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ON GENERALIZATIONS OF STIRLING NUMBERS AND SOME WELL-KNOWN MATRICES*

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Abstract. We introduce a generalization of the Stirling numbers of the first kind and the second kind. By arranging these numbers into matrices, we generalize the Stirling matrices of the first kind and the second kind investigated by Cheon and Kim [Stirling matrix via Pascal matrix, Linear Algebra Appl. 329 (2001) 49–59]. Furthermore, we introduce generalizations of the Pascal matrix and the symmetric Pascal matrix with two real arguments, and generalize earlier results related to the Pascal matrices, Stirling matrices and matrices involving Bell numbers.
Keywords: Pascal matrix, Stirling numbers, Stirling matrix, Generalized hypergeometric function, Cholesky factorization, Bell numbers.

1. Introduction

In the past few decades, there has been an increasing interest in researching specific matrices, primarily those whose entries are binomial coefficients and Stirling numbers. The matrix derived naturally from the Pascal triangle was called the Pascal matrix, and together with its generalizations it was investigated in [3, 10, 12, 14, 27, 30, 38, 39]. The matrices derived from the number triangles consisting of the Stirling numbers of the first kind and the second kind were firstly investigated by Comtet [11], and later by Cheon and Kim [6], Yang and You [35], and Mikkawy [22, 23].

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Since then, many generalizations of the Pascal matrices and the Stirling matrices (of the first kind and the second kind) were suggested [2, 4, 8, 20, 37, 40]. Because the binomial coefficients and the Stirling numbers can be represented in terms of symmetric polynomials, the investigations in [29, 36] (concerning the lower triangular matrices consisting of symmetric polynomials) gave an interesting generalization and unified approach to Pascal and Stirling matrices. In this paper, we give another generalization of the Pascal and Stirling matrices. In addition, we also introduce a new generalization of the Stirling numbers.

Concern the following well-known definitions and a lemma.

Definition 1.1. [3, 38] The generalized Pascal matrix $\mathcal{P}_n[x] = [p_{i,j}[x]]$, $i, j = 1, \dots, n$, is defined as

$$p_{i,j}[x] = \begin{cases} x^{i-j}\binom{i-1}{j-1}, & i \geqslant j \\ 0, & i < j. \end{cases}$$

Definition 1.2. [38] The $n \times n$ matrices $\mathcal{S}_n[x] = [\mathbf{s}_{i,j}[x]]$ and $\mathcal{D}_n[x] = [d_{i,j}[x]]$, $i, j = 1, \dots, n$, are defined by

$$\mathbf{s}_{i,j}[x] = \begin{cases} x^{i-j}, & i \geqslant j \\ 0, & i < j. \end{cases}, \quad d_{i,j}[x] = \begin{cases} 1, & i = j \\ -x, & i = j+1 \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 1.1. [38] *The inverse of the matrix $\mathcal{S}_n[x]$ is equal to*

$$\mathcal{S}_n[x]^{-1} = \mathcal{D}_n[x].$$

The inverse of the Pascal matrix $\mathcal{P}_n[x]$, in a notation $\mathcal{P}_n[x]^{-1} = [p'_{i,j}[x]]$, $i, j = 1, \dots, n$, is equal to

$$\mathcal{P}_n[x]^{-1} = \mathcal{P}_n[-x].$$

Precisely, the elements of the matrix $\mathcal{P}_n[x]^{-1}$ are determined by

$$p'_{i,j}[x] = \begin{cases} (-x)^{i-j}\binom{i-1}{j-1}, & i - j \geqslant 0 \\ 0, & i - j < 0 \end{cases}$$

The factorization of the Pascal matrix via the matrix $\mathcal{S}_n$ was also investigated in [38]. Precisely, if we recall that $\oplus$ denotes the direct sum of two matrices (or in other words, the construction of the block diagonal matrix given two matrices), then consider the following matrices:

$$\begin{aligned} I_n &= \operatorname{diag}(1, 1, \dots, 1), \\ G_k[x] &= I_{n-k} \oplus \mathcal{S}_k[x] = \begin{bmatrix} I_{n-k} & 0 \\ 0 & \mathcal{S}_k[x] \end{bmatrix} \in R^{n \times n}, \quad k = 1, \dots, n-1 \end{aligned}$$

and assuming that $G_n[x] = \mathcal{S}_n[x]$, the following factorization of the Pascal matrix is established

$$\mathcal{P}_n[x] = G_n[x] G_{n-1}[x] \cdots G_1[x]. \tag{1.1}$$

In light of these results, we impose a question: Can we generalize the matrix $\mathcal{S}_n[x]$ so that the matrix product on the right-hand side of (1.1) gives a kind of generalization of the Pascal matrix? The answer is positive. We now describe in detail the way we generalize the matrix $\mathcal{S}_n[x]$. Many authors investigated classes of matrices with the property that the power of each matrix from the observed class is again in that class. Aggarwala and Lamoureux [1], for example, showed that the kth power of the Pascal matrix is again the Pascal matrix, proving the fact

$$\mathcal{P}_n[x]^k = \mathcal{P}_n[kx],$$

for any integer k. In paper [31], the authors introduced the notion of the generalized Catalan matrices $\mathcal{C}_n[a, b; x]$, where $a > 0$ and $b \geqslant 0$ are integers and x is real, and showed that the kth power of the generalized Catalan matrix $\mathcal{C}_n[a, b; x]$ obeys the same property with respect to the first parameter, i.e.

$$\mathcal{C}_n[a, b; x]^k = \mathcal{C}_n[ak, b; x],$$

when $k \geqslant 0$. The goals of the present paper are now clear:

1. Motivated by the results of the Pascal matrix and the generalized Catalan matrices, we introduce the notion of the Binomial matrix, which represents a generalization of the matrix $\mathcal{S}_n[x]$, and prove that the power of the Binomial matrix is again the Binomial matrix.

2. We introduce a generalization of the Pascal matrix with two real parameters x and y, named the (x, y)-generalized Pascal matrix. and establish a factorization of (x, y)-generalized Pascal matrix via the Binomial matrix, analogue to (1.1). Some other properties of the (x, y)-generalized Pascal matrix are also investigated.

