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PARA-SASAKIAN MANIFOLD ADMITTING RICCI-YAMABE SOLITONS WITH QUARTER SYMMETRIC METRIC CONNECTION

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Abstract. In the year 2019, Guler and Crasmareanu [6] conducted an investigation into another geometric flow known as the Ricci-Yamabe map. This map is nothing but a scalar combination of the Ricci and the Yamabe flow [7]. The primary objective of the current paper is to provide a characterization of a Ricci Yamabe soliton on a para-Sasakian manifold [17]. To commence, we prove that a para-Sasakian manifold admits a nearly quasi-Einstein manifold. Moreover, we discuss whether such a manifold is shrinking, expanding, or steady. Subsequently, we generalize these findings to Ricci-Yamabe solitons on para-Sasakian manifolds equipped with a quarter symmetric metric connection. Lastly, we furnish an illustration of a three-dimensional para-Sasakian manifold admitting a Ricci-Yamabe soliton which satisfies our results.
Keywords: Ricci-Yamabe soliton, Para-Sasakian manifold, Quasi-Einstein manifold.

1. Introduction

In differential geometry, the Ricci flow represents an intrinsic flow of geometry.

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Similar to formal terms of the heat diffusion process, the Ricci flows hold a significant position and play a crucial role in theoretical physics. The concept of the Ricci flow was initially introduced by R.S. Hamilton in 1982. This flow constitutes an evolutionary equation for the metric denoted as g on a Riemannian manifold M, defined as follows:

$$\frac{\partial}{\partial t}(g(t)) = -2S, \tag{1.1}$$

where S indicates the Ricci curvature tensor of M.

In Differential Geometry, the concept of a Ricci soliton arises when considering solutions to the Ricci flow, or self-similar solutions to it, which are characterized by evolving solely through a one-parameter family of diffeomorphisms and scaling transformations.
A Ricci soliton (g, U, λ) on a Riemannian manifold (M, g) is a generalization of an Einstein metric. For a Riemannian manifold (M, g) admitting Ricci soliton, it requires the existence of a smooth non-zero potential vector field U along with a constant, denoted as Λ. These satisfy the following equation [7]:

$$\pounds_U g + 2S + 2\Lambda g = 0, \tag{1.2}$$

where $\pounds_U$ represents the Lie derivative along the direction of U. The nature of the Ricci soliton's behavior, whether it is shrinking, steady, or expanding, depends on the sign of λ. Ricci solitons find application across diverse fields such as economics, physics, and biology, sparking growing interest due to their versatile utility.

Following the introduction of the Ricci flow concept, Hamilton [7] introduced the Yamabe flow, a new concept designed to construct the Yamabe metric on a compact Riemannian manifold (M, g). For such a manifold, a time-dependent metric $g(\cdot, t)$ is said to evolve under the Yamabe flow if the metric g satisfies the equation [11]:

$$\frac{\partial}{\partial t}(g(t)) = -rg(t), \quad g(0) = g_0, \tag{1.3}$$

where r signifies the scalar curvature of M.

While the Yamabe flow is equivalent to the Ricci flow in 2-dimensional spaces, they diverge in higher dimensions. The Yamabe flow preserves the conformal class of the metric, unlike the Ricci flow.

A Yamabe soliton [2], representative of a self-similar solution to the Yamabe flow, is characterized on a Riemannian manifold (M, g) by a vector field ξ that follows the equation:

$$\frac{1}{2}\pounds_U g = (r - \Lambda)g, \tag{1.4}$$

where $\pounds_U g$ denotes the Lie derivative of the metric g along the vector field U, r denotes the scalar curvature, and Λ is a constant. The soliton's behavior corresponds to expanding ($\Lambda < 0$), steady ($\Lambda = 0$), or shrinking ($\Lambda > 0$).

In 2019, S. Guler and M. Crasmareanu [6] introduced a novel geometric flow named the Ricci-Yamabe map, which combines the Ricci and Yamabe flows. This flow, also referred to as the (ρ, q)-Ricci-Yamabe flow, gives rise to the notion of a Ricci-Yamabe soliton if it evolves solely through a one-parameter group of diffeomorphisms and scaling. A manifold's metric is classified as a Ricci-Yamabe soliton (RYS) (g, U, Λ, ρ, q) in $n > 2$ dimensions if it satisfies the equation [12]:

$$\pounds_U g + 2\rho S + [2\Lambda - qr]\, g = 0, \tag{1.5}$$

where $\pounds_U g$ denotes the Lie derivative of the metric g along the vector field U, and Λ, ρ, and q are real scalars. Similar to other soliton types, the RYS behavior is categorized as expansion, steadiness, or shrinkage based on the sign of Λ.

Moreover, the authors in [6] established a correspondence between the (ρ, q)-Ricci-Yamabe flow and well-known geometric flows:

1. If $\rho = 1$ and $q = 0$, the (ρ, q)-Ricci-Yamabe flow becomes the Ricci flow.
2. If $\rho = 0$ and $q = 1$, it becomes the Yamabe flow.
3. If $\rho = 1$ and $q = -1$, it becomes the Einstein flow.

Yamabe and Ricci solitons represent special solutions of Hamilton's Yamabe and Ricci flows, respectively. Inspired by Hamilton's work, many mathematicians have explored generalizations of such solitons in recent years, often considering Λ as a variable.

