - \Theta_n^2 = 4(-1)^{n-1}h^{n-1}(k^2+h),\\
&\Gamma_{n+1}\Theta_{n-1} - \Gamma_{n-1}\Theta_{n+1} = 4k(-1)^{n-1}h^{n-1},\\
&\Gamma_{n+1}\Gamma_{n+2} - \Gamma_n\Gamma_{n+3} = 8k^3(-1)^n h^n,\\
&\Theta_{n+1}\Theta_{n+2} - \Theta_n\Theta_{n+3} = 8k(-1)^{n+1}h^n(k^2+h),\\
&\Gamma_{2n+1}^2 - \Gamma_{2n-1}\Gamma_{2n+3} = -\Gamma_2^2 h^{2n-1},\\
&\Theta_{2n+1}^2 - \Theta_{2n-1}\Theta_{2n+3} = 16k^2(k^2+h)h^{2n-1},\\
&\Gamma_{2n}\Gamma_{2n+1} = \frac{k^2}{k^2+h}\left[\Theta_{4n+1} - 2h^{2n}k\right],\\
&\Gamma_n\Theta_n = \Gamma_{2n},\\
&\Theta_{2n}\Theta_{2n+1} = \Theta_{4n+1} + 2h^{2n}k,\\
&\Gamma_n\Gamma_{n+1}\Gamma_{n+2} = \frac{k^2}{k^2+h}[\Gamma_{3n+3} - (-h)^n(h^2\Gamma_{n-1} + k\Gamma n + 2)],\\
&\Theta_n\Theta_{n+1}\Theta_{n+2} = \Theta_{3n+3} + (-h)^n(h^2\Theta_{n-1} + k\Theta n + 2)].
\end{aligned}$$

Proof. We prove the first identity. By lemma 2.1, we have:

$$\begin{aligned}
\Gamma_{n+1}\Gamma_{n-1} - \Gamma_n^2 &= \frac{2k}{p}[\alpha^{n+1} - \beta^{n+1}]\frac{2k}{p}[\alpha^{n-1} - \beta^{n-1}] - [\frac{2k}{p}(\alpha^n - \beta^n)]^2\\
&= (\frac{2k}{p})^2\left[2\alpha^n\beta^n - \alpha^{n+1}\beta^{n-1} - \alpha^{n-1}\beta^{n+1}\right]
\end{aligned}$$

$$
\begin{aligned}
&= (\frac{2k}{p})^2[-\alpha^{n-1}\beta^{n-1}(\alpha^2+\beta^2-2\alpha\beta)] \\
&= \frac{4k^2}{p^2}(-[\alpha\beta]^{n-1})(\alpha-\beta)^2 \\
&= \frac{-4k^2}{p^2}(-h)^{n-1}p^2 = 4k^2(-1)^n h^{n-1}.
\end{aligned}
$$

Thus, the proof of first identity is completed. The other cases can be proved by the similar method. □

Theorem 2.2. *Let Γ_n be a sequence as in 1.2 and Θ_n be a sequence as in 1.3. Then, the generating function of Γ_n is*

$$
f(x) = \frac{2kx}{1-2kx-x^2h}, \tag{2.2}
$$

and the generating function of Θ_n is

$$
g(x) = \frac{2+2kx}{1-2kx-x^2h} \tag{2.3}
$$

Proof. Suppose that the generating function of Γ_n has the formal power series$f(x) = \sum_{n=0}^{\infty}\Gamma_n x^n$. Thus, we get

$$
\begin{aligned}
f(x)-2kxf(x)-x^2hf(x) &= \sum_{n=0}^{\infty}\Gamma_n x^n - 2kx\sum_{n=0}^{\infty}\Gamma_n x^n - x^2h\sum_{n=0}^{\infty}\Gamma_n x^n \\
&= \sum_{n=0}^{\infty}\Gamma_n x^n - 2k\sum_{n=1}^{\infty}\Gamma_{n-1}x^n - h\sum_{n=2}^{\infty}\Gamma_{n-2}x^n \\
&= \Gamma_0 + (\Gamma_1 - 2k\Gamma_0)x + \sum_{n=2}^{\infty}(\Gamma_n - 2k\Gamma_{n-1} - h\Gamma_{n-2})x^n.
\end{aligned}
$$

Since $\Gamma_n - 2k\Gamma_{n-1} - h\Gamma_{n-2} = 0$, by substituting the initial values Γ_0 and Γ_1, we obtain that $f(x)(1-2kx-x^2h) = 2x$. Hence, we get

$$
f(x) = \sum_{m=0}^{\infty}\Gamma_n x^n = \frac{2kx}{1-2kx-x^2h}.
$$

Similarly, we obtain the generationg function of Θ_n. □

Theorem 2.3. *Let Γ_n be a sequence as in 1.2 and i be a natural number. Then, we have*

$$
\sum_{m=0}^{n}\Gamma_{mi} = \frac{\Gamma_i - \Gamma_{(n+1)i} + (-h)^i\Gamma_{ni}}{1-\Theta_i + (-h)^i}. \tag{2.4}
$$