3. The generalized Pascal matrix is further used for factorization of the generalized Stirling matrices.

4. Since factorization $A = GG^T$ of a symmetric positive definite matrix A often occurs in the context of practical computational problems [9], where G is a lower triangular matrix with positive diagonal entries, many authors observed the matrix $\mathcal{P}_n[x]\mathcal{P}_n[x]^T$. Cheon and Mikkawy [9] showed that each entry of $\mathcal{P}_n(x, y)\mathcal{P}_n(x, y)^T$ can be represented by the hypergeometric function, where $\mathcal{P}_n(x, y)$ is the extended generalized Pascal matrix defined by Zhang and Lui [39].
   To that effect, we show that each entry of $\mathcal{P}_n[x, y]\mathcal{P}_n[x, y]^T$, where $P_n[x, y]$ is the (x, y)-generalized Pascal matrix, can also be represented by the hypergeometric function. Later, we show that our result represents an extension of the result by Cheon and Mikkawy [9].

New generalizations of the Stirling numbers and the Stirling matrices are needed to achieve the described goals. Thus, we organize the paper as follows. In Section

2. we recapitulate the past results on generalizations of the Stirling numbers, and by making an analogy with the ordinary Stirling numbers, we introduce the y-generalized Stirling numbers, both of the first and the second kind. Later in the same section, we introduce the lower triangular (x, y)-generalized Stirling matrices, consisting of the mentioned numbers. In Section 3., we establish the goals 1-3, while Section 4. is reserved for establishing the goal 4. Section 5. explores a new factorization of the (x, y)-generalized Stirling matrices. This factorization yields a new generalization of the Bell numbers, the so-called y-generalized Bell numbers. As a consequence, we establish a connection with the (x, y)-generalized Pascal matrix and the matrices with expressions involving the y-generalized Bell numbers. In the final section, we draw the conclusions.

2. Generalizations of Stirling numbers and Stirling matrices

Stirling numbers represent one of the main focuses in combinatorics, and their properties and applications have been studied intensively in the literature. Given $n, k \in \mathbb{N}$ with $n \geqslant k \geqslant 0$, the Stirling numbers of the first kind $s(n, k)$ and the Stirling numbers of the second kind $S(n, k)$ can be defined in the following expansion of a variable x

$$\begin{aligned} [x]_n &= \sum_{k=0}^{n} (-1)^{n-k} s(n,k) x^k, \\ x^n &= \sum_{k=0}^{n} S(n,k)[x]_k, \end{aligned} \tag{2.1}$$

where $[x]_n = x(x-1)\cdots(x-n+1)$ is the falling factorial notation. Let $n, k \in \mathbb{N}_0$. Then we have the following Pascal-type recurrence relations for the Stirling numbers

$$\begin{aligned} s(n,k) &= s(n-1,k-1) + (n-1)s(n-1,k), \\ S(n,k) &= S(n-1,k-1) + kS(n-1,k), \end{aligned} \tag{2.2}$$

where $s(n, 0) = s(0, k) = S(n, 0) = S(0, k) = 0$ and $s(0, 0) = S(0, 0) = 1$. In addition, the Stirling numbers of the first kind satisfy the following formula, known as *horizontal* recurrence relation

$$s(n,k) = \sum_{l=k-1}^{n-1} \binom{l}{k-1} s(n-1,l), \tag{2.3}$$

while the Stirling numbers of the second kind satisfies *vertical* recurrence relation

$$S(n,k) = \sum_{l=k-1}^{n-1} \binom{n-1}{l} S(l,k-1). \tag{2.4}$$

The matrices consisting of the Stirling numbers of the first kind and of the second kind are called *the Stirling matrix of the first kind* and *the Stirling matrix* *of the second kind*, respectively. These matrices have been studied in [6, 11, 35]. Later, Cheon and Kim [6] introduced *the generalized Stirling matrix of the first kind* $s_n[x]$ and *the generalized Stirling matrix of the second kind* $S_n[x]$ with their (i,j)th elements equal to

$$(s_n[x])_{i,j} = \begin{cases} x^{i-j}s(i,j), & i \geqslant j \\ 0, & i < j, \end{cases}$$

$$(S_n[x])_{i,j} = \begin{cases} x^{i-j}S(i,j), & i \geqslant j \\ 0, & i < j. \end{cases}$$

The properties of the (generalized) Stirling matrices of the first kind and of the second kind were examined in detail by many authors, as well as the connections between the Pascal, Stirling and Vandermonde matrices.

Many generalizations of the Stirling numbers have been introduced in literature. Authors in [18] used a generalized factorial of z with increment α, in a notation $(z|\alpha)_n = z(z-\alpha)\cdots(z-(n-1)\alpha)$, as a generalization of the falling factorial symbol. The correlations between the generalized Stirling numbers and the generalized factorial analogical to (2.1) were established in [18]. These generalizations also satisfy relations analogical to (2.2).

Merris [21] examined the p-Striling numbers $S_p(m,n)$, which are reduced to the Stirling numbers of the second kind in the case $p = 1$. They satisfy the following equality

$$S_p(m,n) = \sum_{k=1}^{m} \binom{m}{k} S(k,n)p^{m-k},$$

which is a generalization of the vertical recurrent relation for the Stirling numbers (2.4). The p-Stirling numbers also satisfy the relation analogical to (2.2). Similar generalized Stirling numbers can be found in [17, 19, 28, 32].

The generalizations of the Stirling numbers with applications in analysis, probability and statistics can be found in [5, 13, 26, 33]. Mikkawy and Desouky [24] gave a connection between symmetric polynomials, generalized Stirling numbers and the Newton general divided difference interpolation polynomial.

In addition to many generalizations of the Stirling numbers, we introduce a new generalization of the Stirling numbers.

Definition 2.1. Let $n, k \in \mathbb{N}_0$ and $y \in \mathbb{R}$. The *y-generalized Stirling numbers of the second kind*, in a notation $S(n,k;y)$, are defined by the following recurrent formula

$$S(n,k;y) = \sum_{l=k-1}^{n-1} \binom{n-l+ly+y-2}{ly+y-1} S(l,k-1;y), \tag{2.5}$$

with the starting conditions $S(n,0;y) = S(0,k;y) = 0$ and $S(0,0;y) = 1$.

In the case $y = 1$, the y-generalized Stirling numbers of the second kind reduce to the Stirling numbers of the second kind (that is, $S(n, k; 1) = S(n, k)$), and the recurrence (2.5) reduces to the vertical recurrence relation (2.4).

A matrix generated with y-generalized Stirling number of the second kind is introduced by the following definition.