Very recently, the $*$-η-Ricci-Yamabe soliton was introduced by S. Roy, S. Dey, A. Bhattacharyya, and M.D. Siddiqi [12] as a novel extension of the RYS. Additionally, A.N. Siddiqui and M.D. Siddiqi [14] presented a study concerning the geometric aspects of relativistic perfect fluid spacetime and GRW-spacetime in terms of almost Ricci-Bourguignon solitons with a torse-forming vector field. The introduction of these geometric flows has become a great centre of interest among geometers, leading to innovative approaches for understanding the geometry of diverse Riemannian manifolds and their submanifolds [16].

Furthermore, the concept of a quarter-symmetric connection in a differentiable manifold with an affine connection was introduced by S. Golab in [5]. A linear connection is characterized as a quarter symmetric connection if its torsion tensor T takes the form:

$$T(U, V) = \eta(V)\phi U - \eta(U)\phi V, \tag{1.6}$$

where U and V represent vector fields on M, η is a 1-form, and ϕ is a tensor of type $(1, 1)$. This notion generalizes semi-symmetric connections; when $\phi = I$, a quarter symmetric connection becomes a semi-symmetric connection. Additionally, a quarter symmetric connection $\overline{\nabla}$ is identified as a quarter symmetric metric connection for a Riemannian metric g if it satisfies $\overline{\nabla} g = 0$, otherwise referred to as non-metric. Quarter symmetric metric connections have been extensively studied by various authors from multiple perspectives [1].

Motivated by these discussions, this paper delves into the examination of para-Sasakian manifolds admitting RYS, as well as RYS on para-Sasakian manifolds equipped with a quarter symmetric metric connection. The paper's structure unfolds as follows: Section 2 provides preliminary information and formulas related to para-Sasakian manifolds. Section 3 explores RYS on para-Sasakian manifolds, evaluating the value of Λ and revealing that if $(g, \xi, \Lambda, \rho, q)$ represents an RYS on a para-Sasakian manifold M, then M determines as a nearly quasi-Einstein manifold. Furthermore, it is established that an RYS $(g, \xi, \Lambda, \rho, q)$ on a para-Sasakian manifold is always expanding. Section 4 delves into the study of RYS on para-Sasakian manifolds within the context of a quarter symmetric metric connection, deducing the value of Λ. This section also proves that RYS on a para-Sasakian manifold with a quarter symmetric metric connection can be characterized as a pseudo η-Einstein manifold and always demonstrated expansion. Finally, the last section offers an example to validate the presented results.

2. Preliminaries

An n-dimensional differentiable manifold M is said to be an almost paracontact manifold [17] if it satisfies an almost para-contact structure (ϕ, ξ, η) where ϕ is a $(1,1)$-tensor, ξ a global vector field and η a 1-form, such that

$$\phi^2 U = U - \eta(U)\xi, \tag{2.1}$$

$$\eta(\xi) = 1, \quad \phi(\xi) = 0, \quad \eta(\phi\xi) = 0. \tag{2.2}$$

If g is a compatible pseudo-Riemannian metric with (ϕ, ξ, η), that is,

$$g(\phi U, \phi V) = g(U, V) - \eta(U)\eta(V), \quad \eta(U) = g(U, \xi), \tag{2.3}$$

$$g(\phi U, V) = g(U, \phi V), \tag{2.4}$$

$$\Phi(U, V) = g(\phi U, V) = g(U, \phi V) = \Phi(V, U), \tag{2.5}$$

for all vector fields U, V on M, then M becomes an almost paracontact Riemannian manifold with an almost paracontact Riemannian structure (ϕ, ξ, η, g). Here Φ is the fundamental 2-form associated to the almost paracontact Riemannian structure.

The normality property of a paracontact metric manifold (M, ξ, η, g) is synonymous with the nullification of the $(1,2)$-torsion tensor denoted by

$$N_\phi(U, V) = [\phi, \phi](U, V) - 2d\eta(U, V)\xi,$$

where

$$[\phi, \phi](U, V) = \phi^2[U, V] + [\phi U, \phi V] - \phi[U, \phi V] - \phi[\phi U, V],$$

for any vector fields U and V defined on the manifold M. A paracontact metric manifold is dubbed para-Sasakian metric manifold when it possesses this property of normality. Alternately, an almost paracontact Riemannian connection manifold earns the label of para-Sasakian if it satisfies the following conditions:

$$\nabla_U \xi = \phi U, \tag{2.6}$$

and

$$(\nabla_U \phi)V = 2\eta(U)\eta(V)\xi - g(U,V)\xi - \eta(V)U, \tag{2.7}$$

where ∇ denotes the Riemannian connection of g.

Furthermore, on an n-dimensional para-Sasakian manifold M, the subsequent relationships are valid:

$$\eta(R(U,V)W) = \eta(V)g(U,W) - \eta(U)g(V,W), \tag{2.8}$$

$$R(U,V)\xi = \eta(U)V - \eta(V)U, \tag{2.9}$$

$$R(\xi,U)V = \eta(V)U - g(U,V)\xi, \tag{2.10}$$

$$S(U,\xi) = -(n-1)\eta(U), \quad Q\xi = -(n-1)\xi. \tag{2.11}$$

These relationships are applicable for any vector fields U, V, and W on M. In these equations, R represents the Riemannian curvature tensor, and S is the Ricci tensor of type $(0,2)$ defined as

$$g(QU,V) = S(U,V),$$

where Q denotes the Ricci operator.