Proof. According lemma 2.1 we have

$$\sum_{m=0}^{n} \Gamma_{mi} = \frac{2k}{p} \sum_{m=0}^{n} \left(\alpha^{mi} - \beta^{mi}\right) = \frac{2k}{p} \left[\sum_{m=0}^{n} (\alpha^i)^m - \sum_{m=0}^{n} (\beta^i)^m \right].$$

By 1.4 we get

$$\begin{aligned} \sum_{m=0}^{n} \Gamma_{mi} &= \frac{2k}{p} \left[\frac{1-(\alpha^i)^{n+1}}{1-\alpha^i} - \frac{1-(\beta^i)^{n+1}}{1-(\beta)^i} \right] \\ &= \frac{2k}{p} \left[\frac{(1-(\alpha^i)^{n+1})(1-\beta^i) - (1-(\beta^i)^{n+})(1-\alpha^i)}{(1-\alpha^i)(1-\beta^i)} \right]. \end{aligned}$$

After some computations, we deduce that

$$\begin{aligned} \sum_{m=0}^{n} \Gamma_{mi} &= \frac{2k}{p} \left[\frac{(\alpha^i - \beta^i) - ((\alpha^i)^{n+1} - (\beta^i)^{n+1}) + \alpha^i \beta^i ((\alpha^i)^n - (\beta^i)^n)}{1 - (\alpha^i + \beta^i) + \alpha^i \beta^i} \right] \\ &= \frac{\frac{2k}{p}(\alpha^i - \beta^i) - \frac{2k}{p}((\alpha^i)^{n+1} - (\beta^i)^{n+1}) + \frac{2k}{p}\alpha^i \beta^i ((\alpha^i)^n - (\beta^i)^n)}{1 - (\alpha^i + \beta^i) + \alpha^i \beta^i} \\ &= \frac{\Gamma_i - \Gamma_{(n+1)i} + (-h)^i \Gamma_{ni}}{1 - \Theta_i + (-h)^i}. \end{aligned}$$

Thus, the proof is completed. □

Theorem 2.4. *Let Θ_n be a sequence as in 1.3 and i be a natural number. Then, we have*

$$\sum_{m=0}^{n} \Theta_{mi} = \frac{2 - \Theta_i - \Theta_{(n+1)i} + (-h)^i \Theta_{ni}}{1 - \Theta_i + (-h)^i}. \tag{2.5}$$

Proof. The proof is similar to Theorem 2.3. □

Example 2.1. For $i = 1, 2$ by theorems 2.3 and 2.4 we have the following summation formulas.

$$\sum_{m=0}^{n} \Gamma_m = \frac{2k - \Gamma_{(n+1)} - h\Gamma_n}{1 - 2k - h} \quad , \quad \sum_{m=0}^{n} \Gamma_{2m} = \frac{\Gamma_2 - \Gamma_{2n+2} + h^2 \Gamma_{2n}}{1 - \Theta_2 + h^2},$$

$$\sum_{m=0}^{n} \Theta_m = \frac{2 - 2k - \Theta_{n+1} - h\Theta_n}{1 - 2k - h} \quad , \quad \sum_{m=0}^{n} \Theta_{2m} = \frac{2 - \Theta_2 - \Theta_{2n+2} + h^2 \Theta_{2n}}{1 - \Theta_2 + h^2},$$

Also, for $i = 3$ we have

$$\sum_{m=0}^{n} \Gamma_{3m} = \frac{\Gamma_3 - \Gamma_{3n+3} - h^3 \Gamma_{3n}}{1 - \Theta_3 - h^3} \quad , \quad \sum_{m=0}^{n} \Theta_{3m} = \frac{2 - \Theta_3 - \Theta_{3n+3} - h^3 \Theta_{3n}}{1 - \Theta_3 - h^3}.$$

Theorem 2.5. *Let Γ_n be as in 1.2 and Θ_n be as in 1.3. Then, we have*

$$\sum_{m=0}^{n-1}\Gamma_m^2=\frac{4k^2}{p^2}\left[\frac{h^2\Theta_{2n-2}-\Theta_{2n}-\Theta_2+2}{h^2-\Theta_2+1}+2\frac{(-h)^n-1}{h+1}\right],$$

$$\sum_{m=0}^{n-1}\Theta_m^2=\left(\frac{2-\Theta_2+h^2\Theta_{2n-2}-\Theta_{2n}}{1-\Theta_2+h^2}\right)+2\left(\frac{1-(-h)^n}{1+h}\right).$$

Proof. See [12] □

3. More summation formulas

In this section, similar to the last section, inspiration by the work of Cerin [6], we obtain formulas for the sums of a finite number of consecutive terms of (k,h)-Pell sequence and (k,h)-Pell-Lucas sequence. Furthermore, we give some theorems that state formulas for the sums of squares of consecutive terms and product of consecutive terms of these sequences. Also, we present formulas, for the alternating sums of consecutive terms and alternating sums of product of consecutive terms of (k,h)-Pell sequence and (k,h)-Pell-Lucas sequence.