Definition 2.2. Let $x, y \in \mathbb{R}$. The (x, y)-*generalized Stirling matrix of the second kind* of dimensions $n \times n$, in a notation $S_n[x, y]$, is element-wise defined by

$$(S_n[x,y])_{i,j} = \begin{cases} x^{i-j} S(i,j;y), & i \geqslant j \\ 0, & i < j, \end{cases} \tag{2.6}$$

where $i, j = 1, \ldots, n$.

Example 2.1. The (x, y)-generalized Stirling matrix of the second kind $S_5[x, y]$ is equal to

$$S_5[x,y] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ xy & 1 & 0 & 0 & 0 \\ \frac{1}{2}x^2 y(y+1) & 3xy & 1 & 0 & 0 \\ \frac{1}{6}x^3 y(y+1)(y+2) & \frac{1}{2}x^2 y(11y+3) & 6xy & 1 & 0 \\ \frac{1}{24}x^4 y(y+1)(y+2)(y+3) & x^3 y\left(8y^2+6y+1\right) & x^2 y(22y+3) & 10xy & 1 \end{bmatrix}.$$

It is easy to conclude that the relation $S_n[x, 1] = S_n[x]$, where $S_n[x]$ is the generalized Stirling matrix of the second kind introduced in [6], holds.

Next, we introduce a generalization of the Stirling numbers of the first kind.

Definition 2.3. Let $n, k \in \mathbb{N}_0$ and $y \in \mathbb{R}$. The *y-generalized Stirling numbers of the first kind*, in a notation $s(n, k; y)$, are defined by the recurrent formula

$$s(n,k;y) = \sum_{l=k-1}^{n-1} \frac{k}{l+1} \binom{y(l+1)}{l+1-k} s(n-1,l;y), \tag{2.7}$$

with the starting conditions $s(n, 0; y) = s(0, k; y) = 0$ and $s(0, 0; y) = 1$.

Analogously, in the case $y = 1$, the y-generalized Stirling numbers of the first kind reduce to the Stirling numbers of the first kind, i.e. $s(n, k; 1) = s(n, k)$, and the recurrence (2.7) reduces to the horizontal recurrence relation (2.3).

Definition 2.4. Let $x, y \in \mathbb{R}$. The (x, y)-*generalized Stirling matrix of the first kind* of dimension $n \times n$, in a notation $s_n[x, y]$, has the elements

$$(s_n[x,y])_{i,j} = \begin{cases} x^{i-j} s(i,j;y), & i \geqslant j \\ 0, & i < j, \end{cases} \tag{2.8}$$

where $i, j = 1, \ldots, n$.

Example 2.2. The (x,y)-generalized Stirling matrix of the first kind $s_5[x,y]$ is of the form

$$s_5[x,y] =$$
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ xy & 1 & 0 & 0 & 0 \\ \frac{1}{2}x^2y(5y-1) & 3xy & 1 & 0 & 0 \\ \frac{1}{3}x^3y\left(29y^2-12y+1\right) & \frac{1}{2}x^2y(25y-3) & 6xy & 1 & 0 \\ \frac{1}{8}x^4y\left(397y^3-246y^2+43y-2\right) & x^3y\left(67y^2-18y+1\right) & x^2y(38y-3) & 10xy & 1 \end{bmatrix}.$$

Recalling the definition of the generalized Stirling matrix of the first kind introduced in [6] in a notation $s_n[x]$, we easily conclude that $s_n[x,1] = s_n[x]$.

In the following section, we first generalize some well-known matrices, such as the Pascal matrix, and later we find their connections with the (x,y)-generalized Stirling matrices.

3. The (x,y)-generalized Pascal matrix and matrix factorizations

We introduce a new lower triangular Toeplitz matrix of two real arguments, with entries involving the binomial coefficients.

Definition 3.1. The Binomial matrix $\mathcal{B}_n[x,y] = [b_{i,j}[x,y]]$, $i,j = 1,\dots,n$, $x,y \in \mathbb{R}$, is defined by

$$b_{i,j}[x,y] = \begin{cases} x^{i-j}\binom{i-j+y-1}{i-j}, & i \geqslant j \\ 0, & i < j. \end{cases}$$

Example 3.1. The Binomial matrix $\mathcal{B}_5[x,y]$ has the form

$$\mathcal{B}_5[x,y] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ xy & 1 & 0 & 0 & 0 \\ \frac{1}{2}x^2y(y+1) & xy & 1 & 0 & 0 \\ \frac{1}{6}x^3y(y+1)(y+2) & \frac{1}{2}x^2y(y+1) & xy & 1 & 0 \\ \frac{1}{24}x^4y(y+1)(y+2)(y+3) & \frac{1}{6}x^3y(y+1)(y+2) & \frac{1}{2}x^2y(y+1) & xy & 1 \end{bmatrix}.$$

As we can see, the relation $\mathcal{B}_n[x,1] = \mathcal{S}_n[x]$ holds.

The following theorem establishes the relation between the Binomial matrix and its kth power.

Theorem 3.1. *For $x \in \mathbb{R}$ and $k, y \in \mathbb{Z}$ we have*

$$\mathcal{B}_n[x,y]^k = \mathcal{B}_n[x,ky].$$

Proof. In the case $k \geqslant 0$ we employ the principle of the mathematical induction, which leads us to

$$\begin{aligned}(\mathcal{B}_n[x,y]^{k+1})_{i,j} &= \sum_{l=j}^{i}(\mathcal{B}_n[x,y])_{i,l}(\mathcal{B}_n[x,ky])_{l,j} \\ &= x^{i-j}\sum_{l=0}^{i-j}\binom{i-j+y-1-l}{y-1}\binom{ky-1+l}{ky-1}.\end{aligned}$$

After applying the identity for the sum of products of binomial coefficients

$$\sum_k \binom{l-k}{m}\binom{q+k}{n} = \binom{l+q+1}{m+n+1} \tag{3.1}$$

(see [15], eq. (5.26)), we get

$$(\mathcal{B}_n[x,y]^{k+1})_{i,j} = x^{i-j}\binom{i-j+(k+1)y-1}{(k+1)y-1},$$

which completes the proof in the first case. The case $k < 0$ follows after verifying $\mathcal{B}_n[x,y]^{-1} = \mathcal{B}_n[x,-y]$ by doing some multiplications. $\square$

In the particular case when $y = 1$, Theorem 3.1 can be restated as follows.