In the work presented in [4], De and Gazi introduced the concept of a nearly quasi-Einstein manifold. They established this notion within the framework of para-Sasakian manifolds. Specifically, a para-Sasakian manifold M is labeled as nearly quasi-Einstein when its Ricci tensor S is non-trivial and fulfills the condition:

$$S(U,V) = ag(U,V) + bD(U,V), \tag{2.12}$$

where a and b stand as non-zero scalar coefficients, and D denotes a symmetric non-zero $(0,2)$-tensor.

Taking inspiration from the concept of a pseudo quasi-Einstein manifold introduced by A. Shaikh in [13], Hui and Chakraborty [9] extended the idea to define a "pseudo η-Einstein manifold" within the context of a para-Sasakian manifold M. According to their definition, a para-Sasakian manifold M qualifies as a pseudo η-Einstein manifold when its Ricci tensor S of type $(0,2)$ is non-trivial and satisfies the following condition:

$$S(U,V) = ag(U,V) + b\eta(U)\eta(V) + cD(U,V). \tag{2.13}$$

Here, a, b, and c are scalar coefficients, with the condition that c cannot be equal to zero, and D represents a symmetric non-zero $(0,2)$-tensor. Moreover, the tensor D adheres to the additional constraint $D(U,\phi) = 0$ for a given vector field U. When these conditions are met, the manifold M is denoted as a pseudo η-Einstein manifold.

3. Results on Para-Sasakian Manifolds Admitting Ricci-Yamabe Soliton

In this section, we state some interesting results on para-Sasakian manifold admitting RYS. So, we consider a RYS $(g, \xi, \Lambda, \rho, q)$ on a para-Sasakian manifold then from (1.5), we have

$$(\pounds_\xi g)(U, V) + 2\rho S(U, V) + (2\Lambda - qr)g(U, V) = 0. \tag{3.1}$$

From (2.5) and (2.6), we get

$$\begin{aligned}(\pounds_\xi g)(U, V) &= g(\nabla_U \xi, V) + g(U, \nabla_V \xi)\\ &= g(\phi U, V) + g(\phi V, U) = 2\Phi(U, V).\end{aligned} \tag{3.2}$$

Substituting (3.2) into (3.1), we have

$$S(U, V) = -\frac{1}{2\rho}(2\Lambda - qr)g(U, V) - \frac{1}{\rho}\Phi(U, V). \tag{3.3}$$

Taking $V = \xi$ and using (2.5), (2.11) in (3.3), we have

$$\Lambda = \rho(n-1) - \frac{qr}{2}. \tag{3.4}$$

Thus, equation (3.4) leads the following:

Theorem 3.1. *If the metric of an n-dimensional para-Sasakian manifold M admits a RYS then the soliton constant* Λ *is given by (3.4).*

From equation (3.3), we can state the following:

Theorem 3.2. *A para-Sasakian manifold M is nearly quasi-Einstein if M admitting RYS.*

In particular, we take $\rho = 1$ and $q = 0$ in (3.3), then we have

$$S(U, V) = -\Lambda g(U, V) - \Phi(U, V), \tag{3.5}$$

which implies that the manifold is a nearly quasi-Einstein manifold.

Putting $V = \xi$ in (3.5), we have

$$S(U, \xi) = -\Lambda \eta(U). \tag{3.6}$$

From (2.11) and (3.6), we get $\Lambda = (n-1) > 0$.

Thus we can conclude the following:

Theorem 3.3. *A RS on a para-Sasakian manifold M is always expanding.*

Remark 3.1. In the work by Hui and Chakraborty [9], it was demonstrated that a RS on the same manifold M always displays an expanding behavios.

Example 3.1. In the example put forth in [2], the authors examined the Euclidean space $M = \mathbb{R}^3$ utilizing standard Cartesian coordinates (u, v, w).

They established the definition of vector fields as follows:

$$\mathcal{E}_1 = e^u \frac{\partial}{\partial v}, \quad \mathcal{E}_2 = e^u \left(\frac{\partial}{\partial v} - \frac{\partial}{\partial w} \right), \quad \mathcal{E}_3 = -\frac{\partial}{\partial u}. \tag{3.7}$$

These vector fields are linearly independent at each point of M.

The Riemannian metric g is then defined as follows:

$$g(\mathcal{E}_i, \mathcal{E}_j) = 0, i \neq j, \quad i, j = 1, 2, 3, g(\mathcal{E}_1, \mathcal{E}_1) = g(\mathcal{E}_2, \mathcal{E}_2) = g(\mathcal{E}_3, \mathcal{E}_3) = 1. \tag{3.8}$$

Similarly, the 1-form η is defined by

$$\eta(W) = g(W, \mathcal{E}_3),$$

where W is a vector field on M. The $(1,1)$ tensor field ϕ is given by

$$\phi(\mathcal{E}_1) = \mathcal{E}_2, \quad \phi(\mathcal{E}_2) = \mathcal{E}_1, \quad \phi(\mathcal{E}_3) = 0$$

with $\xi = \mathcal{E}_3$. Hence, the set (ϕ, ξ, η, g) establishes an almost paracontact structure on M.