Theorem 3.1. *Let Γ_n be as in 1.2 and Θ_n be as in 1.3. Then, we have*

$$\sum_{i=0}^{m}\Gamma_{n+i}=\frac{h\Gamma_{m+n}+\Gamma_{m+n+1}-h\Gamma_{n-1}-\Gamma_n}{h+2k+1},$$

$$\sum_{i=0}^{m}\Theta_{n+i}=\frac{h(\Theta_{m+n}-\Theta_{n-1})+\Theta_{m+n+1}-\Theta_n}{h+2k-1},$$

$$\sum_{i=0}^{m}(-1)^i\Gamma_{n+i}=\frac{(-1)^{m+1}h\Gamma_{m+n}+(-1)^m\Gamma_{m+n+1}-h\Gamma_{n-1}+\Gamma_n}{h+2k+1},$$

$$\sum_{i=0}^{m}(-1)^i\Theta_{n+i}=\frac{(-1)^{m+1}h\Theta_{m+n}-h\Theta_{n-1}+(-1)^m\Theta_{m+n+1}+\Theta_n}{2k-h+1}.$$

$$\sum_{i=0}^{m}\Gamma_{2n+2i}=\frac{h^2(\Gamma_{2m+2n}-\Gamma_{2n-2})-\Gamma_{2m+2n+2}+\Gamma_{2n}}{h^2-\Theta_2+1},$$

$$\sum_{i=0}^{m}\Theta_{2n+2i}=\frac{h^2(\Theta_{2m+2n}-\Theta_{2n-2})-\Theta_{2m+2n+2}+\Theta_{2n}}{h^2-\Theta_2+1},$$

$$\sum_{i=0}^{m}(-1)^i\Gamma_{2n+2i}=\frac{h^2\left[(-1)^m\Gamma_{2m+2n}+\Gamma_{2n-2}\right]+(-1)^m\Gamma_{2m+2n+2}+\Gamma_{2n}}{h^2+\Theta_2+1},$$

$$\sum_{i=0}^{m}(-1)^i\Theta_{2n+2i}=\frac{h^2\left[(-1)^m\Theta_{2m+2n}+\Theta_{2n-2}\right]+(-1)^m\Theta_{2m+2n+2}+\Theta_{2n}}{h^2+\Theta_2+1}.$$

Proof. By definition of Γ_n, we have $\Gamma_{n+i} = \frac{2k}{p}(\alpha^{n+i} - \beta^{n+i})$. Thus, we get

$$\begin{aligned}\sum_{i=0}^{m}\Gamma_{n+i} &= \sum_{i=0}^{m}\frac{2k}{p}(\alpha^{n+i}-\beta^{n+i})\\ &= \frac{2k}{p}\sum_{i=0}^{m}\alpha^{n+i} - \frac{2k}{p}\sum_{i=0}^{m}\beta^{n+i}\\ &= \frac{2k}{p}\alpha^{n}\sum_{i=0}^{m}\alpha^{i} - \frac{2k}{p}\beta^{n}\sum_{i=0}^{m}\beta^{i}.\end{aligned}$$

By using 1.4, we obtain

$$\begin{aligned}\sum_{i=0}^{m}\Gamma_{n+i} &= \frac{2k}{p}\left[\alpha^{n}(\frac{1-\alpha^{m+1}}{1-\alpha}) - \beta^{n}(\frac{1-\beta^{n+1}}{1-\beta})\right]\\ &= \frac{2k}{p}\left[\frac{(\alpha^{n}-\alpha^{n+m+1})(1-\beta)-(\beta^{n}-\beta^{n+m+1})(1-\alpha)}{(1-\alpha)(1-\beta)}\right].\end{aligned}$$

Hence, by doing a process similar to theorem 2.3, we get the following result

$$\begin{aligned}\sum_{i=0}^{m}\Gamma_{n+i} &= \frac{-h\Gamma_{m+n} - \Gamma_{m+n+1} + h\Gamma_{n-1} + \Gamma_{n}}{-h-2k+1}\\ &= \frac{h\Gamma_{m+n} + \Gamma_{m+n+1} - h\Gamma_{n-1} - \Gamma_{n}}{h+2k-1}.\end{aligned}$$

Other summation formulas can similarly be proved. □

The following theorems present new summation formulas about the consecutive terms and alternative consecutive terms of (k, h)-Pell sequence and (k, h)-Pell-Lucas sequence, which we have presented without any proofs.

