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CLASSIFICATIONS OF UNITARY KRASNER HYPERRINGS OF SMALL ORDER

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Abstract. In this article, we investigate the distributability of the binary operation of monoids with zero compared to the hyperoperation of canonical hypergroups of order 2 and 3 with the help of analytical and algebraic methods and without using computer calculations. Then, the unitary Krasner hyperrings of orders 2 and 3 are counted and classified up to isomorphism. In the end, we show the Krasner hyperfields of small order with their Cayley tables.
Keywords: algebra, hypergroups, Cayley table.

1. Introduction

The theory of hyperstructures was introduced by Marty in 1934 at the eighth congress of Scandinavian mathematicians. He introduced hypergroups as a generalization of groups to solve non-invertible algebraic problems. A hypergroup is a set equipped with a hyperoperation $\circ : H \times H \longrightarrow P^*(H)$ that satisfies the associative law and the reproduction law. Hyperstructures have applications in various fields such as optimization theory, formal language theory, geometry, graph theory, fuzzy sets, cryptography, automata, binary networks, computer program analysis, codes, and artificial intelligence. Krasner was the first to introduce the concept of

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hyperrings through the creation of structures containing operations and hyperoperations. He also replaced a collective group of a field with a special hypergroup and introduced a hyperfield [13]. In reference [10], various types of hyperrings and their applications can be seen. Among the introduced types of hyperrings, Krasner's hyperrings have received more attention from researchers due to their proximity to ordinary rings. The classification and counting of rings and fields to a certain order and within the isomorphism class are fundamental topics in ring and field theory. Dealing with this subject in hyperstructure theory has also been underway by researchers for some time. Due to the large number of hyperoperations compared to operations on a set, the classification and counting of hyperstructures cannot be compared to the classification and counting of algebraic structures.

Vaziri and et. al. [16] enumerated and classified Krasner hyperrings and new classes of hyperrings (weak distributive, left inclusion distributive, right inclusion distributive, left near, right near) up to isomorphism of order less than 4 and then determined their automorphism groups by computer calculations. Ameri et al. [5] have counted the number of hyperfields (of Krasner type) up to order 6 using computer programs. Iranmanesh et al. [12] have counted polygroups and Krasner hyperfields of order 4. In article [6], only the number of third-order hyperrings has been counted without any classification. Also, in articles [5, 12], Krasner's hyperfields have been counted up to small orders. The idea of obtaining the Cayley table of hyperrings without computer calculations for certain Krasner hyperrings leads us to classify and count unitary Krasner hyperrings up to isomorphism. One of the direct applications of classifying Krasner hyperrings and hyperfields is to study and gain a deeper understanding of hypervector spaces over Krasner hyperfields and hypermodules over Krasner hyperrings. In addition, counting complete hyperrings, polysymmetrical hyperrings made from Krasner hyperrings, and (H, R)-hyperrings using the classification of Krasner hyperrings and having their automorphism groups will be possible at small orders.

Vaziri and et. al. [16] computed Krasner hyperrings by computer programming. In this article, by using canonical hypergroups and monoids with zero of order 3 to in vestigate the types of distributivity of the semigroup operation on the corresponding canonical hypergroup, and calculations have been performed. We have obtained all unitary Krasner hyperrings and Krasner hyperfields of order 3 by analytical and algebraic methods without using computer calculations. Here, in addition to counting, the structure of the unitary Krasner hyperring and its Cayley tables have been determined precisely. In section 2, the necessary definitions are presented. In section 3, the exact number (within the isomorphism class) of all unitary Krasner hyperrings and Krasner hyperfields with orders less than 4 are determined and classified.

Definition 1.1. [11] Let $M = (M, \cdot, 1, 0)$ be a monoid $(M, \cdot, 1)$ together with an distinguished absorbing element $0 \in M$, that is such that $\forall x \in M : \ 0 \cdot x = 0 = x \cdot 0$.