The Levi-Civita connection ∇ corresponding to the metric g leads to the following relations:

$$[\mathcal{E}_1, \mathcal{E}_2] = 0, \quad [\mathcal{E}_1, \mathcal{E}_3] = \mathcal{E}_1, \quad [\mathcal{E}_2, \mathcal{E}_3] = \mathcal{E}_2. \tag{3.9}$$

By setting $\mathcal{E}_3 = \xi$ and applying Koszul's formula, the following expressions are obtained:

$$\begin{aligned}
&\nabla_{\mathcal{E}_1} \mathcal{E}_2 = 0, \quad \nabla_{\mathcal{E}_1} \mathcal{E}_3 = \mathcal{E}_1, \quad \nabla_{\mathcal{E}_1} \mathcal{E}_1 = -\mathcal{E}_3, \\
&\nabla_{\mathcal{E}_2} \mathcal{E}_3 = \mathcal{E}_2, \quad \nabla_{\mathcal{E}_2} \mathcal{E}_2 = -\mathcal{E}_3, \quad \nabla_{\mathcal{E}_2} \mathcal{E}_1 = 0, \\
&\nabla_{\mathcal{E}_3} \mathcal{E}_3 = 0, \quad \nabla_{\mathcal{E}_3} \mathcal{E}_2 = 0, \quad \nabla_{\mathcal{E}_3} \mathcal{E}_1 = 0.
\end{aligned} \tag{3.10}$$

Based on the above results, it becomes evident that the manifold M is indeed para-Sasakian.

Additionally, the expressions for the curvature tensor and the Ricci tensor are deduced as follows:

$$\begin{aligned}
&R(\mathcal{E}_1, \mathcal{E}_2)\mathcal{E}_2 = -\mathcal{E}_1, \quad R(\mathcal{E}_1, \mathcal{E}_3)\mathcal{E}_3 = -\mathcal{E}_1, \quad R(\mathcal{E}_2, \mathcal{E}_1)\mathcal{E}_1 = -\mathcal{E}_2, \\
&R(\mathcal{E}_2, \mathcal{E}_3)\mathcal{E}_3 = -\mathcal{E}_2, \quad R(\mathcal{E}_3, \mathcal{E}_1)\mathcal{E}_1 = -\mathcal{E}_3, \quad R(\mathcal{E}_3, \mathcal{E}_2)\mathcal{E}_2 = -\mathcal{E}_3, \\
&R(\mathcal{E}_1, \mathcal{E}_2)\mathcal{E}_3 = 0, \quad R(\mathcal{E}_3, \mathcal{E}_2)\mathcal{E}_3 = \mathcal{E}_2, \quad R(\mathcal{E}_3, \mathcal{E}_1)\mathcal{E}_2 = 0.
\end{aligned} \tag{3.11}$$

The Ricci tensor is computed as:

$$\begin{aligned}
&S(\mathcal{E}_1, \mathcal{E}_1) = -2, S(\mathcal{E}_2, \mathcal{E}_2) = -2, S(\mathcal{E}_3, \mathcal{E}_3) = -2, \\
&S(\mathcal{E}_1, \mathcal{E}_2) = 0, S(\mathcal{E}_1, \mathcal{E}_3) = 0, S(\mathcal{E}_2, \mathcal{E}_3) = 0.
\end{aligned} \tag{3.12}$$

The scalar curvature is found to be $r=-6$. Substituting these values into the expression (3.3), the following relation is obtained:

$$S(\mathcal{E}_1,\mathcal{E}_1)=-\frac{\Lambda}{\rho}-\frac{qr}{2\rho},\quad S(\mathcal{E}_2,\mathcal{E}_2)=-\frac{\Lambda}{\rho}-\frac{qr}{2\rho},\quad S(\mathcal{E}_3,\mathcal{E}_3)=-\frac{\Lambda}{\rho}-\frac{qr}{2\rho}. \tag{3.13}$$

Combining equations (3.12) and (3.13), it yields:

$$\Lambda=2\rho-\frac{qr}{2}. \tag{3.14}$$

This relation satisfies the condition in equation (3.4). Consequently, it is evident that Theorem 3.1 is valid. Also, (3.14) can be rewritten as $\Lambda=2\rho+3q$. As we can see that distinct values of Λ emerge as ρ and q vary. However, when substituting $\rho=1$ and $q=0$ into equation (3.14), then $\Lambda=2>0$. This confirms that the three-dimensional manifold M is perpetually expanding. Hence, it is established that g characterizes a Ricci soliton on a three-dimensional para-Sasakian manifold M, confirming the validity of Theorem 3.3.

4. Results on Para-Sasakian Manifolds with Quarter Symmetric Metric Connection Admitting Ricci-Yamabe Soliton

In this section, we study para-Sasakian manifold with quarter symmetric metric connection admitting RYS. Let $\overline{\nabla}$ be a linear connection on n-dimensional differentiable manifold M which is known as quarter symmetric connection [5] if its torsion tensor T of the connection $\overline{\nabla}$ is the form of

$$\begin{aligned} T(U,V) &= \overline{\nabla}_U V-\overline{\nabla}_V U-[U,V] \\ &= \eta(V)\phi U-\eta(U)\phi V, \end{aligned} \tag{4.1}$$

where ϕ is a tensor of type $(1,1)$ and η is a 1-form.
Also, if the quarter symmetric connection $\overline{\nabla}$ satisfies the following condition:

$$(\overline{\nabla}_U g)(V,W)=0, \tag{4.2}$$

for all vector fields U,V,W on M, then $\overline{\nabla}$ is said to be a quarter symmetric metric connection.