Theorem 3.2. *Let Γ_n be as in 1.2 and Θ_n be as in 1.3. Then, we have*

$$\sum_{i=0}^{m}\Gamma_{2n+2i+1} = \frac{h^2[\Gamma_{2m+2n+1} - \Gamma_{2n-1}] - \Gamma_{2m+2n+3} + \Gamma_{2n+1}}{h^2 - \Theta_2 + 1},$$

$$\sum_{i=0}^{m}\Theta_{2n+2i+1} = \frac{h^2[\Theta_{2m+2n+1} - \Theta_{2n-1}] - \Theta_{2m+2n+3} + \Theta_{2n+1}}{h^2 - \Theta_2 + 1},$$

$$\sum_{i=0}^{m}(-1)^i\Gamma_{2n+2i+1} = \frac{h^2[(-1)^m\Gamma_{2m+2n+1} + \Gamma_{2n-1}] + (-1)^m\Gamma_{2m+2n+3} + \Gamma_{2n+1}}{h^2 + \Theta_2 + 1}.$$

$$\sum_{i=0}^{m}(-1)^i\Theta_{2n+2i+1} = \frac{h^2[(-1)^m\Theta_{2m+2n+1} + \Theta_{2n-1}] + (-1)^m\Theta_{2m+2n+3} + \Theta_{2n+1}}{h^2 + \Theta_2 + 1}.$$

Theorem 3.3. *Let Γ_n be as in 1.2 and Θ_n be as in 1.3. Then, we have*

$$\sum_{i=0}^{m}\Gamma_{n+i}^2 = \frac{k^2}{k^2+h}\left[\frac{\Theta_{2n}+h^2(\Theta_{2m+2n}-\Theta_{2n-2})-\Theta_{2m+2n+2}}{h^2-\Theta_2+1} - 2(-h)^n\left(\frac{1-(-h)^{m+1}}{1+h}\right)\right],$$

$$\sum_{i=0}^{m}\Theta_{n+i}^2 = \left(\frac{\Theta_{2n}+h^2(\Theta_{2m+2n}-\Theta_{2n-2})-\Theta_{2m+2n+2}}{h^2-\Theta_2+1}\right) + 2(-h)^n\left(\frac{(-h)^{m+1}-1}{1+h}\right),$$

$$\sum_{i=0}^{m}\Gamma_{2n+2i}^2 = \frac{k^2}{k^2+h}\left[\frac{h^4(\Theta_{4m+4n}-\Theta_{4n-4})-\Theta_{4m+4n+4}+\Theta_{4n}}{h^4-\Theta_4+1} - (2h^{2n})\left(\frac{h^{2m+2}-1}{h^2-1}\right)\right],$$

$$\sum_{i=0}^{m}\Theta_{2n+2i}^2 = \left[\frac{h^4(\Theta_{4m+4n}-\Theta_{4n-4})-\Theta_{4m+4n+4}+\Theta_{4n}}{h^4-\Theta_4+1}\right] + (2h^{2n})\left(\frac{h^{2m+2}-1}{h^2-1}\right),$$

$$\sum_{i=0}^{m}\Gamma_{2n+2i+1}^2 = \frac{k^2}{k^2+h}\left[\frac{h^4(\Theta_{4m+4n+2}-\Theta_{4n-2})-\Theta_{4m+4n+6}+\Theta_{4n+2}}{h^4-\Theta_4+1} + (2h^{2n+1})\left(\frac{h^{2m+2}-1}{h^2-1}\right)\right],$$

$$\sum_{i=0}^{m}\Theta_{2n+2i+1}^2 = \left[\frac{h^4(\Theta_{4m+4n+2}-\Theta_{4n-2})-\Theta_{4m+4n+6}+\Theta_{4n+2}}{h^4-\Theta_4+1}\right] - (2h^{2n+1})\left(\frac{h^{2m+2}-1}{h^2-1}\right).$$

Theorem 3.4. *Let Γ_n be as in 1.2 and Θ_n be as in 1.3. Then, we have*

$$\sum_{i=0}^{m}(-1)^i\Gamma_{n+i}^2 = \frac{k^2}{k^2+h}\left[\frac{\Theta_{2n}+h^2((-1)^m\Theta_{2m+2n}+\Theta_{2n-2})+(-1)^m\Theta_{2m+2n+2}}{h^2+\Theta_2+1} - 2(-h)^n\left(\frac{1-h^{m+1}}{1-h}\right)\right],$$

$$\sum_{i=0}^{m}(-1)^i\Theta_{n+i}^2 = \left(\frac{\Theta_{2n}+h^2((-1)^m\Theta_{2m+2n}+\Theta_{2n-2})+(-1)^m\Theta_{2m+2n+2}}{h^2+\Theta_2+1}\right) + 2(-h)^n\left(\frac{1-h^{m+1}}{1-h}\right),$$

$$\sum_{i=0}^{m}(-1)^i\Gamma_{2n+2i}^2 = \frac{k^2}{k^2+h}\left[\frac{h^4((-1)^m\Theta_{4m+4n}+\Theta_{4n-4})+(-1)^m\Theta_{4m+4n+4}+\Theta_{4n}}{h^4+\Theta_4+1} - (2h^{2n})\left(\frac{h^{2m+2}(-1)^m+1}{h^2+1}\right)\right],$$