A monoid with an distinguished absorbing element 0 is called a monoid with zero in the literature.

Definition 1.2. [10] A Krasner hyperring is an algebraic hyperstructure $(R, +, \cdot)$ $(R, +)$ is a commutative polygroup(canonical hypergroup) and $(R, \cdot)$ is a semigroup with zero element 0 such that $\cdot$ is strongly distributive with respect to $+$.

Definition 1.3. A unitary Krasner hyperring is a Krasner hyperring such that $(R, \cdot)$ is a monoid(identity element is called 1), that means

$$1 \cdot x = x \cdot 1 = x, \ \forall x \in R.$$

Definition 1.4. [10] A Krasner hypefield is a Krasner hyperring such that $(R - \{0\}, \cdot)$ is a group.

2. Computation of unitary Krasner hyperrings of order 2

According to the preliminary concepts in algebraic hyperstructures and with direct calculations, we have the following Theorem:

Theorem 2.1. *There are 8 hypergroups $(R, +_i)$ of order 2, where $R = \{0, 1\}$ and $i = 1, 2, \ldots, 8$, up to isomorphism. We have listed these eight hypergroups with their Cayley's tables below:*

$+_1$	0	1
0	0	1
1	1	0

$+_2$	0	1
0	0	1
1	1	R

$+_3$	0	1
0	0	R
1	R	R

$+_4$	0	1
0	R	R
1	R	0

$+_5$	0	1
0	R	R
1	0	1

$+_6$	0	1
0	0	R
1	1	R

$+_7$	0	1
0	0	R
1	R	1

$+_8$	0	1
0	R	R
1	R	R

Table 1: All hypergroups of order 2 up to isomorphism

By using Theorem 2.1, we obtain:

Corollary 2.1. *There are 2 canonical hypergroups $(R, +_1)$ and $(R, +_2)$ of order 2.*

Theorem 2.2. *There exist 5 non isomorphic semigroups $(R, \cdot_j)$ of order 2, where $j = 1, 2, \ldots, 5$, as following Cayley's Tables:*

$\cdot_1$	0	1
0	0	0
1	0	0

$\cdot_2$	0	1
0	0	0
1	0	1

$\cdot_3$	0	1
0	0	0
1	1	1

$\cdot_4$	0	1
0	0	1
1	0	1

$\cdot_5$	0	1
0	0	1
1	1	0

Table 2: All semigroups of order 2 up to isomorphism

Corollary 2.2. *There are 2 non-isomorphic monoids $(R, \cdot_2)$ and $(R, \cdot_5)$ of order 2. There exists only 1 non-isomorphic monoid with zero $(R, \cdot_2)$.*

Let $R = \{0, 1\}$. By check of distributive law for Krasner hyperrings, we obtain:

Theorem 2.3. *There exist 4 non-isomorphic Krasner hyperrings of order 2 as follows:*

$(R, +_1, \cdot_1)$:

$+_1$	0	1
0	0	1
1	1	0

$\cdot_1$	0	1
0	0	0
1	0	0

$(R, +_1, \cdot_2)$:

$+_1$	0	1
0	0	1
1	1	0

$\cdot_2$	0	1
0	0	0
1	0	1

$(R, +_2, \cdot_1)$:

$+_2$	0	1
0	0	1
1	1	R

$\cdot_1$	0	1
0	0	0
1	0	0

$(R, +_2, \cdot_2)$:

$+_2$	0	1
0	0	1
1	1	R

$\cdot_2$	0	1
0	0	0
1	0	1

Table 3: All Krasner hyperrings of order 2 up to isomorphism

Proof. It is straightforward. □

Corollary 2.3. *There are 2 non-isomorphic unitary Krasner hyperrings* $(R, +_1, \cdot_2)$ *and* $(R, +_2, \cdot_2)$*. Moreover,* $(R, +_1, \cdot_2)$ *and* $(R, +_2, \cdot_2)$ *are hyperfields.*

3. Computation of unitary Krasner hyperrings of order 3

In this section let $R = \{0, 1, 2\}$. To obtain all unitary Krasner hyperrings, we consider the all non-isomorphic canonical hypergroups and all monoids with zero element 0, of order 3.

