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ON CONFORMAL-MATSUMOTO CHANGE OF m-TH ROOT FINSLER METRICS

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Abstract. In this paper, we have considered conformal-Matsumoto change of the class of m-th root Finsler metrics. We have established the necessary and sufficient condition for the transformed metric to be projectively flat or locally dually flat. Further, we have proved the non-existence of the concerned metric which is projectively flat with non-zero flag curvature.
Keywords: Locally dually flat metric, projectively flat metric, Matsumoto change, conformal change, projective change, m-th root metric.

1. Introduction

Investigation of geometrical structures of family of probability distribution leads to the appearance of information geometry and applied to various fields like, multi-terminal information theory, statistical inference and control system. Finsler information geometry includes a unique and essential class of Finsler metrics called dually flat Finsler metrics. These metrics play an important role for understanding flat Finsler information structure. While studying the information geometry of Riemannian manifold, Amari and Nagaoka [1] introduced the concept of locally dually flat Riemannian metrics. Later, Shen [4] extended this concept of flatness to Finsler geometry.

The m-th root metric was first introduced by H. Shimada in 1979 [10]. Mathematicians as well as Physicists become interested in the study of m-th root Finsler

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metrics as it has numerous applications in Ecology, Biology and in Physics including the study possible model of space time as well as unified gauge theory. The special m-th root metric $F = \sqrt[m]{y^{i_1}y^{i_2}\dots y^{i_m}}$ is called Berwald-Moór metric which plays a very important role in theory of space-time structure, gravitation and general relativity.

Let M be an n-dimensional manifold and TM be its tangent bundle. Let $F : TM \to \mathbb{R}$ be defined as $F = A^{\frac{1}{m}}$, where $A := a_{i_1i_2\dots i_m}(x)y^{i_1}y^{i_2}...y^{i_m}$ and $a_{i_1i_2\dots i_m}$ is symmetric in all indices. Then F is called an m-th root Finsler metric on M. For $m = 3, 4$, it is called cubic and quatric metric, respectively. Many authors have studied and characterized the Randers change [11], Conformal Kropina change [6], and generalized Kropina change [13] of m-th root Finsler metrics. Tayebi *et al.* [11] and M. Kumar [7] have studied generalized m-th root Finsler metrics and discussed conditions for a Finsler metric to be locally dually flat and locally projectively flat. In the paper, we have considered the Conformal-Matsumoto change of Finsler metric defined by following

$$\overline{F} = e^{\sigma(x)}\frac{F^2}{F-\beta}, \tag{1.1}$$

where $\beta = b_i(x)y^i$ is a one-form and the Finsler metric F is an m-th root metric given by $F = A^{\frac{1}{m}}$.

Hilbert's fourth problem in the regular case is to characterize the metrics on an open domain $V \subset \mathbb{R}^n$ such that the straight line segment is the shortest curve joining two points, *i.e.,* geodesic are straight line, and such Finsler metrics are called projectively flat. It is well known that Riemannian metric is locally projectively flat if and only if it is of constant sectional curvature. Thus, one can say that the problem is completely solved for Riemannian metric. Now the question aries, whether the problem can be solved for Finsler metric? In Finsler spaces, the flag curvature is natural extension of the sectional curvature and every locally projectively flat Finsler metric is of scalar flag curvature. But the converse is not true, *i.e.,* there exist Finsler metrics with scalar or constant flag curvature that are not locally projectively flat [8]. Thus, it is natural to study Finsler metrics of scalar (respectively constant) flag curvature, as the problem is still open in Finsler geometry. Here, we have characterized the locally projectively flatness and locally dually flatness of Conformal Matsumoto m-th root Finsler metric and gave the following results as

Theorem 1.1. *Let $F = A^{\frac{1}{m}}$ be m-th root Finsler metric. Assume that $\overline{F} = \overline{F}(x,y)$ be the Conformal-Matsumoto change of F. Then the necessary and sufficient condition for $\overline{F}$ to be projectively flat is $A_{x^j} = 0$, $b_j =$ constant and the change is homothetic.*

Theorem 1.2. *Let $F = A^{\frac{1}{m}}$ be m-th root Finsler metric. Assume that $\overline{F} = \overline{F}(x,y)$ be the Conformal-Matsumoto change of F. Then the necessary and sufficient condition for $\overline{F}$ to be locally dually flat is $A_{x^i} = 0$, $b_i =$ constant and the change is homothetic.*