Corollary 3.1. *For every $k \in \mathbb{Z}$ and $x \in \mathbb{R}$,*

$$\mathcal{S}_n[x]^k = \mathcal{B}_n[x,k].$$

Also, we have $\mathcal{B}_n[x,-1] = \mathcal{D}_n[x]$, so the result $\mathcal{S}_n[x]^{-1} = \mathcal{D}_n[x]$ from [38] is immediately obtained.

Next, we introduce a generalization of the Pascal matrix of two real arguments.

Definition 3.2. The (x,y)-generalized Pascal matrix $\mathcal{P}_n[x,y] = [p_{i,j}[x,y]]$, where $x,y \in \mathbb{R}$ and $i,j = 1,\dots,n$, is defined by

$$p_{i,j}[x,y] = \begin{cases} x^{i-j}\binom{i-j+jy-1}{i-j}, & i \geqslant j \\ 0, & i < j. \end{cases} \tag{3.2}$$

Example 3.2. The (x,y)-generalized Pascal matrix $\mathcal{P}_5[x,y]$ possesses the form

$$\mathcal{P}_5[x,y] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ xy & 1 & 0 & 0 & 0 \\ \frac{1}{2}x^2y(y+1) & 2xy & 1 & 0 & 0 \\ \frac{1}{6}x^3y(y+1)(y+2) & x^2y(2y+1) & 3xy & 1 & 0 \\ \frac{1}{24}x^4y(y+1)(y+2)(y+3) & \frac{1}{3}x^3y(2y+1)(2y+2) & \frac{3}{2}x^2y(3y+1) & 4xy & 1 \end{bmatrix}.$$

The relation $\mathcal{P}_n[x,1] = \mathcal{P}_n[x]$ evidently holds. Next, define matrices

$$G_n^{(k)}[x,y] = I_k \oplus \mathcal{B}_{n-k}[x,y] = \begin{bmatrix} I_k & 0 \\ 0 & \mathcal{B}_{n-k}[x,y] \end{bmatrix}, \quad 0 \leqslant k < n.$$

Then we have the following statement.

Lemma 3.1. *For $x \in \mathbb{R}$ and $y \in \mathbb{Z}$,*

$$\mathcal{P}_n[x,y] = \mathcal{B}_n[x,y](I_1 \oplus \mathcal{P}_{n-1}[x,y]).$$

Proof. After applying the following steps

$$
\begin{aligned}
&(\mathcal{B}_n[x,y](I_1 \oplus \mathcal{P}_{n-1}[x,y]))_{i,j} \\
&= \sum_{k=j}^{i} b_{i,k}[x,y] p_{k-1,j-1}[x,y] \\
&= x^{i-j} \sum_{k=j}^{i} \binom{i-k+y-1}{y-1} \binom{(j-1)y+k-j-1}{(j-1)y-1},
\end{aligned}
$$

and relation (3.1), the Proposition is proved. $\square$

By recursively applying Lemma 3.1, we get a factorization of the (x,y)-generalized Pascal matrix.

Proposition 3.1. *For $x \in \mathbb{R}$ and $y \in \mathbb{Z}$ we have*

$$
\mathcal{P}_n[x,y] = \prod_{k=0}^{n-1} G_n^{(k)}[x,y]. \tag{3.3}
$$

Remark 3.1. Relation (3.3) presents a generalization of the result

$$
\mathcal{P}_n[x,1] = \prod_{k=0}^{n-1} G_n^{(k)}[x,1]
$$

from [38].

We are now in a position to evaluate the inverse of the (x,y)-generalized Pascal matrix.

Theorem 3.2. *The inverse of the (x,y)-generalized Pascal matrix, which we denote by $\mathcal{P}_n[x,y]^{-1} = [p'_{i,j}[x,y]]$, $i,j = 1,\ldots,n$, $x \in \mathbb{R}$, $y \in \mathbb{R}$, has the elements equal to*

$$
p'_{i,j}[x,y] = \begin{cases} x^{i-j} \cdot \frac{j}{i} \cdot \binom{i-j-iy-1}{i-j}, & i \geqslant j \\ 0, & i < j. \end{cases} \tag{3.4}
$$

Proof. According to Proposition 3.1, we have

$$
\mathcal{P}_n[x,y]^{-1} = \prod_{k=0}^{n-1} \left(G_n^{(n-1-k)}[x,y] \right)^{-1} = \prod_{k=0}^{n-1} F_n^{(n-1-k)}[x,y],
$$

where

$$
F_n^{(k)}[x,y] = \left(G_n^{(k)}[x,y] \right)^{-1} = \begin{bmatrix} I_k & 0 \\ 0 & \mathcal{B}_{n-k}[x,-y] \end{bmatrix}.
$$

After some computations, the desired result is obtained. $\square$

When $y \in \mathbb{N}$, for $i \geqslant j$ we have $i-j-iy-1<0$. Thus, we use the generalization of the binomial coefficient to negative integers in (3.4). Consequently, Theorem 3.2 can be restated in the case $y \in \mathbb{N}$ as follows.

Theorem 3.3. *The elements of the matrix $\mathcal{P}_n[x,y]^{-1}$ are equal to*

$$p'_{i,j}[x,y] = \begin{cases} (-x)^{i-j} \cdot \frac{j}{i} \cdot \binom{iy}{i-j}, & i \geqslant j \\ 0, & i<j. \end{cases} \tag{3.5}$$

in the case $x \in \mathbb{R},\ y \in \mathbb{N}$.

It is now straightforward to conclude that $\mathcal{P}_n[x,1]^{-1} = \mathcal{P}_n[-x,1]$. In this way, we obtain the well-known result for the inverse of the Pascal matrix (see Proposition 1.1).