$$\sum_{i=0}^{m}(-1)^i\Theta_{2n+2i}^2 = \left[\frac{h^4((-1)^m\Theta_{4m+4n}-\Theta_{4n-4})+(-1)^m\Theta_{4m+4n+4}+\Theta_{4n}}{h^4+\Theta_4+1}\right] + (2h^{2n})\left(\frac{h^{2m+2}(-1)^m+1}{h^2+1}\right),$$

$$\sum_{i=0}^{m}(-1)^i\Gamma_{2n+2i+1}^2$$
$$=\frac{k^2}{k^2+h}\left[\frac{h^4((-1)^m\Theta_{4m+4n+2}+\Theta_{4n-2})+(-1)^m\Theta_{4m+4n+6}+\Theta_{4n+2}}{h^4+\Theta_4+1}+(2h^{2n+1})\left(\frac{h^{2m}(-1)^m+1}{h^2+1}\right)\right],$$
$$\sum_{i=0}^{m}(-1)^i\Theta_{2n+2i+1}^2$$
$$=\left[\frac{h^4((-1)^m\Theta_{4m+4n+2}+\Theta_{4n-2})+(-1)^m\Theta_{4m+4n+6}+\Theta_{4n+2}}{h^4+\Theta_4+1}\right]-(2h^{2n+1})\left(\frac{h^{2m+2}(-1)^m+1}{h^2+1}\right).$$

Theorem 3.5. *Let Γ_n be as in 1.2 and Θ_n be as in 1.3. Then, we have*

$$\sum_{i=0}^{m}\Gamma_i\Gamma_{i+1}=\frac{k^2}{k^2+h}[\frac{\Theta_1(1+h)+h^2\Theta_{2m+1}-\Theta_{2m+3}}{1-\Theta_2+h^2}-(\frac{1+h^{m+1}(-1)^m}{1+h})\Theta_1],$$
$$\sum_{i=0}^{m}\Theta_i\Theta_{i+1}=[\frac{\Theta_1(1+h)+h^2\Theta_{2m+1}-\Theta_{2m+3}}{1-\Theta_2+h^2}]+(\frac{1+h^{m+1}(-1)^m}{1+h})\Theta_1,$$
$$\sum_{i=0}^{m}\Gamma_{2n+2i}\Gamma_{2n+2i+1}$$
$$=\frac{k^2}{k^2+h}[\frac{\Theta_{4n+1}+h^4(\Theta_{4m+4n+1}-\Theta_{4n-3})-\Theta_{4n+4m+5}}{1-\Theta_4+h^4}-(\frac{1-h^{2m+2}}{1-h^2})h^{2n}\Theta_1],$$
$$\sum_{i=0}^{m}\Theta_{2n+2i}\Theta_{2n+2i+1}$$
$$=[\frac{\Theta_{4n+4m+5}-h^4(\Theta_{4m+4n+1}-\Theta_{4n-3})-\Theta_{4n+1}}{1-\Theta_4+h^4}-(\frac{1-h^{2m+2}}{1-h^2})h^{2n}\Theta_1].$$

4. Circulant matrices aspects

In this section, we consider particular circulant matrices involving (k,h)-Pell sequence and (k,h)-Pell-Lucas sequence. We obtain the eigenvalues and determinants of these circulant matrices. Then, we find some upper and lower bounds for the spectral norm of these circulant matrices.

Definition 4.1. [11] A matrix $C=[c_{i,j}]\in M_n$ is called a Circulant matrix if it is of the form $c_{i,j}=a_{j-i}$ for $j\geq i$, and $c_{i,j}=a_{n+j-i}$ for $j<i$.

Now, we define the $n\times n$ circulant matrices C_n and D_n with (k,h)-Pell sequence and (k,h)-Pell-Lucas sequence respectively by

$$C_n=circ(\Gamma_0,\Gamma_1,\Gamma_2,\cdots,\Gamma_{n-1}), \tag{4.1}$$

and

$$D_n=circ(\Theta_0,\Theta_1,\Theta_2,\cdots,\Theta_{n-1}), \tag{4.2}$$

where, Γ_n is the n^{th} term of (k,h)-Pell sequence and Θ_n is the n^{th} term of (k,h)-Pell-Lucas sequence.

From [3] we have the following theorem that presents a formula to compute the eigenvalues of a circulant matrix.

Theorem 4.1. *Let $C = circ(c_0, c_1, c_2, \cdots, c_{n-1})$ is an $n \times n$ circulant matrix. Then, the eigenvalues of C are*

$$\lambda_j = \sum_{k=0}^{n-1} c_k w^{jk}, \quad j = 0, 1, \cdots, n-1$$

where, $w = \exp(\frac{2\pi i}{n})$ *and* $i = \sqrt{-1}$.

By applying this theorem, we have the following corollary about the eigenvalues of circulant matrix 4.1.