Riemannian curvature is a significant and central concept of Riemannian geometry and in 1926, it was extended to Finsler geometry by L. Berwald. The Riemannian curvature of Finsler space is a family of linear transformation on a tangent plane. The Riemann curvature $\mathbf{R}_y : T_xM \to T_xM$ is defined by $\mathbf{R}_y(u) = R^i_j(x,y)u^j\partial/\partial x^i$, $u = u^j\partial/\partial x^j$, where

$$R^i_j(x,y) := 2\frac{\partial G^i}{\partial x^j} - y^k\frac{\partial^2 G^i}{\partial x^k y^j} + 2G^k\frac{\partial^2 G^i}{\partial y^k x^j} - \frac{\partial G^i}{\partial x^k}\frac{\partial G^k}{\partial x^j}\,.$$

The flag curvature $\mathbf{K}$ at a point x is a function of tangent plane $P \subset T_xM$ and a non-zero vector $y \in P$ which tells us about how curved the space is, namely, $\mathbf{K} = \mathbf{K}(x,y,P)$. For each tangent plane $P \subset T_xM$ containing $y \in P\backslash\{0\}$, the flag curvature is defined as [4]

$$\mathbf{K}(x,y,P) = \frac{\mathbf{g}_y(\mathbf{R}_y(u),u)}{\mathbf{g}_y(y,y)\mathbf{g}_y(u,u) - [\mathbf{g}_y(y,u)]^2}\,,$$

where $u \in P$ such that $P = span\{y,u\}$. If $\mathbf{K}(x,y,P) = \mathbf{K}(x,y)$, then the Finsler metric is called scalar flag curvature and if $\mathbf{K}(x,y,P) = constant$, then the Finsler metric is called constant flag curvature. The Finsler metric is said to have isotropic flag curvature if $\mathbf{K}(x,y,P) = \mathbf{K}(x)$ is a function of $x \in M$ and by Schur's lemma, if the flag curvature is isotropic and $dim(M) \geq 3$, then $\mathbf{K} =$ constant. There are many non-Riemannian projectively flat Finsler metric with constant flag curvature. For example, Funk metric and Hilbert metric [4] are projectively flat as well as of constant flag curvature on strongly convex domain. This makes it worth considering the investigation of constant flag curvature for various Finsler metrics. In this case, we have taken into account the Conformal Matsumoto metric and provided the following result as

Theorem 1.3. *Let $\overline{F} = \overline{F}(x,y)$ be the conformal-Matsumoto change of m-th root ($m > 4$) metric $F = \sqrt[m]{A}$ such that A is irreducible. Suppose that $\overline{F}$ is projectively flat with constant flag curvature. Then $\overline{F}$ has vanishing flag curvature* $\mathbf{K} = 0$ *and it also satisfies the three condition mentioned in the equations (4.4)-(4.6).*

2. Projectively Flat Finsler Metrics

Projectively flat Finsler metrics of constant flag curvatures can be described by using algebraic equation or by using Taylor expression. The projective factor of projectively flat Finsler metric can be calculated using the general formula provided by Hamel [5] as: A Finsler metric $F = F(x,y)$ on an open subset $V \subset \mathbb{R}^n$ is projectively flat if and only if it satisfies $F_{x^iy^j}y^i = F_{x^j}$.

Proof of theorem 1.1: For an m-th root metric $F = A^{\frac{1}{m}}$, we have used the following notations

$$A_i = \frac{\partial A}{\partial y^i}, \quad A_{x^i} = \frac{\partial A}{\partial x^i}, \quad A_{0l} = A_{x^ky^l}y^k = \frac{\partial^2 A}{\partial x^k y^l}y^k\,.$$

Differentiating 1.1 with respect to x^i, we get

$$[\,\overline{F}\,]_{x^i} = \frac{e^{\sigma(x)}}{m(A^{\frac{1}{m}}-\beta)^2}\Big\{A^{\frac{3}{m}-1}A_{x^i} + m\sigma_{x^i}A^{\frac{3}{m}} - 2\beta A^{\frac{2}{m}-1}A_{x^i} + m\left(\beta_{x^i}-\beta\sigma_{x^i}\right)A^{\frac{2}{m}-1}\Big\}. \tag{2.1}$$