Now, a quarter symmetric metric connection $\overline{\nabla}$ on a para-Sasakian manifold is defined by

$$\overline{\nabla}_U V=\nabla_U V+H(U,V), \tag{4.3}$$

where H is a tensor of type $(1,1)$.

For $\overline{\nabla}$ to be a quarter-symmetric metric connection on M such that

$$H(U,V)=\frac{1}{2}\left[T(U,V)+T^{'}(U,V)+T^{'}(V,U)\right], \tag{4.4}$$

where the tensor $T^{'}$ and T are related by

$$g(T^{'}(U,V),W)=g(T(W,U),V). \tag{4.5}$$

From (4.1) and (4.5), we have

$$T'(U,V) = \eta(U)\phi V - g(\phi U, V)\xi. \tag{4.6}$$

Using (4.1) and (4.6) in (4.4), we obtain

$$H(U,V) = \eta(V)\phi U - g(\phi U, V)\xi. \tag{4.7}$$

Hence, a quarter symmetric metric connection on M is given by

$$\overline{\nabla}_U V = \nabla_U V + \eta(V)\phi U - g(\phi U, V)\xi. \tag{4.8}$$

Now, suppose that R and $\overline{R}$ are the curvature tensor of Levi-Civita connection ∇ and the quarter symmetric metric connection $\overline{\nabla}$ on a para-Sasakian manifold, respectively

$$\begin{aligned} \overline{R}(U,V)W &= R(U,V)W + 3g(\phi U, W)\phi V - 3g(\phi V, W)\phi U \\ &\quad + [\eta(U)V - \eta(V)U]\,\eta(W) \\ &\quad - [\eta(U)g(V,W) - \eta(V)g(U,W)]\,\xi. \end{aligned} \tag{4.9}$$

From above equation, we get

$$\begin{aligned} \overline{S}(U,V) &= S(U,V) + 2g(U,V) - (n+1)\eta(U)\eta(V) \\ &\quad -3trace(\phi)g(\phi U, V), \end{aligned} \tag{4.10}$$

where S and $\overline{S}$ defined as the Ricci tensor of para-Saskian manifold with respect to Levi-Civita connection and quarter symmetric metric connection, respectively.

Also, we have

$$\overline{S}(U,\xi) = -2(n-1)\eta(U). \tag{4.11}$$

Contracting (4.10), we obtain

$$\overline{r} = r + (n-1) - 3(trace(\phi))^2, \tag{4.12}$$

where r and $\overline{r}$ are the scalar curvature of para-Saskian manifold with respect to Levi-Civita connection and quarter symmetric metric connection, respectively.

Now, let $(g, \xi, \Lambda, \rho, q)$ be a RYS on a para-Sasakian manifold M with respect to $\overline{\nabla}$. Then we have

$$(\overline{\pounds}_\xi g)(U,V) + 2\rho\overline{S}(U,V) + (2\Lambda - q\overline{r})g(U,V) = 0. \tag{4.13}$$

Now, from (2.2), (2.5) and (4.8), we obtain

$$\begin{aligned} (\overline{\pounds}_\xi g)(U,V) &= g(\overline{\nabla}_U \xi, V) + g(U, \overline{\nabla}_V \xi) \\ &= g(\nabla_U \xi, V) + \eta(\xi)g(\phi U, V) - g(\phi U, \xi)g(\xi, V) \\ &\quad + g(\nabla_V \xi, U) + \eta(\xi)g(\phi V, U) - g(\phi V, \xi)g(\xi, U) \\ &= (\pounds_\xi g)(U,V) + 2\Phi(U,V). \end{aligned} \tag{4.14}$$

Use (4.14), (3.2), (4.12) and (4.10) in (4.13), we have

$$\rho\overline{S}(U,V) = -\left[\Lambda - \frac{qr}{2} - \frac{(n-1)q}{2} + \frac{3q}{2}(trace(\phi))^2\right] g(U,V) \tag{4.15}$$

$$-2\Phi(U,V), \tag{4.16}$$

and

$$S(U,V) = \frac{1}{\rho}\left[-\Lambda + \frac{qr}{2} + \frac{(n-1)q}{2} - \frac{3q}{2}(trace(\phi))^2 - 2\rho\right] g(U,V)$$
$$+(n+1)\eta(U)\eta(V) + \frac{1}{\rho}(3\rho\ trace(\phi) - 2)\Phi(U,V). \tag{4.17}$$

Putting $U = \xi$ and using (2.5), (2.11) in (4.17), we obtain

$$\Lambda = 2(n-1)\rho + \frac{qr}{2} + \frac{(n-1)q}{2} - \frac{3q}{2}(trace(\phi))^2. \tag{4.18}$$

Thus, (4.18) leads the following:

Theorem 4.1. *If the metric of an n-dimensional para-sasakian manifold admits a RYS with quarter symmetric metric connection then soliton constant Λ is given by (4.18).*

We conclude from (4.17):

Theorem 4.2. *A para-Sasakian manifold M with quarter symmetric metric connection is a pseudo η-Einstein manifold if M admits RYS.*

If we take $\rho = 1$ and $q = 0$ in (4.17), then we have

$$S(U,V) = -(\Lambda + 2)g(U,V) + (n+1)\eta(U)\eta(V)$$
$$+(3\ trace(\phi) - 2)\Phi(U,V). \tag{4.19}$$

which shows that the manifold is a pseudo η-Einstein.