Corollary 4.1. *Let C_n be a circulant matrix as in 4.1 and Λ_j for $j = 0, 1, 2, \cdots, n-1$, be the eigenvalues of C_n. Then, we have*

$$\Lambda_0 = \sum_{m=0}^{n-1} \Gamma_m = \frac{\Gamma_n + h\Gamma_{n-1} - 2k}{2k + h - 1},$$

and for $j \geq 1$ we have

$$\Lambda_j = \frac{(2k - h\Gamma_{n-1})w^j - \Gamma_n}{1 - (2k + hw^j)w^j}.$$

where, $w = exp(\frac{2\pi i}{n})$ *and* $i = \sqrt{-1}$.

Proof. For $j = 0$ the result follows from Corollary2.1 and theorem4.1.
For $j \geq 1$, by lemma2.1 and theorem4.1 we have

$$\begin{aligned} \Lambda_j &= \sum_{m=0}^{n-1} \Gamma_m w^{jm} = \sum_{m=0}^{n-1} \frac{2k}{p}(\alpha^m - \beta^m)\left(e^{\frac{2\pi i j}{n}}\right)^m \\ &= \frac{2k}{p} \sum_{m=0}^{n-1} \left((\alpha e^{\frac{2\pi i j}{n}})^m - (\beta e^{\frac{2\pi i j}{n}})^m\right). \end{aligned}$$

According to 1.4, we get

$$\Lambda_j = \frac{2k}{p}\left[\frac{1 - \alpha^n \left(e^{\frac{2\pi i j}{n}}\right)^n}{1 - \alpha e^{\frac{2\pi i j}{n}}} - \frac{1 - \beta^n \left(e^{\frac{2\pi i j}{n}}\right)^n}{1 - \beta e^{\frac{2\pi i j}{n}}}\right] = \frac{2k}{p}\left[\frac{1 - \alpha^n}{1 - \alpha w^j} - \frac{1 - \beta^n}{1 - \beta w^j}\right],$$

where, $w = \exp(\frac{2\pi i}{n})$ and $i = \sqrt{-1}$.
By some computations, we obtain

$$\begin{aligned} \Lambda_j &= \frac{2k}{p}\left[\frac{(\alpha - \beta)w^j - (\alpha^n - \beta^n) + \alpha\beta(\alpha^{n-1} - \beta^{n-1})w^j}{(1 - \alpha w^j)(1 - \beta w^j)}\right] \\ &= \frac{2k}{p}\left[\frac{(\alpha - \beta)w^j - (\alpha^n - \beta^n) + \alpha\beta(\alpha^{n-1} - \beta^{n-1})w^j}{1 - (\alpha + \beta)w^j + (\alpha\beta)w^{2j}}\right]. \end{aligned}$$

Consequently, by lemma 2.1, we deduce that

$$\Lambda_j = \frac{\Gamma_1 w^j - \Gamma_n - h\Gamma_{n-1}w^j}{1 - 2kw^j - hw^{2j}} = \frac{(2k - h\Gamma_{n-1})w^j - \Gamma_n}{1 - (2k + hw^j)w^j}.$$

Thus, the proof is completed. □

Now, we obtain the determinant of circulant matrix 4.1. Firstly, we need the following lemma from [3].

Lemma 4.1. *Let x and y are real variables and $w = \exp(\frac{2\pi i}{n})$ where, $i = \sqrt{-1}$. Then*

$$\prod_{j=0}^{n-1}(x - yw^j) = x^n - y^n.$$

As a result of the lemma 4.1 we have the following corollary.

Corollary 4.2. *Let C_n be a circulant matrix as in 4.1. Then, determinant of C_n is given by*

$$\det(C_n) = \frac{(2k - h\Gamma_{n-1})^n - (\Gamma_n)^n}{1 - \Theta_n + (-h)^n}.$$

Proof. By corollary 4.1, we have

$$\det(C_n) = \prod_{j=0}^{n-1}\Lambda_j = \prod_{j=0}^{n-1}\frac{(2k - h\Gamma_{n-1})w^j - \Gamma_n}{(1 - \alpha w^j)(1 - \beta w^j)} = \frac{\prod_{j=0}^{n-1}\left((2k - h\Gamma_{n-1})w^j - \Gamma_n\right)}{\prod_{j=0}^{n-1}(1 - \alpha w^j)\prod_{j=0}^{n-1}(1 - \beta w^j)}.$$

Consequently, according to lemma 4.1 we get

$$det(C_n) = \frac{(2k - h\Gamma_{n-1})^n - (\Gamma_n)^n}{(1 - \alpha^n)(1 - \beta^n)} = \frac{(2k - h\Gamma_{n-1})^n - (\Gamma_n)^n}{1 - \Theta_n + (-h)^n}.$$

□

Also, we have the following corollaries about the eigenvalues and determinant of circulant matrix 4.2.

Corollary 4.3. *Let D_n be a circulant matrix as in 4.2 and ν_j for $j = 0, 1, 2, \cdots, n-1$ be the eigenvalues of D_n. Then, we have*

$$\nu_0 = \sum_{m=0}^{n-1}\Theta_m = \frac{\Theta_n + h\Theta_{n-1} + 2k - 2}{2k + h - 1}.$$

and for $j \geq 1$ we have

$$\nu_j = \frac{2 - \Theta_n - (2k + h\Theta_{n-1})w^j}{1 - (2k + hw^j)w^j}.$$

where, $w = \exp(\frac{2\pi i}{n})$ and $i = \sqrt{-1}$.