Example 3.3. The inverse of the (x,y)-generalized Pascal matrix $\mathcal{P}_5[x,y]^{-1}$ has the form

$$\mathcal{P}_5[x,y]^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -xy & 1 & 0 & 0 & 0 \\ -\frac{1}{2}x^2(1-3y)y & -2xy & 1 & 0 & 0 \\ -\frac{1}{6}x^3(1-4y)(2-4y)y & -x^2(1-4y)y & -3xy & 1 & 0 \\ -\frac{1}{24}x^4(1-5y)(2-5y)(3-5y)y & -\frac{1}{3}x^3(1-5y)(2-5y)y & -\frac{3}{2}x^2(1-5y)y & -4xy & 1 \end{bmatrix}.$$

Similarly, as we did for matrices $G_n^{(k)}[x,y]$, now we define the matrices $\widetilde{P}_n^{(k)}[x,y]$ as

$$\widetilde{P}_n^{(k)}[x,y] = I_k \oplus \mathcal{P}_{n-k}[x,y], \quad 0 \leqslant k < n.$$

Lemma 3.2. *For $x \in \mathbb{R}$ and $y \in \mathbb{N}$, the (x,y)-generalized Stirling matrix of the second kind can be factorized as follows*

$$S_n[x,y] = \mathcal{P}_n[x,y] \cdot (I_1 \oplus S_{n-1}[x,y]). \tag{3.6}$$

Proof. Straightaway from the definition of the y-generalized Stirling numbers of the second kind we conclude

$$\begin{aligned} (\mathcal{P}_n[x,y] \cdot (I_1 \oplus S_{n-1}[x,y]))_{i,j} &= \sum_{l=j-1}^{i-1} p_{i,l+1}[x,y](S_n[x,y])_{l,j-1} \\ &= x^{i-j} \sum_{l=j-1}^{i-1} \binom{i-l+ly+y-2}{ly+y-1} S(l,j-1;y) \\ &= x^{i-j} S(i,j;y) = (S_n[x,y])_{i,j}\,. \end{aligned}$$

This equality is valid for all $i,j = 1,\ldots,n$, and therefore the proof is completed. $\square$

As a consequence of Lemma 3.2 we get a factorization of the (x, y)-generalized Stirling matrix of the second kind.

Proposition 3.2. *For $x \in \mathbb{R}$ and $y \in \mathbb{N}$ we have*

$$\mathcal{S}_n[x, y] = \prod_{k=0}^{n-1} \widetilde{P}_n^{(k)}[x, y]. \tag{3.7}$$

Remark 3.2. The identity (3.7) represents a generalization of the result

$$\mathcal{S}_n[x, 1] = \prod_{k=0}^{n-1} \widetilde{P}_n^{(k)}[x, 1]$$

from [6].

Now we focus on a factorization of the (x, y)-generalized Stirling matrix of the first kind. Consider matrices

$$\widetilde{P'}_n^{(k)}[x, y] = I_k \oplus \mathcal{P}_{n-k}[x, y]^{-1}, \quad 0 \leqslant k < n.$$

Lemma 3.3. *For $x \in \mathbb{R}$ and $y \in \mathbb{N}$ we have*

$$s_n[-x, y] = (I_1 \oplus s_{n-1}[-x, y]) \cdot \mathcal{P}_n[x, y]^{-1}. \tag{3.8}$$

Proof. Similarly as in Lemma 3.2, we obtain

$$\begin{aligned}\left((I_1 \oplus s_{n-1}[-x, y]) \cdot \mathcal{P}_n[x, y]^{-1}\right)_{i,j} &= (-x)^{i-j} \sum_{l=j-1}^{i-1} \frac{j}{l+1} \binom{(l+1)y}{l+1-j} s(i-1, l; y) \\ &= (-x)^{i-j} s(i, j; y) = \left(s_n[-x, y]\right)_{i,j},\end{aligned}$$

and the proof is completed. $\square$

Proposition 3.3. *For $x \in \mathbb{R}$ and $y \in \mathbb{N}$ we have*

$$s_n[-x, y] = \prod_{k=0}^{n-1} \widetilde{P'}_n^{(n-1-k)}[x, y]. \tag{3.9}$$

Proof. By recursively applying the previous lemma, we obtain the Proposition. $\square$

Remark 3.3. The identity (3.9) presents a generalization of the result

$$s_n[-x, 1] = \prod_{k=0}^{n-1} \widetilde{P'}_n^{(n-1-k)}[x, 1]$$

established in [7].

Propositions 3.2 and 3.3 yield an important relationship between the (x, y)-generalized Stirling matrices.

Proposition 3.4. *For $x \in \mathbb{R}$ and $y \in \mathbb{N}$,*

$$\begin{aligned} S_n[x,y]^{-1} &= s_n[-x,y], \\ s_n[x,y]^{-1} &= S_n[-x,y]. \end{aligned} \tag{3.10}$$

Again, setting $y = 1$ in (3.10), we regain the inverses for the generalized Stirling matrices of the first kind and of the second kind (See [6], Lemma 4.1.):

$$\begin{aligned} S_n[x,1]^{-1} = s_n[-x,1] &\Rightarrow S_n[x]^{-1} = s_n[-x], \\ s_n[x,1]^{-1} = S_n[-x,1] &\Rightarrow s_n[x]^{-1} = S_n[-x]. \end{aligned}$$

Example 3.4. By putting $n = 4$ in (3.7), we get

$$S_4[x,y] = \widetilde{P}_4^{(0)}[x,y]\widetilde{P}_4^{(1)}[x,y]\widetilde{P}_4^{(2)}[x,y]\widetilde{P}_4^{(3)}[x,y],$$

or in the expanded form,

$$S_4[x,y] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ xy & 1 & 0 & 0 \\ \frac{1}{2}x^2y(y+1) & 3xy & 1 & 0 \\ \frac{1}{6}x^3y(y+1)(y+2) & \frac{1}{2}x^2y(11y+3) & 6xy & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ xy & 1 & 0 & 0 \\ \frac{1}{2}x^2y(y+1) & 2xy & 1 & 0 \\ \frac{1}{6}x^3y(y+1)(y+2) & x^2y(2y+1) & 3xy & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & xy & 1 & 0 \\ 0 & \frac{1}{2}x^2y(y+1) & 2xy & 1 \end{bmatrix}$$

$$\cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & xy & 1 \end{bmatrix} \cdot I_4.$$

For the inverse of the (x, y)-generalized Stirling matrix of the second kind, we have relations

$$S_4[x,y]^{-1} = \widetilde{P'}_4^{(3)}[x,y]\widetilde{P'}_4^{(2)}[x,y]\widetilde{P'}_4^{(1)}[x,y]\widetilde{P'}_4^{(0)}[x,y]$$

and $S_4[x,y]^{-1} = s_4[-x,y]$, or in the expanded form,

$$S_4[x,y]^{-1} = s_4[-x,y] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -xy & 1 & 0 & 0 \\ \frac{1}{2}x^2y(5y-1) & -3xy & 1 & 0 \\ -\frac{1}{3}x^3y\left(29y^2-12y+1\right) & \frac{1}{2}x^2y(25y-3) & -6xy & 1 \end{bmatrix}$$

$$= I_4 \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -xy & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -xy & 1 & 0 \\ 0 & \frac{1}{2}x^2y(3y-1) & -2xy & 1 \end{bmatrix}$$

$$\cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ -xy & 1 & 0 & 0 \\ \frac{1}{2}x^2y(3y-1) & -2xy & 1 & 0 \\ -\frac{1}{6}x^3y(4y-2)(4y-1) & x^2y(4y-1) & -3xy & 1 \end{bmatrix}.$$