Substitute $V = \xi$ in (4.19), we get

$$S(U,\xi) = (-\Lambda + n - 1)\eta(U). \tag{4.20}$$

From (2.11), (2.6) and (4.20), we have

$$\Lambda = 2(n-1) > 0. \tag{4.21}$$

Thus, from (4.21) we can state that

Theorem 4.3. *A RS on a para-Sasakian manifold M with quarter symmetric metric connection is always expanding.*

Example 4.1. In the example [10], the authors examined the Euclidean space $M = \mathbb{R}^5$ utilizing standard Cartesian coordinates $(u_1, u_2, u_3, u_4, u_5)$.

They define the vector fields as follows:

$$\begin{aligned} &\mathcal{E}_1 = \frac{\partial}{\partial u_1}, \quad \mathcal{E}_2 = e^{-u_1}\frac{\partial}{\partial u_2}, \quad \mathcal{E}_3 = e^{-u_1}\frac{\partial}{\partial u_3}, \\ &\mathcal{E}_4 = e^{-u_1}\frac{\partial}{\partial u_4}, \quad \mathcal{E}_5 = e^{-u_1}\frac{\partial}{\partial u_5}. \end{aligned} \tag{4.22}$$

These vector fields are linearly independent at each point of M.
The Riemannian metric g is then defined as follows:

$$\begin{aligned} g(\mathcal{E}_i, \mathcal{E}_j) &= 0, i \neq j, \quad i, j = 1, 2, 3, 4, 5 \\ g(\mathcal{E}_1, \mathcal{E}_1) &= g(\mathcal{E}_2, \mathcal{E}_2) = g(\mathcal{E}_3, \mathcal{E}_3) = 1 \\ g(\mathcal{E}_4, \mathcal{E}_4) &= g(\mathcal{E}_5, \mathcal{E}_5) = 1. \end{aligned} \tag{4.23}$$

Similarly, the 1-form η is defined by

$$\eta(W) = g(W, \mathcal{E}_1),$$

where W is a vector field on M. The $(1, 1)$ tensor field ϕ is given by

$$\phi(\mathcal{E}_1) = 0, \quad \phi(\mathcal{E}_2) = \mathcal{E}_2, \quad \phi(\mathcal{E}_3) = \mathcal{E}_3, \quad \phi(\mathcal{E}_4) = \mathcal{E}_4, \quad \phi(\mathcal{E}_5) = \mathcal{E}_5$$

with $\xi = \mathcal{E}_1$. Hence, the set (ϕ, ξ, η, g) establishes an almost paracontact structure on M.

Then they define the following relations:

$$\begin{aligned} &[\mathcal{E}_1, \mathcal{E}_2] = -\mathcal{E}_2, \ [\mathcal{E}_1, \mathcal{E}_3] = -\mathcal{E}_3, \ [\mathcal{E}_1, \mathcal{E}_4] = -\mathcal{E}_4, \ [\mathcal{E}_1, \mathcal{E}_5] = -\mathcal{E}_5 \\ &[\mathcal{E}_2, \mathcal{E}_3] = [\mathcal{E}_2, \mathcal{E}_4] = [\mathcal{E}_2, \mathcal{E}_5] = [\mathcal{E}_3, \mathcal{E}_4] = [\mathcal{E}_3, \mathcal{E}_5] = [\mathcal{E}_4, \mathcal{E}_5] = 0. \end{aligned} \tag{4.24}$$

By taking $\mathcal{E}_1 = \xi$ and applying Koszul's formula, the following expressions are obtained:

$$\begin{aligned} &\nabla_{\mathcal{E}_1}\mathcal{E}_1 = 0, \ \nabla_{\mathcal{E}_1}\mathcal{E}_2 = 0, \ \nabla_{\mathcal{E}_1}\mathcal{E}_3 = 0, \ \nabla_{\mathcal{E}_1}\mathcal{E}_4 = 0, \nabla_{\mathcal{E}_1}\mathcal{E}_5 = 0, \\ &\nabla_{\mathcal{E}_2}\mathcal{E}_1 = \mathcal{E}_2, \ \nabla_{\mathcal{E}_2}\mathcal{E}_2 = -\mathcal{E}_1, \ \nabla_{\mathcal{E}_2}\mathcal{E}_3 = 0, \ \nabla_{\mathcal{E}_2}\mathcal{E}_4 = 0, \ \nabla_{\mathcal{E}_2}\mathcal{E}_5 = 0, \\ &\nabla_{\mathcal{E}_3}\mathcal{E}_1 = \mathcal{E}_3, \ \nabla_{\mathcal{E}_3}\mathcal{E}_2 = 0, \ \nabla_{\mathcal{E}_3}\mathcal{E}_3 = -\mathcal{E}_1, \ \nabla_{\mathcal{E}_3}\mathcal{E}_4 = 0, \ \nabla_{\mathcal{E}_3}\mathcal{E}_5 = 0, \\ &\nabla_{\mathcal{E}_4}\mathcal{E}_1 = \mathcal{E}_4, \ \nabla_{\mathcal{E}_4}\mathcal{E}_2 = 0, \ \nabla_{\mathcal{E}_4}\mathcal{E}_3 = 0, \ \nabla_{\mathcal{E}_4}\mathcal{E}_4 = -\mathcal{E}_1, \ \nabla_{\mathcal{E}_4}\mathcal{E}_5 = 0, \\ &\nabla_{\mathcal{E}_5}\mathcal{E}_1 = \mathcal{E}_5, \ \nabla_{\mathcal{E}_5}\mathcal{E}_2 = 0, \ \nabla_{\mathcal{E}_5}\mathcal{E}_3 = 0, \ \nabla_{\mathcal{E}_5}\mathcal{E}_4 = 0, \ \nabla_{\mathcal{E}_5}\mathcal{E}_5 = -\mathcal{E}_1. \end{aligned} \tag{4.25}$$