Proof. The proof is similar to corollary4.1. □

Corollary 4.4. *Let D_n be a circulant matrix as in 4.2. Then, determinant of of D_n is given by*

$$\det(D_n) = |D_n| = \frac{(2-\Theta_n)^n - (2k + h\Theta_{n-1})^n}{1 - \Theta_n + (-h)^n}.$$

Proof. The proof is similar to corollary 4.2. □

By next theorem, we give the Euclidean norm of circulant matrix C_n.

Theorem 4.2. *Let C_n be a circulant matrix as in 4.1. Then, the Euclidean norm of C_n is*

$$\|C_n\|_E = \frac{2\sqrt{n}k}{p}\sqrt{\frac{h^2\Theta_{2n-2} - \Theta_{2n} - \Theta_2 + 2}{h^2 - \Theta_2 + 1} + 2\left(\frac{(-h)^n - 1}{h+1}\right)}.$$

Proof. By definition of Euclidean norm, we have

$$\|C_n\|_E^2 = n\left(\Gamma_0^2 + \Gamma_1^2 + \Gamma_2^2 + \cdots + \Gamma_{n-1}^2\right) = n\sum_{k=0}^{n-1}\Gamma_k^2.$$

By theorem2.5 we obtain

$$\|C_n\|_E^2 = n\frac{4k^2}{p^2}\left[\frac{h^2\Theta_{2n-2} - \Theta_{2n} - \Theta_2 + 2}{h^2 - \Theta_2 + 1} + 2\frac{(-h)^n - 1}{h+1}\right].$$

Conseqently, by taking $(\frac{1}{2})^{th}$ power from the both sides of the above equality, we get

$$\|C_n\|_E^2 = \frac{2\sqrt{n}k}{p}\sqrt{\frac{h^2\Theta_{2n-2} - \Theta_{2n} - \Theta_2 + 2}{h^2 - \Theta_2 + 1} + 2\left(\frac{(-h)^n - 1}{h+1}\right)}.$$

□

Corollary 4.5. *Let C_n be a circulant matrix as in 4.1. Then, we have the following upper bound and lower bound for the spectral norm of C_n*

$$\frac{2k}{p}\sqrt{\frac{h^2\Theta_{2n-2} - \Theta_{2n} - \Theta_2 + 2}{h^2 - \Theta_2 + 1} + 2\left(\frac{(-h)^n - 1}{h+1}\right)} \le \|C_n\|_2$$

$$\le \frac{2\sqrt{n}k}{p}\sqrt{\frac{h^2\Theta_{2n-2} - \Theta_{2n} - \Theta_2 + 2}{h^2 - \Theta_2 + 1} + 2\left(\frac{(-h)^n - 1}{h+1}\right)}.$$

Proof. It follows from 1.8 and theorem 4.2. □

Now, we find the Euclidean norm of circulant matrix D_n.

Theorem 4.3. *Let D_n be a circulant matrix as in 4.2. Then, the Euclidean norm of D_n is*

$$\|D_n\|_E = \sqrt{n}\sqrt{\frac{2-\Theta_2+h^2\Theta_{2n-2}-\Theta_{2n}}{1-\Theta_2+h^2}+2\left(\frac{1-(-h)^n}{1+h}\right)}.$$

Proof. The proof is similar to theorem4.2. □

Corollary 4.6. *Let D_n be a circulant matrix as in 4.2. Then, we have the following upper bound and lower bound for the spectral norm of D_n*

$$\sqrt{\frac{2-\Theta_2+h^2\Theta_{2n-2}-\Theta_{2n}}{1-\Theta_2+h^2}+2\left(\frac{1-(-h)^n}{1+h}\right)} \le \|D_n\|_2$$
$$\le \sqrt{n}\sqrt{\frac{2-\Theta_2+h^2\Theta_{2n-2}-\Theta_{2n}}{1-\Theta_2+h^2}+2\left(\frac{1-(-h)^n}{1+h}\right)}.$$

Proof. It follows from 1.8 and theorem4.3. □

By using the definition of Hadamard product, we obtain an upper bound for the spectral norm circulant matrix 4.1.