Theorem 3.1. *[9] There are only 10 non-isomorphic canonical hypergroups of* *order 3 as following:*

$+_1$	0	1	2
0	0	1	2
1	1	0	2
2	2	2	0, 1

$+_2$	0	1	2
0	0	1	2
1	1	0	2
2	2	2	R

$+_3$	0	1	2
0	0	1	2
1	1	0, 1	2
2	2	2	0, 1

$+_4$	0	1	2
0	0	1	2
1	1	0, 1	2
2	2	2	R

$+_5$	0	1	2
0	0	1	2
1	1	0, 2	1, 2
2	2	1, 2	R

$+_6$	0	1	2
0	0	1	2
1	1	0, 2	1, 2
2	2	1, 2	0, 1

$+_7$	0	1	2
0	0	1	2
1	1	R	1, 2
2	2	1, 2	R

$+_8$	0	1	2
0	0	1	2
1	1	1	R
2	2	R	2

$+_9$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

$+_{10}$	0	1	2
0	0	1	2
1	1	1, 2	R
2	2	R	1, 2

Table 4: All canonical hypergroups of order 3 up to isomorphism

Theorem 3.2. *[11] There are only 3 non-isomorphic monoids with zero of order 3:*

$\cdot_1$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	0

$\cdot_2$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

$\cdot_3$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	2

Table 5: All monoids with zero of order 3 up to isomorphism

For obtain the all hyperrnigs we need to all permutations of members respect to multiplications $\cdot_i$ (or $+_i$) in Theorem 3.2 (Theorem 3.1). Since for every isomorphism η of a semigroup with zero element 0 we have $\eta(0) = 0$, therefore we have two isomorphisms $id = \begin{cases} 0 \to 0 \\ 1 \to 1 \\ 2 \to 2 \end{cases}$ and $\eta = (ab) = \begin{cases} 0 \to 0 \\ 1 \to 2 \\ 2 \to 1 \end{cases}$. So by applying $\eta = (12)$ on three semigroups $(R, \cdot_j)$, where $j = 1, 2, 3$, in Theorem 3.2 we obtain 3 semigroups $(R, \cdot_j)$, where $j = 4, 5, 6$, as following that are isomorph with semigroups in Theorem 3.2, but we need to them for obtain the all states.

$\cdot_4$	0	1	2
0	0	0	0
1	0	0	1
2	0	1	2

$\cdot_5$	0	1	2
0	0	0	0
1	0	2	1
2	0	1	2

$\cdot_6$	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

In fact, for all $j = 1, 2, 3$, $(R, \cdot_j) \cong (R, \cdot_{j+3})$.

In order to check of distributive law for Krasner hyperrings, note that $(R, +_i)$, $i = 1, 2, \ldots, n$, are commutative and so $a \cdot_j (b +_i c) = (b +_i c) \cdot_j a$, where $j = 1, 2, \ldots, 6$. The number of all states for $a \cdot_j (b +_i c)$ is 27 states(for $(R, +_i, \cdot_j)$). We obtain:

Lemma 3.1. *For all $a, b, c \in R$, we have*

(1) If $a \in \{0, 1\}$ then $a \cdot_j (b +_i c) = b +_i c = (b +_i c) \cdot_j a$,(18 states)

(2) If $b = 0$ or $c = 0$ then $a \cdot_j (b +_i c) = (b +_i c) \cdot_j a$.(5 states)

Remark 3.1. According to the above points and Lemma 3.1, it is enough to check the following three situations:

$$2 \cdot_j (2 +_i 2); \qquad 2 \cdot_j (1 +_i 2); \qquad 2 \cdot_j (1 +_i 1). \qquad (*)$$

Note that $2 \cdot_j (1 +_i 2) = 2 \cdot_j (2 +_i 1)$.