Differentiating 2.1 with respect to y^j and multiplying the result by y^i implies that

$$\begin{aligned}[\,\overline{F}\,]_{x^iy^j}y^i = {}& \frac{e^{\sigma(x)}}{m(A^{\frac{1}{m}}-\beta)^3}\Bigg[\Big(\frac{1}{m}-1\Big)A_0A_jA^{\frac{4}{m}-2} + \left(A_{0j}+\sigma_0A_j\right)A^{\frac{4}{m}-1}\\ &+3\beta\Big(1-\frac{1}{m}\Big)A_0A_jA^{\frac{3}{m}-2} - 3\beta A_0A_jA^{\frac{3}{m}-1} + m\left(\beta_{0j}+b_j\sigma_0\right)A^{\frac{3}{m}}\\ &+2\beta^2\Big(\frac{2}{m}-1\Big)A_0A_jA^{\frac{2}{m}-2} + 2\beta\left(\beta A_{0j}+\beta\sigma_0A_j-b_jA_0-\beta_0A_j\right)A^{\frac{2}{m}-1}\\ &+m\big(2b_j\beta_0-2\beta\sigma_0b_j-\beta\beta_{0j}+\beta b_j\sigma_0\big)A^{\frac{2}{m}}\Bigg].\end{aligned} \tag{2.2}$$

Now, if we consider $\overline{F}$ to be projectively flat Finsler metric, we have $[\,\overline{F}\,]_{x^iy^j}y^i - [\,\overline{F}\,]_{x^j} = 0$, which in view of equation 2.1 and 2.2 gives us

$$\begin{aligned}&(\frac{1}{m}-1)A_0A_jA^{\frac{4}{m}-2} + \left(A_{0j}+\sigma_0A_j-A_{x^j}\right)A^{\frac{4}{m}-1} + 3\beta(1-\frac{1}{m})A_0A_jA^{\frac{3}{m}-2}\\ &-3\beta(A_0A_j+\sigma_0A_{0j}-A_{x^j})A^{\frac{3}{m}-1}+\Big\{m(\beta_{0j}+b_j\sigma_0)-m(\beta_{x^j}-\beta\sigma_{x^j})+m\beta\sigma_{x^j}\Big\}A^{\frac{3}{m}}\\ &+2\beta^2(\frac{2}{m}-1)A_0A_jA^{\frac{2}{m}-2}+\Big\{2\beta(\beta A_{0j}+\beta\sigma_0A_j-b_jA_0-\beta_0A_j)-2\beta^2A_{x^j}\Big\}A^{\frac{2}{m}-1}\\ &+\Big\{m(2b_j\beta_0-2\beta\sigma_0b_j-\beta\beta_{0j}+\beta b_j\sigma_0)+m\beta(\beta_{x^j}-\beta\sigma_{x^j})\Big\}A^{\frac{2}{m}} = 0,\end{aligned}$$

which can be simplified as

$$\begin{aligned}&(\frac{1}{m}-1)A_0A_jA^{\frac{2}{m}-1} + \left(A_{0j}+\sigma_0A_j-A_{x_j}\right)A^{\frac{2}{m}} + 3\beta(1-\frac{1}{m})A_0A_jA^{\frac{1}{m}-1}\\ &-3\beta\left(A_0A_j+\sigma_0A_{0j}-A_{x^j}\right)A^{\frac{1}{m}} + m(\beta_{0j}+b_j\sigma_0-\beta_{x^j}+2\beta\sigma_{x^j})A^{\frac{3}{m}}\\ &+2\beta^2(\frac{2}{m}-1)A_0A_jA^{\frac{2}{m}-2} + 2\beta(\beta A_{0j}+\beta\sigma_0A_j-b_jA_0-\beta_0A_j-\beta A_{x^j})\\ &+m(2b_j\beta_0-\beta\sigma_0b_j-\beta\beta_{0j}+\beta\beta_{x^j}-\beta^2\sigma_{x^j})A = 0.\end{aligned} \tag{2.3}$$

To calculate the above equation, we recall the following Lemma.

Lemma 2.1. ([12]) *Let $F = A^{\frac{1}{m}} (m > 2)$ be an m-th root Finsler metric on an open subset $V \subset R^n$. Suppose that the equation $\psi A^{\frac{2}{m}-1} + \Xi A^{\frac{2}{m}} + \Phi A^{\frac{1}{m}+1} + \Theta A^{\frac{1}{m}} + \Upsilon A^{\frac{1}{m}-1} + \Omega = 0$ holds, where $\psi, \Xi, \Phi, \Theta, \Upsilon, \Omega$ are homogeneous polynomials in y. Then $\psi = \Xi = \Phi = \Theta = \Upsilon = \Omega = 0$.*