Theorem 4.4. *Let C_n be a matrix as in 4.1. Then, we have the following upper bound for the spectral norm of C_n.*

$$\|C_n\|_2 \le \frac{4k^2}{p^2}\left[\frac{h^2\Theta_{2n-2}-\Theta_{2n}-\Theta_2+2}{h^2-\Theta_2+1}+2\left(\frac{(-h)^n-1}{h+1}\right)\right].$$

Proof. By definition of Hadamard product for C_n, we have

$$\begin{aligned}
C_n &= \begin{bmatrix} \Gamma_0 & \Gamma_1 & \Gamma_2 & \cdots & \Gamma_{n-1} \\ \Gamma_1 & \Gamma_2 & \Gamma_3 & \cdots & \Gamma_0 \\ \Gamma_2 & \Gamma_3 & \Gamma_4 & \cdots & \Gamma_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \Gamma_{n-1} & \Gamma_0 & \Gamma_1 & \cdots & \Gamma_{n-2} \end{bmatrix} \\
&= \begin{bmatrix} 1 & \Gamma_1 & \Gamma_2 & \cdots & \Gamma_{n-1} \\ 1 & 1 & \Gamma_3 & \cdots & \Gamma_0 \\ 1 & 1 & 1 & \cdots & \Gamma_1 \\ \vdots & \vdots & \vdots & \vdots & \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix} \circ \begin{bmatrix} \Gamma_0 & 1 & 1 & \cdots & 1 \\ \Gamma_1 & \Gamma_2 & 1 & \cdots & 1 \\ \Gamma_2 & \Gamma_3 & \Gamma_4 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \\ \Gamma_{n-1} & \Gamma_0 & \Gamma_1 & \cdots & \Gamma_{n-2} \end{bmatrix} \\
&= A \circ B.
\end{aligned}$$

By definition of maximum row length norm and maximum column length norm, we have

$$\begin{aligned} r_1(A) &= \max_i \sqrt{\sum_j |a_{ij}|^2} = \sqrt{\sum_{j=0}^{n-1} \Gamma_j^2} \\ &= \sqrt{\frac{4k^2}{p^2}\left[\frac{h^2\Theta_{2n-2} - \Theta_{2n} - \Theta_2 + 2}{h^2 - \Theta_2 + 1} + 2\left(\frac{(-h)^n - 1}{h+1}\right)\right]}. \end{aligned}$$

Also, we have

$$\begin{aligned} c_1(B) &= \max_j \sqrt{\sum_i |b_{ij}|^2} = \sqrt{\sum_{j=0}^{n-1} \Gamma_j^2} \\ &= \sqrt{\frac{4k^2}{p^2}\left[\frac{h^2\Theta_{2n-2} - \Theta_{2n} - \Theta_2 + 2}{h^2 - \Theta_2 + 1} + 2\left(\frac{(-h)^n - 1}{h+1}\right)\right]}. \end{aligned}$$

According to 1.9, we obtain

$$\begin{aligned} \|C_n\|_2 &\leq r_1(A)c_1(B) \\ &= \left(\sqrt{\frac{4k^2}{p^2}\left[\frac{h^2\Theta_{2n-2} - \Theta_{2n} - \Theta_2 + 2}{h^2 - \Theta_2 + 1} + 2\left(\frac{(-h)^n - 1}{h+1}\right)\right]}\right)^2 \\ &= \frac{4k^2}{p^2}\left[\frac{h^2\Theta_{2n-2} - \Theta_{2n} - \Theta_2 + 2}{h^2 - \Theta_2 + 1} + 2\left(\frac{(-h)^n - 1}{h+1}\right)\right]. \end{aligned}$$

□

Now, we obtain an upper bound for the spectral norm of 4.2.

Theorem 4.5. *Let D_n be a matrix as in 4.2. Then, we have the following upper bound for the spectral norm of D_n.*

$$\|D_n\|_2 \leq \left(\frac{2 - \Theta_2 + h^2\Theta_{2n-2} - \Theta_{2n}}{1 - \Theta_2 + h^2}\right) + 2\left(\frac{1-(-h)^n}{1+h}\right).$$

Proof. The proof is similar to theorem4.4. □

5. Numerical example

In this section, by using **MATLAB software**, we present two numerical examples about the eigenvalues, determinants and Euclidean norm of circulant matrices C_n and D_n for some values of n.

Example 5.1. Following table shows the eignvalues, determinants and Euclidean norm of circulant matrix C_n associated with (k,h)-Pell sequence for some values of n (for $h = 1, k = 1$).

n	det(C_n)	$\|C_n\|_E$	Eigenvalues of circulant matrix C_n
2	-4	2.8284	-2, 2
3	-72	7.7460	-3.4641, 3.4641, 6
4	10240	21.9089	-8.9443, -8, 8.9443, 16
5	8067200	58.9915	-24.67.6, -18.2034, 24.6706, 18.2034, 40
6	-3.8259+e10	156.0769	-66.0908, 46.1303, 66.0908, 46.1303, -42, 98

Example 5.2. Following table shows the eignvalues, determinants and Euclidean norm of circulant matrix D_n associated with (k,h)-Pell-Lucass sequence for some values of n (for $h = 1, k = 1$).

n	det(D_n)	$\|D_n\|_E$	Eigenvalues of circulant matrix D_n
2	-4	2.8284	-2, 2
3	-224	10.9545	-5.2915, 5.2915, 8
4	39600	30.7246	-13.4164, -10, 13.4164, 22
5	4.02+e7	83.4266	-35.4980 , -25.5322 , 35.4980 , 25.5322 , 56
6	-3.0554+e11	220.6717	-93.8723 , -65.8179, 93.8723, 65.8179 , -58, 138

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