Theorem 3.3. *For all $i = 1, 2, \ldots, 10$ and $j = 1, 3, 6$, the hyperstructure $(R, +_i, \cdot_j)$ is not a unitary Krasner hyperring.*

Proof. For $i \neq 8$, we have $1 \notin 2 \cdot_j (1 +_i 1)$ but $1 \in 2 \cdot_j 1 +_i 2 \cdot_j 1$ and so distributive law is not holds.

For $i = 8$, $2 \cdot_3 (1 +_8 2) = \{0, 2\} \neq \{2\} = 2 \cdot_3 1 +_8 2 \cdot_3 2$ and $1 \cdot_6 (2 +_8 1) = \{0, 1\} \neq \{1\} = 1 \cdot_6 2 +_8 1 \cdot_6 1$. □

Theorem 3.4. *For all $i = 1, 2, \ldots, 5$, the hyperstructures $(R, +_i, \cdot_2)$ and $(R, +_i, \cdot_5)$ are not unitary Krasner hyperrings.*

Proof. For $i \neq 5$, we have $1 \notin 2 \cdot_2 (1 +_i 1)$ but $1 \in 2 \cdot_2 1 +_i 2 \cdot_2 1$ and for $i = 5$, $2 \notin 2 \cdot_2 (1 +_i 1)$ but $2 \in 2 \cdot_2 1 +_i 2 \cdot_3 1$. Therefor $(R, +_i, \cdot_2)$ is not a Krasner hyperring.

For $i \neq 5$, we have $2 \in 2 \cdot_5 (2 +_i 2)$ but $2 \notin 2 \cdot_5 2 +_i 2 \cdot_5 2$ and for $i = 5$, $2 \notin 2 \cdot_5 (2 +_i 2)$ but $2 \in 2 \cdot_5 2 +_i 2 \cdot_5 2$. Therefor $(R, +_i, \cdot_5)$ is not a Krasner hyperring. □

Theorem 3.5. *For all $i = 6, 7, \ldots, 10$, the hyperstructure $(R, +_i, \cdot_2)$ is a unitary Krasner hyperring.*

Proof. According to Lemma 3.1 and Remark 3.1, it is sufficient to check only the following three conditions for distributability

$$2 \cdot_2 (2 +_i 2); \qquad 2 \cdot_2 (1 +_i 2); \qquad 2 \cdot_2 (1 +_i 1).$$

For $i = 6$, we have:

√ $2 \cdot_2 (2 +_6 2) = \{0, 2\} = 2 \cdot_2 2 +_6 2 \cdot_2 2$.

√ $2 \cdot_2 (1 +_6 2) = \{1, 2\} = 2 \cdot_2 1 +_6 2 \cdot_2 2$.

$\surd$ $2 \cdot_2 (1 +_6 1) = \{0, 1\} = 2 \cdot_2 1 +_6 2 \cdot_2 1.$

For $i = 7$, we have:

$\surd$ $2 \cdot_2 (2 +_7 2) = \{0, 1, 2\} = 2 \cdot_2 2 +_7 2 \cdot_2 2.$

$\surd$ $2 \cdot_2 (1 +_7 2) = \{1, 2\} = 2 \cdot_2 1 +_7 2 \cdot_2 2.$

$\surd$ $2 \cdot_2 (1 +_7 1) = \{0, 1, 2\} = 2 \cdot_2 1 +_7 2 \cdot_2 1.$

For $i = 8$, we have:

$\surd$ $2 \cdot_2 (2 +_8 2) = \{1\} = 2 \cdot_3 2 +_8 2 \cdot_2 2.$

$\surd$ $2 \cdot_2 (1 +_8 2) = \{0, 1, 2\} = 2 \cdot_2 1 +_8 2 \cdot_2 2.$

$\surd$ $2 \cdot_2 (1 +_8 1) = \{1\} = 2 \cdot_2 1 +_8 2 \cdot_2 1.$

For $i = 9$, we have:

$\surd$ $2 \cdot_2 (2 +_9 2) = \{2\} = 2 \cdot_3 2 +_9 2 \cdot_2 2.$

$\surd$ $2 \cdot_2 (1 +_9 2) = \{0\} = 2 \cdot_2 1 +_9 2 \cdot_2 2.$

$\surd$ $2 \cdot_2 (1 +_9 1) = \{1\} = 2 \cdot_2 1 +_9 2 \cdot_2 1.$

For $i = 10$, we have:

$\surd$ $2 \cdot_2 (2 +_{10} 2) = \{1, 2\} = 2 \cdot_2 2 +_{10} 2 \cdot_2 2.$

$\surd$ $2 \cdot_2 (1 +_{10} 2) = \{0, 1, 2\} = 2 \cdot_2 1 +_{10} 2 \cdot_2 2.$

$\surd$ $2 \cdot_2 (1 +_{10} 1) = \{1, 2\} = 2 \cdot_2 1 +_{10} 2 \cdot_2 1.$

Therefor proof is complete. □

Theorem 3.6. *For all $i = 6, 7, \ldots, 10$, the hyperstructure $(R, +_i, \cdot_5)$ is a unitary Krasner hyperring.*

Proof. According to Lemma 3.1 and Remark 3.1, it is sufficient to check only the following three conditions for distributivity

$$1 \cdot_5 (2 +_i 2); \qquad 1 \cdot_5 (1 +_i 2); \qquad 1 \cdot_5 (1 +_i 1).$$

(Note that in $(R, \cdot_5)$, identity element is 2).

For $i = 6$, we have:

$\surd$ $1 \cdot_5 (2 +_6 2) = \{0, 2\} = 1 \cdot_5 2 +_6 1 \cdot_5 2.$

$\surd$ $1 \cdot_5 (1 +_6 2) = \{1, 2\} = 1 \cdot_5 1 +_6 1 \cdot_5 2.$

$\surd$ $1 \cdot_5 (1 +_6 1) = \{0, 1\} = 1 \cdot_5 1 +_6 1 \cdot_5 1$.

For $i = 7$, we have:

$\surd$ $1 \cdot_5 (2 +_7 2) = \{0, 1, 2\} = 1 \cdot_5 2 +_7 1 \cdot_5 2$.

$\surd$ $1 \cdot_5 (1 +_7 2) = \{1, 2\} = 1 \cdot_5 1 +_7 1 \cdot_5 2$.

$\surd$ $1 \cdot_5 (1 +_7 1) == \{0, 1, 2\} = 1 \cdot_5 1 +_7 1 \cdot_5 1$.

For $i = 8$, we have:

$\surd$ $1 \cdot_5 (2 +_8 2) = \{1\} = 1 \cdot_5 2 +_8 1 \cdot_5 2$.

$\surd$ $1 \cdot_5 (1 +_8 2) = \{0, 1, 2\} = 1 \cdot_5 1 +_8 1 \cdot_5 2$.

$\surd$ $1 \cdot_5 (1 +_8 1) = \{2\} = 1 \cdot_5 1 +_8 1 \cdot_5 1$.

For $i = 9$, we have:

$\surd$ $1 \cdot_5 (2 +_9 2) = \{2\} = 1 \cdot_5 2 +_9 1 \cdot_5 2$.

$\surd$ $1 \cdot_5 (1 +_9 2) = \{0\} = 1 \cdot_5 1 +_9 1 \cdot_5 2$.

$\surd$ $1 \cdot_5 (1 +_9 1) = \{1\} = 1 \cdot_5 1 +_9 1 \cdot_5 1$.

For $i = 10$, we have:

$\surd$ $1 \cdot_5 (2 +_{10} 2) = \{1, 2\} = 1 \cdot_5 2 +_{10} 1 \cdot_5 2$.