After applying (3.10) on (3.6) and (3.8) respectively, the following two matrix identities are obtained

$$\begin{aligned}\mathcal{P}_n[x,y] &= S_n[x,y]\left(I_1 \oplus s_{n-1}[-x,y]\right),\\ \mathcal{P}_n[x,y]^{-1} &= \left(I_1 \oplus S_{n-1}[x,y]\right) s_n[-x,y].\end{aligned}$$

By putting $x = 1$ in the previous two identities and equalizing the (n,k)th elements of matrices on the both sides, we get the additional identities for the y-generalized Stirling numbers

$$\begin{aligned}\binom{n-k+ky-1}{ky-1} &= \sum_{l=k}^{n}(-1)^{l-k}S(n,l;y)s(l-1,k-1;y),\\ (-1)^{n-k}\frac{n}{k}\binom{ny}{n-k} &= \sum_{l=k}^{n}(-1)^{l-k}S(n-1,l-1;y)s(l,k;y),\end{aligned}$$

which generalize relations established in [6] for the ordinary Stirling numbers.

4. The (x, y)-symmetric Pascal matrix

We restate the definition of the generalized hypergeometric function

$$(4.1)\qquad {}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;\lambda) = \sum_{l=0}^{\infty}\frac{(a_1)_l\cdots(a_p)_l}{(b_1)_l\cdots(b_q)_l}\cdot\frac{\lambda^l}{l!},$$

where

$$(a)_n = a(a+1)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}$$

is the well-known Pochhammer function (also known as rising factorial notation, an opposite to the falling factorial notation restated in Section 2.), and $\Gamma(n)$ is the Euler gamma function.

For the sake of simplicity, given tuples $A_p = (a_1,\ldots,a_p)$ and $B_q = (b_1,\ldots,b_q)$, we use a shortened notation for the generalized hypergeometric function

$${}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;\lambda) = {}_pF_q(A_p;B_q;\lambda).$$

For any $x, y \in \mathbb{R}$, we define the $n \times n$ matrix $\Pi_n[x,y]$ by

$$(4.2)\qquad \Pi_n[x,y] = \mathcal{P}_n[x,y]\mathcal{P}_n[x,y]^T,$$

where $\mathcal{P}_n[x,y]$ is the (x,y)-generalized Pascal matrix (3.2). Our goal in this section is to show that each entry of $\Pi_n[x,y]$ can be represented by the generalized hypergeometric function (4.1) when $y \in \mathbb{N}$. To that effort, we introduce the following tuples

$$I_n = \begin{cases} (1-i), & n=1 \\ (1-i, 1+\frac{i}{n-1}, 1+\frac{i+1}{n-1}, \ldots, 1+\frac{i+(n-2)}{n-1}), & n>1, \end{cases}$$

$$J_n = \begin{cases} (1-j), & n=1 \\ (1-j, 1+\frac{j}{n-1}, 1+\frac{j+1}{n-1}, \ldots, 1+\frac{j+(n-2)}{n-1}), & n>1, \end{cases}$$

$$K_{2n-1} = \begin{cases} (1), & n=1 \\ (1, \frac{n+1}{n}, \frac{n+1}{n}, \frac{n+2}{n}, \frac{n+2}{n}, \ldots, \frac{n+(n-1)}{n}, \frac{n+(n-1)}{n}), & n>1, \end{cases}$$

defined for all $n \in \mathbb{N}$. Let $M_{2n} = I_n \star J_n$ denotes the $(2n)$-tuple such that the first n elements are from I_n, while the last n elements are from J_n. Notice that the length of the list M_{2n} can only be an even number, while the length of the list K_{2n-1} can only be an odd number. Finally, let a_n be the array of integers defined by

$$a_n = \begin{cases} 1, & n=0 \\ n^{2n}, & n>0. \end{cases}$$

We are now in a position to state the main result of this section.

Theorem 4.1. *Let $\Pi_n[x,y]$ be the (x,y)-symmetric Pascal matrix, defined by* (4.2), *for $x \in \mathbb{R}$ and $y \in \mathbb{N}$. Then its (i,j)th entry can be represented by*

$$(\Pi_n[x,y])_{i,j} = x^{i+j-2} \frac{(i)_{y-1}}{(y-1)!} \frac{(j)_{y-1}}{(y-1)!} \; {}_{2y}F_{2y-1}\left(M_{2y}; K_{2y-1}; \frac{a_{y-1}}{a_y} \cdot \frac{1}{x^2} \right). \tag{4.3}$$

Proof. We start from the right-hand side of (4.3), which we denote by R for the sake of simplicity. The trick of the proof is to immediately apply (4.1) on R, and then simplify the products of the Pochhammer functions that occur. What we obtain after applying (4.1) is

$$R = x^{i+j-2} \frac{(i)_{y-1}(j)_{y-1}}{((y-1)!)^2}$$
$$\cdot \sum_{k=0}^{\infty} \frac{(1-i)_k(1+\frac{i}{y-1})_k \cdots (1+\frac{i+y-2}{y-1})_k (1-j)_k (1+\frac{j}{y-1})_k \cdots (1+\frac{j+y-2}{y-1})_k}{(1)_k (\frac{y+1}{y})_k^2 \cdots (\frac{2y-1}{y})_k^2} \frac{\frac{a_{y-1}}{a_y} \cdot \frac{1}{x^2}}{k!}.$$

First, we simplify the Pochhammer functions in the numerators of ${}_{2y}F_{2y-1}$ from R:

(4.4)

$$(1-l)_k = (-1)^k \frac{(l-1)!}{(l-k-1)!}, \quad \text{for } l=i \text{ and } l=j,$$

$$\left(1+\frac{l+m}{y-1}\right)_k = \frac{\prod_{p=1}^{k}(l+m+p(y-1))}{(y-1)^k}, \quad \text{for } l=i,\ l=j, \text{ and } 0 \leqslant m \leqslant y-2,$$

and then, we simplify the Pochhammer functions in the denominators of ${}_{2y}F_{2y-1}$ from R:

$$\left(\frac{y+m}{y}\right)_k = \frac{\prod_{p=1}^{k}(py+m)}{y^k}, \quad \text{for } 0 \leqslant m \leqslant y-1. \tag{4.5}$$