Using Lemma 2.1, the equation 2.3 reduces to

$$A_0A_j = 0, \tag{2.4}$$

(2.5) $$A_{0j} + \sigma_0 A_j - A_{x_j} = 0,$$

(2.6) $$\beta_{0j} + b_j\sigma_0 - \beta_{x^j} + 2\beta\sigma_{x^j} = 0,$$

(2.7) $$\beta(A_{0j} + \sigma_0 A_j - A_{x^j}) - b_j A_0 - \beta_0 A_j = 0,$$

(2.8) $$2b_j\beta_0 - \beta\sigma_0 b_j - \beta\beta_{0j} + \beta\beta_{x^j} - \beta^2\sigma_{x^j} = 0.$$

By 2.5 and 2.7, we get

(2.9) $$b_j A_0 + \beta_0 A_j = 0\,.$$

In view of equation 2.4, we have $A_0 = 0$ as $A_j \neq 0$. From equation 2.9 and $A_0 = 0$ it follows that $\beta_0 = 0$. Taking partial derivative of $\beta_0 = 0$, we obtain

(2.10) $$\beta_{0j} + \beta_{x^j} = 0.$$

Substituting equation 2.6 and $\beta_0 = 0$ in equation 2.8, we obtain $\sigma_{x^j} = 0$, *i.e.,* the conformal change is homothetic and then we have $\sigma_0 = 0$. Again from equation 2.6 and using $\sigma_0 = 0$, we have $\beta_{0j} - \beta_{x^j} = 0$. By considering 2.10, we get $\beta_{x^j} = 0$, which implies b_j = constant. This proofs our theorem 1.1.

3. Locally Dually Flat Finsler Metrics

Proof of Theorem 1.2: A Finsler metric $F = F(x, y)$ on an open subset $U \subset \mathbb{R}^n$ is locally dually flat if and only if it satisfies the following equations [4]:

$$[\overline{F}^2]_{x^i y^j} y^i - 2[\overline{F}^2]_{x^j} = 0.$$

From equation 1.1, we have

$$[\overline{F}^2]_{x^i} = 2\frac{e^{\sigma(x)}}{m(A^{\frac{1}{m}} - \beta)^3}\Big\{A^{\frac{5}{m}-1}A_{x^i} + m\sigma_{x^i}A^{\frac{5}{m}} - 2\beta A_{x^i}A^{\frac{4}{m}-1} + m\left(\beta_{x^i} - \beta\sigma_{x^i}\right)A^{\frac{4}{m}-1}\Big\}.$$

(3.1)

Again on differentiating 3.1 with respect to y^j and contracting by y^i, gives us

$$\begin{aligned}[\overline{F}^2]_{x^i y^j} y^i = \frac{e^{\sigma(x)}}{m(A^{\frac{1}{m}} - \beta)^4}\Bigg\{&(\frac{2}{m} - 1)A_0 A_j A^{\frac{6}{m}-2} + (A_{0j} + 2\sigma_0 A_j)\,A^{\frac{6}{m}-1} \\ &+3\beta(1 - \frac{3}{m})A_0 A_j A^{\frac{5}{m}-2} + (\beta_0 A_j - 3\beta A_{0j} - 6\beta\sigma_0 A_j + b_j A_0)A^{\frac{5}{m}-1} \\ &+m(\beta_{0j} + 2\sigma_0 b_j)A^{\frac{5}{m}} + \beta(2\beta A_{oj} - 4\beta A_j - 4A_0 b_j + 4\beta\sigma_0 A_j)A^{\frac{4}{m}-1} \\ &+2\beta^2(\frac{4}{m} - 1)A_0 A_j A^{\frac{4}{m}-2} + m(3\beta_0 b_j - \beta\beta_{0j} - 2\beta\sigma_0 b_j)A^{\frac{4}{m}}\Bigg\}.\end{aligned}$$

If $\overline{F}$ is locally dually flat Finsler metric, then $[\overline{F}^2]_{x^i y^j} y^i - 2[\overline{F}^2]_{x^j} = 0$. From these, we get

$$(\frac{2}{m} - 1)A_0 A_j A^{\frac{6}{m}-2} + (A_{0j} + 2\sigma_0 A_j - A_{x^j})\,A^{\frac{6}{m}-1} + 3\beta(1 - \frac{3}{m})A_0 A_j A^{\frac{5}{m}-2}$$