$\surd$ $1 \cdot_5 (1 +_{10} 2) = \{0, 1, 2\} = 1 \cdot_5 1 +_{10} 1 \cdot_5 2$.

$\surd$ $1 \cdot_5 (1 +_{10} 1) = \{1, 2\} = 1 \cdot_5 1 +_{10} 1 \cdot_5 1$.

Therefor proof is complete. □

Lemma 3.2. *For $i = 6, 7, 8, 9, 10$, $(R, +_i, \cdot_2) \cong (R, +_i, \cdot_5)$.*

Proof. Since for $i = 6, 7, 8, 9, 10$ set $\xi = (1\ 2)$. It is not difficult to see that ξ is an isomorphism. □

Theorem 3.7. *For all $i \in \{2, 3, 5, 6, \ldots, 10\}$, the hyperstructure $(R, +_i, \cdot_4)$ is not a Krasner hyperring.*

Proof. For $i \in \{2, 3, 5, 6, 7, 9, 10\}$, we have $1 \cdot_4 (2 +_i 2) \neq 1 \cdot_4 2 +_i 1 \cdot_4 2$ and for $i = 8$, $2 \cdot_4 (1 +_i 1) \neq 2 \cdot_4 1 +_i 2 \cdot_4 1$. Therefor $(R, +_i, \cdot_4)$ is not a Krasner hyperring. □

Theorem 3.8. *For all $i \in \{1, 4\}$, the hyperstructure $(R, +_i, \cdot_4)$ is a unitary Krasner hyperring.*

Proof. According to Lemma 3.1 and Remark 3.1, it is sufficient to check only the following three conditions for distributivity

$$1 \cdot_j (2 +_i 2); \qquad 1 \cdot_j (1 +_i 2); \qquad 1 \cdot_j (1 +_i 1).$$

Where $j = 2, 5$.(Note that in $(R, \cdot_4)$, identity element is 2).

For $i = 1$ and $j = 4$, we have:

√ $1 \cdot_4 (2 +_1 2) = \{0\} = 1 \cdot_4 2 +_1 1 \cdot_4 2$.

√ $1 \cdot_4 (1 +_1 2) = \{1\} = 1 \cdot_4 1 +_1 1 \cdot_4 2$.

√ $1 \cdot_4 (1 +_1 1) = \{0\} = 1 \cdot_4 1 +_1 1 \cdot_4 1$.

For $i = 4$ and $j = 4$, we have:

√ $1 \cdot_4 (2 +_4 2) = \{0, 1\} = 1 \cdot_4 2 +_4 1 \cdot_4 2$.

√ $1 \cdot_4 (1 +_4 2) = \{1\} = 1 \cdot_4 1 +_4 1 \cdot_4 2$.

√ $1 \cdot_4 (1 +_4 1) == \{0\} = 1 \cdot_4 1 +_4 1 \cdot_4 1$.

Therefor proof is complete. □

Theorem 3.9. *Let $R = \{0, 1, 2\}$ then we obtain 7 non-isomorphic unitary Krasner hyperrings as follows:*

$+_1$	0	1	2
0	0	1	2
1	1	0	2
2	2	2	0, 1

$\cdot_4$	0	1	2
0	0	0	0
1	0	0	1
2	0	1	2

$+_4$	0	1	2
0	0	1	2
1	1	0, 1	2
2	2	2	R

$\cdot_4$	0	1	2
0	0	0	0
1	0	0	1
2	0	1	2

$+_6$	0	1	2
0	0	1	2
1	1	0, 2	1, 2
2	2	1, 2	0, 1

$\cdot_2$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

$+_7$	0	1	2
0	0	1	2
1	1	R	1, 2
2	2	1, 2	R

$\cdot_2$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

$+_8$	0	1	2
0	0	1	2
1	1	1	R
2	2	R	2

$\cdot_2$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

$+_9$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

$\cdot_2$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

$+_{10}$	0	1	2
0	0	1	2
1	1	1, 2	R
2	2	R	1, 2

$\cdot_2$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Table 6: All unitary Krasner hyperrings of order 3 up to isomorphism