Upon observing that $k! = (1)_k$ and making use of (4.4) and (4.5), we find that the right-hand side of (4.3) reduces to

$$R = x^{i+j-2} \sum_{k=0}^{\min(i,j)-1} \frac{(i+y-2+k(y-1))!}{(i-k-1)!} \frac{(j+y-2+k(y-1))!}{(j-k-1)!} \frac{1}{((ky+y-1)!)^2} \frac{1}{x^{2k}}.$$

Finally, after the substitution $k \to k-1$, we obtain

$$\begin{aligned} R &= x^{i+j} \sum_{k=1}^{\min(i,j)} \frac{(i-k+ky-1)!}{(i-k)!(ky-1)!} \frac{(j-k+ky-1)!}{(j-k)!(ky-1)!} \frac{1}{x^{2k}} \\ &= x^{i+j} \sum_{k=1}^{\min(i,j)} \binom{i-k+ky-1}{i-k} \binom{j-k+ky-1}{j-k} x^{-2k}, \end{aligned}$$

which is exactly the left-hand side of (4.3), since

$$(\Pi_n[x,y])_{i,j} = \sum_{k=1}^{\min(i,j)} p_{i,k}[x,y] p_{j,k}[x,y],$$

which completes the proof. $\square$

Remark 4.1. Setting $x=1$ and $y=1$ in (4.2) and (4.3), we regain the well-known result

$$(\mathcal{P}_n[1]\mathcal{P}_n[1]^T)_{i,j} = \binom{i+j-2}{j-1} \tag{4.6}$$

(see [16, 25]). By putting $y=1$ for an arbitrary $x \in \mathbb{R}$, we also regain the well-known formula (see [9], Theorems 3.1 and 3.3). One should notice that formula (4.3) and the result from [9] present different ways in generalizing formula (4.6).

After applying the Cauchy – Binet formula, we immediately obtain the following result.

Corollary 4.1. *For every $x, y \in \mathbb{R}$,*

$$\det \Pi_n[x,y] = 1.$$

5. The (x, y)-generalized Stirling matrices and their relation with generalized Bell numbers

In this section, we investigate another factorization of the (x, y)-generalized Stirling matrices. Let us define matrices $\widehat{S}_n[x, y]$ and $\widehat{s}_n[x, y]$ of order n with elements equal to

$$\begin{aligned}
(\widehat{S}_n[x,y])_{i,j} &= \begin{cases} x^{i-j}S(i+1,j+1;y), & i \geqslant j \\ 0, & i < j \end{cases}, \\
(\widehat{s}_n[x,y])_{i,j} &= \begin{cases} x^{i-j}s(i+1,j+1;y), & i \geqslant j \\ 0, & i < j \end{cases},
\end{aligned}$$

and let us consider the matrices $N^{(k)}_{1,n}[x, y]$ and $N^{(k)}_{2,n}[x, y]$, both of order n for each $k = 1, 2, \dots, n$, with entries defined by

$$\begin{aligned}
\left(N^{(k)}_{1,n}[x,y]\right)_{i,j} &= \begin{cases} x^{i-j}S_{i-j}, & j = k, i \geqslant k \\ 0, & j = k, i < k \\ \delta_{i,j}, & \text{otherwise} \end{cases}, \\
\left(N^{(k)}_{2,n}[x,y]\right)_{i,j} &= \begin{cases} x^{i-j}s_{i-j}, & j = k, i \geqslant k \\ 0, & j = k, i < k \\ \delta_{i,j}, & \text{otherwise} \end{cases}.
\end{aligned}$$

We are now in a position to propose the following statement.

Lemma 5.1. *For $x, y \in \mathbb{R}$, we have*

$$\begin{aligned}
S_n[x,y] &= N^{(1)}_{1,n}[x,y]\left(I_1 \oplus \widehat{S}_{n-1}[x,y]\right), \\
s_n[x,y] &= N^{(1)}_{2,n}[x,y]\left(I_1 \oplus \widehat{s}_{n-1}[x,y]\right).
\end{aligned}$$

Proof. The first equality follows after applying the following transformations

$$\begin{aligned}
\left(N^{(1)}_{1,n}[x,y]\left(I_1 \oplus \widehat{S}_{n-1}[x,y]\right)\right)_{i,j} &= \sum_{k=1}^{i}(N^{(1)}_{1,n}[x,y])_{i,k}\left(\widehat{S}_{n-1}[x,y]\right)_{k-1,j-1} \\
&= \begin{cases} x^{i-1}S(i,1;y), & j = 1 \\ \left(\widehat{S}_{n-1}[x,y]\right)_{i-1,j-1}, & j > 1 \end{cases} \\
&= x^{i-j}S(i,j;y) = (S_n[x,y])_{i,j}.
\end{aligned}$$

Likewise, we prove the second equality, and the proof is therefore finished. □

By recursively applying Lemma 5.1, we get the following result.

Theorem 5.1. *For $x, y \in \mathbb{R}$ we have*

$$S_n[x,y] = \prod_{k=1}^{n} N_{1,n}^{(k)}[x,y], \tag{5.1}$$

$$s_n[x,y] = \prod_{k=1}^{n} N_{2,n}^{(k)}[x,y]. \tag{5.2}$$

Example 5.1. Setting $n = 4$ in equality (5.1) we obtain

$$S_4[x,y] = N_{1,4}^{(1)}[x,y] \cdot N_{1,4}^{(2)}[x,y] \cdot N_{1,4}^{(3)}[x,y] \cdot N_{1,4}^{(4)}[x,y],$$

or in an expanded form

$$S_4[x,y] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ xy & 1 & 0 & 0 \\ \frac{1}{2}x^2y(y+1) & 0 & 1 & 0 \\ \frac{1}{6}x^3y(y+1)(y+2) & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 3xy & 1 & 0 \\ 0 & \frac{1}{2}x^2y(11y+3) & 0 & 1 \end{bmatrix}$$
$$\cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 6xy & 1 \end{bmatrix} \cdot I_4.$$