Using the above equations in (4.8), we have

$$\begin{aligned} &\overline{\nabla}_{\mathcal{E}_1}\mathcal{E}_1 = 0, \ \overline{\nabla}_{\mathcal{E}_1}\mathcal{E}_2 = 0, \ \overline{\nabla}_{\mathcal{E}_1}\mathcal{E}_3 = 0, \ \overline{\nabla}_{\mathcal{E}_1}\mathcal{E}_4 = 0, \overline{\nabla}_{\mathcal{E}_1}\mathcal{E}_5 = 0, \\ &\overline{\nabla}_{\mathcal{E}_2}\mathcal{E}_1 = 2\mathcal{E}_2, \ \overline{\nabla}_{\mathcal{E}_2}\mathcal{E}_2 = -2\mathcal{E}_1, \ \overline{\nabla}_{\mathcal{E}_2}\mathcal{E}_3 = 0, \ \overline{\nabla}_{\mathcal{E}_2}\mathcal{E}_4 = 0, \ \overline{\nabla}_{\mathcal{E}_2}\mathcal{E}_5 = 0, \\ &\overline{\nabla}_{\mathcal{E}_3}\mathcal{E}_1 = 2\mathcal{E}_3, \ \overline{\nabla}_{\mathcal{E}_3}\mathcal{E}_2 = 0, \ \overline{\nabla}_{\mathcal{E}_3}\mathcal{E}_3 = -2\mathcal{E}_1, \ \overline{\nabla}_{\mathcal{E}_3}\mathcal{E}_4 = 0, \ \overline{\nabla}_{\mathcal{E}_3}\mathcal{E}_5 = 0, \\ &\overline{\nabla}_{\mathcal{E}_4}\mathcal{E}_1 = 2\mathcal{E}_4, \ \overline{\nabla}_{\mathcal{E}_4}\mathcal{E}_2 = 0, \ \nabla_{\mathcal{E}_4}\mathcal{E}_3 = 0, \ \overline{\nabla}_{\mathcal{E}_4}\mathcal{E}_4 = -2\mathcal{E}_1, \ \overline{\nabla}_{\mathcal{E}_4}\mathcal{E}_5 = 0, \\ &\overline{\nabla}_{\mathcal{E}_5}\mathcal{E}_1 = 2\mathcal{E}_5, \ \overline{\nabla}_{\mathcal{E}_5}\mathcal{E}_2 = 0, \ \overline{\nabla}_{\mathcal{E}_5}\mathcal{E}_3 = 0, \ \overline{\nabla}_{\mathcal{E}_5}\mathcal{E}_4 = 0, \ \overline{\nabla}_{\mathcal{E}_5}\mathcal{E}_5 = -2\mathcal{E}_1. \end{aligned} \tag{4.26}$$

Additionally, the components of the curvature tensor with respect to Levi-Civita connection and quarter symmetric metric connection are obtained as follows:

$$\begin{aligned}
&R(\mathcal{E}_1,\mathcal{E}_2)\mathcal{E}_1=\mathcal{E}_2, R(\mathcal{E}_1,\mathcal{E}_2)\mathcal{E}_2=-\mathcal{E}_1, R(\mathcal{E}_1,\mathcal{E}_3)\mathcal{E}_1=\mathcal{E}_3, R(\mathcal{E}_1,\mathcal{E}_3)\mathcal{E}_3=-\mathcal{E}_1,\\
&R(\mathcal{E}_1,\mathcal{E}_4)\mathcal{E}_1=\mathcal{E}_4, R(\mathcal{E}_1,\mathcal{E}_4)\mathcal{E}_4=-\mathcal{E}_1, R(\mathcal{E}_1,\mathcal{E}_5)\mathcal{E}_1=\mathcal{E}_5, R(\mathcal{E}_1,\mathcal{E}_5)\mathcal{E}_5=-\mathcal{E}_1,\\
&R(\mathcal{E}_2,\mathcal{E}_3)\mathcal{E}_2=\mathcal{E}_3, R(\mathcal{E}_2,\mathcal{E}_3)\mathcal{E}_3=-\mathcal{E}_2, R(\mathcal{E}_2,\mathcal{E}_4)\mathcal{E}_2=\mathcal{E}_4, R(\mathcal{E}_2,\mathcal{E}_4)\mathcal{E}_4=-\mathcal{E}_2,\\
&R(\mathcal{E}_2,\mathcal{E}_5)\mathcal{E}_2=\mathcal{E}_5, R(\mathcal{E}_2,\mathcal{E}_5)\mathcal{E}_5=-\mathcal{E}_2, R(\mathcal{E}_3,\mathcal{E}_4)\mathcal{E}_3=\mathcal{E}_4, R(\mathcal{E}_3,\mathcal{E}_4)\mathcal{E}_4=-\mathcal{E}_3,\\
&R(\mathcal{E}_3,\mathcal{E}_5)\mathcal{E}_3=\mathcal{E}_5, R(\mathcal{E}_3,\mathcal{E}_5)\mathcal{E}_5=-\mathcal{E}_3, R(\mathcal{E}_4,\mathcal{E}_5)\mathcal{E}_4=\mathcal{E}_5, R(\mathcal{E}_4,\mathcal{E}_5)\mathcal{E}_5=-\mathcal{E}_4.
\end{aligned}\tag{4.27}$$