Proof. By Theorems 3.6 and 3.8, all of hyperstructures $(R, +_1, \cdot_4)$, $(R, +_1, \cdot_4)$, $(R, +_4, \cdot_4)$, $(R, +_6, \cdot_2)$,$(R, +_7, \cdot_2)$,$(R, +_8, \cdot_2)$,$(R, +_9, \cdot_2)$ and $(R, +_{10}, \cdot_2)$ are unitary Krasner hyperrings. Moreover, None of the above unitary Krasner hyperrings are isomorphism. □

Theorem 3.10. *For $i = 6, 7, \ldots, 10$, the unitary Krasner hyperring $(R, +_i, \cdot_2)$ is a Krasner hyperfield.*

Proof. The structure $(R - \{0\}, \cdot_2)$ is a group. In fact, $(R - \{0\}, \cdot_2) \cong (\mathbb{Z}_2, +)$. □

Theorem 3.10, has already been proved independently by Vaziri, Ghadiri and Mirokili[16] and Ameri, Eyvazi and Hoskova-Mayerova[5], by computer calculations and with the quotient hyperfields.

Ameri, Eyvazi and Hoskova-Mayerova[5], show that there are 5 hyperfields of order 3 up to isomorphism which all of them are quotient hyperfields.

Remark 3.2. [5] Note that the underling multiplicative in all cases is isomorphic to $\mathbb{Z}_2$. At the following by $HFmn$, we mean n^{th} hyperfield of order m.

1. $HF31 \cong (S, \oplus, \odot) \cong (R, +_8, \cdot_4)$

+	0	1	−1
0	0	1	−1
1	1	1	S
−1	−1	S	−1

2. $HF32 \cong \mathbb{Z}_3 \cong (R, +_9, \cdot_4)$

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

3. $HF33 \cong \mathbb{Z}5/ < 4 >\cong (R, +_6, \cdot_4)$

+	0	1	a
0	0	1	a
1	1	$\{0, a\}$	$\{1, a\}$
a	a	$\{1, a\}$	$\{0, 1\}$

4. $HF34 \cong \mathbb{Z}_7/ < 4 >\cong (R, +_{10}, \cdot_4)$

+	0	1	a
0	0	1	a
1	1	$1, a$	$\{0, 1, a\}$
a	a	$\{0, 1, a\}$	$\{1, a\}$

5. $HF35 \cong (R, +_7, \cdot_4)$ quotient (see Proposition 3.17 in [14] for hyperfield $HF33$).

+	0	1	a
0	0	1	a
1	1	$\{0, 1, a\}$	$\{1, a\}$
a	a	$\{1, a\}$	$\{0, 1, a\}$

Corollary 3.1. *Every unitary Krasner hyperring of order less that 4 is commutative.*

4. conclusion

In this article, we first obtained the unitary Krasner hyperrings of order 2 with their Cayley tables. Then, with the help of canonical hypergroups of order 3 and monoids with zero order 3, we calculated (by analytical and algebraic methods) the number of unitary Krasner hyperrings. The number of 7 non-isomorph unitary Krasner hyperrings with Cayley tables were presented. Also, Krasner hyperfields of order 3 have been obtained.

For the future works, Krasner hyperrings of order 3 and 4 or a special class of them can be characterized by a similar method. It should be noted that Krasner hyperrings of order 3 have been counted with the help of computer calculations[16], but they have not been investigated by analytical and algebraic methods. Also, considering that vector hyperspaces are defined on Krasner hyperfields and hypermodules are defined on Krasner hyperrings, the results stated in this article are useful in counting and classifying these hyperstructures. Also, it will be possible to characterize hyperrings made of Krasner and M-polysymmetrical hyperrings using their automorphism groups in small orders.

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