Setting $n = 4$ in equality (5.2) we obtain

$$s_4[x,y] = N_{2,4}^{(1)}[x,y] \cdot N_{2,4}^{(2)}[x,y] \cdot N_{2,4}^{(3)}[x,y] \cdot N_{2,4}^{(4)}[x,y],$$

or in an expanded form

$$s_4[x,y] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ xy & 1 & 0 & 0 \\ \frac{1}{2}x^2y(5y-1) & 0 & 1 & 0 \\ \frac{1}{3}x^3y\left(29y^2-12y+1\right) & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 3xy & 1 & 0 \\ 0 & \frac{1}{2}x^2y(25y-3) & 0 & 1 \end{bmatrix}$$
$$\cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 6xy & 1 \end{bmatrix} \cdot I_4.$$

Finally, we can define the *y-generalized Bell numbers* $\omega(n;y)$ by

$$\omega(n;y) = \sum_{k=1}^{n} S(n,k;y), \tag{5.3}$$

with the condition $\omega(0;y) = 1$. By setting $y = 1$ in (5.3) we obtain the Bell numbers $\omega(n)$. Since the ith Bell number $\omega(i;y)$ is sum of the entries in the ith row of the matrix $S_n[1,y]$, what we obtain after multiplying (3.6) in the case $x = 1$ from the right with $E = [1 \quad 1 \quad \dots \quad 1]^T$ is

$$\mathcal{P}_n[1,y] \cdot [\omega(0;y) \quad \omega(1;y) \quad \dots \quad \omega(n-1;y)]^T = [\omega(1;y) \quad \omega(2;y) \quad \dots \quad \omega(n;y)]^T.$$

Similarly, as for the Bell numbers (which was shown in [6]), the following formula is valid for the y-generalized Bell numbers, as it is stated in the following theorem.

Theorem 5.2. *For arbitrary $n, k \in \mathbb{N}$, we have the following matrix identity*

$$\mathcal{P}_n[-1] \cdot \begin{bmatrix} \omega(0;y) & \omega(1;y) & \dots & \omega(n-1;y) \\ \omega(1;y) & \omega(2;y) & \dots & \omega(n;y) \\ \vdots & \vdots & \ddots & \vdots \\ \omega(n-1;y) & \omega(n;y) & \dots & \omega(2n-2;y) \end{bmatrix} \tag{5.4}$$

$$= \begin{bmatrix} \omega(0;y) & \omega(1;y) & \dots & \omega(n-1;y) \\ \triangle\omega(0;y) & \triangle\omega(1;y) & \dots & \triangle\omega(n-1;y) \\ \vdots & \vdots & \ddots & \vdots \\ \triangle^{n-1}\omega(0;y) & \triangle^{n-1}\omega(1;y) & \dots & \triangle^{n-1}\omega(n-1;y) \end{bmatrix}, \tag{5.5}$$

where $\omega(0;y) = 1$ and $\triangle$ is the difference operator defined by

$$\triangle\omega(n;y) = \omega(n+1;y) - \omega(n;y), \quad \triangle^m\omega(n;y) = \triangle(\triangle^{m-1}\omega(n;y)),\ m = 2, 3, \dots$$

Proof. First, note that, by the definition of the difference operator $\triangle$, the following formula can be easily proved by means of the mathematical induction:

$$\triangle^m\omega(n;y) = \sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\omega(n+k;y), \tag{5.6}$$

for every $m \in \mathbb{N}$. Now, if we start from the matrix product given in (5.4), we get that the (i,j)th element of this product is equal to

$$\sum_{k=1}^{i}(-1)^{i-k}\binom{i-1}{k-1}\omega(k+j-2;y) = \sum_{k=0}^{i-1}(-1)^{i-1-k}\binom{i-1}{k}\omega(j-1+k;y).$$

Now, by employing (5.6) we further get

$$\sum_{k=1}^{i}(-1)^{i-k}\binom{i-1}{k-1}\omega(k+j-2;y) = \triangle^{i-1}\omega(j-1;y)$$

which is exactly the (i,j)th element of the matrix given in (5.5), and the proof is completed. $\square$

6. Conclusion

The present paper gives new insights into generalizations of matrices containing particular types of numbers. First, we introduce a new generalization of the Stirling numbers of the first kind and the second kind. Later, we pay attention to various lower triangular square matrices. After naturally deriving a new generalization of the Stirling matrices, both of the first kind and of the second kind, we introduce the notion of the Binomial matrix $\mathcal{B}_n[x,y]$ and show that $\mathcal{B}_n[x,y]^k = \mathcal{B}_n[x,ky]$. In addition, we use the Binomial matrix to construct a generalization of the Pascal matrix, which we call the (x, y)-generalized Pascal matrix, in a notation $\mathcal{P}_n[x, y]$. The inverse of $\mathcal{P}_n[x, y]$ is also found using the Binomial matrix.

The (x, y)-generalized Pascal matrix is further used for various matrix factorizations:

- First, we use $\mathcal{P}_n[x, y]$ and its inverse for factorizations of the (x, y)-generalized Stirling matrices.
- Motivated by the practical computational problems, we find a positive definite symmetric matrix $\Pi_n[x, y]$ which satisfy the relation $\Pi_n[x, y] = \mathcal{P}_n[x, y]$ $\mathcal{P}_n[x, y]^T$. We show that each element of the matrix $\Pi_n[x, y]$ can be represented by the generalized hypergeometric function ${}_pF_q$.
- Using generalizations of the Stirling numbers, we have defined a generalization of the Bell numbers, named the *y-generalized Bell numbers*. We have established the connection between the (x, y)-generalized Pascal matrix and the matrices with expressions involving the y-generalized Bell numbers.

In this way, we generalize the results from papers [6, 7, 9, 16, 25, 38]. Future work on this subject may include the following topics:

- For matrices introduced in this paper, one may obtain many other factorizations, especially for the (x, y)-symmetric Pascal matrix, due to its importance and applicability in practical problems. Some combinatorial identities may also be derived from these matrix factorizations.
- Investigations of the y-generalized Stirling numbers of the first kind and the second kind from both combinatorial and algebraic points of view may be of much interest.
- Finally, we have defined the y-generalized Bell numbers $\omega(n; y)$ by (5.3), with the condition $\omega(0; y) = 1$. Note that the y-generalized Bell numbers reduce to the ordinary Bell numbers in the case $y = 1$. The investigations and factorizations of the matrix generated with the y-generalized Bell numbers may bring many results, particularly some combinatorial identities, as was the case with the ordinary Bell matrix exploited in [34].

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