and

$$\begin{aligned}
&\overline{R}(\mathcal{E}_1,\mathcal{E}_2)\mathcal{E}_1=2\mathcal{E}_2, \overline{R}(\mathcal{E}_1,\mathcal{E}_2)\mathcal{E}_2=-2\mathcal{E}_1, \overline{R}(\mathcal{E}_1,\mathcal{E}_3)\mathcal{E}_1=2\mathcal{E}_3,\\
&\overline{R}(\mathcal{E}_1,\mathcal{E}_3)\mathcal{E}_3=-2\mathcal{E}_1, \overline{R}(\mathcal{E}_1,\mathcal{E}_4)\mathcal{E}_1=2\mathcal{E}_4, \overline{R}(\mathcal{E}_1,\mathcal{E}_4)\mathcal{E}_4=-2\mathcal{E}_1,\\
&\overline{R}(\mathcal{E}_1,\mathcal{E}_5)\mathcal{E}_1=2\mathcal{E}_5, \overline{R}(\mathcal{E}_1,\mathcal{E}_5)\mathcal{E}_5=-2\mathcal{E}_1, \overline{R}(\mathcal{E}_2,\mathcal{E}_3)\mathcal{E}_2=2\mathcal{E}_3,\\
&\overline{R}(\mathcal{E}_2,\mathcal{E}_3)\mathcal{E}_3=-2\mathcal{E}_2, \overline{R}(\mathcal{E}_2,\mathcal{E}_4)\mathcal{E}_2=2\mathcal{E}_4, \overline{R}(\mathcal{E}_2,\mathcal{E}_4)\mathcal{E}_4=-2\mathcal{E}_2,\\
&\overline{R}(\mathcal{E}_2,\mathcal{E}_5)\mathcal{E}_2=2\mathcal{E}_5, \overline{R}(\mathcal{E}_2,\mathcal{E}_5)\mathcal{E}_5=-2\mathcal{E}_2, \overline{R}(\mathcal{E}_3,\mathcal{E}_4)\mathcal{E}_3=2\mathcal{E}_4,\\
&\overline{R}(\mathcal{E}_3,\mathcal{E}_4)\mathcal{E}_4=-2\mathcal{E}_3, \overline{R}(\mathcal{E}_3,\mathcal{E}_5)\mathcal{E}_3=2\mathcal{E}_5, \overline{R}(\mathcal{E}_3,\mathcal{E}_5)\mathcal{E}_5=-2\mathcal{E}_3,\\
&\overline{R}(\mathcal{E}_4,\mathcal{E}_5)\mathcal{E}_4=2\mathcal{E}_5, \overline{R}(\mathcal{E}_4,\mathcal{E}_5)\mathcal{E}_5=-2\mathcal{E}_4.
\end{aligned}\tag{4.28}$$

The Ricci tensor for Levi-Civita connection and quarter symmetric metric connection are computed by using above expression:

$$S(\mathcal{E}_1,\mathcal{E}_1)=-4,\quad S(\mathcal{E}_2,\mathcal{E}_2)=S(\mathcal{E}_3,\mathcal{E}_3)=S(\mathcal{E}_4,\mathcal{E}_4)=S(\mathcal{E}_5,\mathcal{E}_5)=2,\tag{4.29}$$

and

$$\overline{S}(\mathcal{E}_1,\mathcal{E}_1)=\overline{S}(\mathcal{E}_2,\mathcal{E}_2)=\overline{S}(\mathcal{E}_3,\mathcal{E}_3)=\overline{S}(\mathcal{E}_4,\mathcal{E}_4)=\overline{S}(\mathcal{E}_5,\mathcal{E}_5)=-8.\tag{4.30}$$

The scalar curvature is found to be $r=4$ and $\overline{r}=-40$, which can be verified by (4.12). Substituting these values into the expression (4.17), the following relation is obtained:

$$\overline{S}(\mathcal{E}_i,\mathcal{E}_i)=-\frac{1}{\rho}\left[\Lambda+20q\right]g(\mathcal{E}_i,\mathcal{E}_i)-\frac{2}{\rho}g(\phi\mathcal{E}_i,\mathcal{E}_i).\tag{4.31}$$

For $\xi=\mathcal{E}_1$, the above relation reduces to $\overline{S}(\xi,\xi)=-\frac{1}{\rho}\left[\Lambda+20q\right]$. Since, $\overline{S}(\mathcal{E}_1,\mathcal{E}_1)=\overline{S}(\xi,\xi)=-8$, then we have $\Lambda=8\rho-20q$, which satisfies our relation (4.18). Thus, Theorem 4.1 exists. For $\rho=1$ and $q=0$, Theorem 4.3 is satisfied.

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