$$+(\beta_0 A_j - 3\beta A_{0j} - 6\beta\sigma_0 A_j + b_j A_0 + 3\beta A_{x^j})A^{\frac{5}{m}-1} + m(\beta_{0j} + 2\sigma_0 b_j - \beta_{x^i} + \beta\sigma_{x^j})A^{\frac{5}{m}}$$
$$+2\beta^2\big(\frac{4}{m} - 1\big)A_0 A_j A^{\frac{4}{m}-2} + \beta(2\beta A_{oj} - 4\beta_0 A_j - 4A_0 b_j + 4\beta\sigma_0 A_j - 2\beta A_{x^i})A^{\frac{4}{m}-1}$$
$$+\Big\{m(3\beta_0 b_j - \beta\beta_{0j} - 2\beta\sigma_0 b_j) + m\beta(\beta_{x^j} - \beta\sigma_{x^j})\Big\}A^{\frac{4}{m}} = 0\,,$$

which can be simplified as

$$\begin{aligned}
&\big(\frac{2}{m} - 1\big)A_0 A_j A^{\frac{2}{m}} + (A_{0j} + 2\sigma_0 A_j - A_{x^j})\, A^{\frac{2}{m}+1} + 3\beta\big(1 - \frac{3}{m}\big)A_0 A_j A^{\frac{5}{m}-2}\\
&+(\beta_0 A_j - 3\beta A_{0j} - 6\beta\sigma_0 A_j + b_j A_0 + 3\beta A_{x^j})A^{\frac{1}{m}}\\
&+m(\beta_{0j} + 2\sigma_0 b_j - \beta_{x^i} + \beta\sigma_{x^j})A^{\frac{1}{m}+1} + 2\beta^2\big(\frac{4}{m} - 1\big)A_0 A_j\\
&+\beta(2\beta A_{oj} - 4\beta_0 A_j - 4A_0 b_j + 4\beta\sigma_0 A_j - 2\beta A_{x^j})A\\
&+m\,\big(3\beta_0 b_j - \beta\beta_{0j} - 2\beta\sigma_0 b_j + \beta\beta_{x^j} - \beta^2\sigma_{x^j}\big)\, A^2 = 0\,.
\end{aligned} \tag{3.2}$$

To solve the above equation, we use the following lemma.

Lemma 3.1. ([12]) *Let $F = A^{\frac{1}{m}}$ $(m > 2)$ be an m-th root Finsler metric on an open subset $V \subset \mathbb{R}^n$. Suppose that the equation $\psi A^{\frac{2}{m}+1} + \Xi A^{\frac{2}{m}} + \Phi A^{\frac{1}{m}+2} + \Theta A^{\frac{1}{m}+1} + \Upsilon A^{\frac{1}{m}} + \Omega = 0$ holds, where $\psi, \Xi, \Phi, \Theta, \Upsilon, \Omega$ are homogeneous polynomials in y. Then $\psi = \Xi = \Phi = \Theta = \Upsilon = \Omega = 0$.*

Using Lemma 3.1 in equation 3.2, it reduces to

$$A_0 A_j = 0, \tag{3.3}$$
$$A_{0j} + 2\sigma_0 A_j - A_{x_j} = 0, \tag{3.4}$$
$$\beta_{0j} + 2b_j\sigma_0 - \beta_{x^j} + 2\beta\sigma_{x^j} = 0, \tag{3.5}$$
$$\beta_0 A_j - 3\beta A_{0j} - 6\beta\sigma_0 A_j + b_j A_0 + 3\beta A_{x^j} = 0, \tag{3.6}$$
$$2\beta A_{oj} - 4\beta_0 A_j - 4A_0 b_j + 4\beta\sigma_0 A_j - 2\beta A_{x^j} = 0, \tag{3.7}$$
$$3\beta_0 b_j - \beta\beta_{0j} - 2\beta\sigma_0 b_j + \beta\beta_{x^j} - \beta^2\sigma_{x^j} = 0\,. \tag{3.8}$$

Since $A_j \neq 0$, therefore 3.3 gives $A_0 = 0$. Substituting 3.4 and $A_0 = 0$ in 3.6, we get $\beta_0 = 0$, which on partial differentiating gives us

$$\beta_{0j} + \beta_{x^j} = 0. \tag{3.9}$$

Solving 3.8, and using 3.5 and $\beta_0 = 0$, we obtain $\sigma_{x^i} = 0$. This means that the conformal change is homothetic and then we have $\sigma_0 = 0$. By 3.5 and $\sigma_0 = 0$, we get $\beta_{0j} = \beta_{x^j}$. By considering 3.9, we obtain $\beta_{x^j} = 0$, which implies $\beta_j =$ constant. This concludes our proof.

By Theorems 1.1 and 1.2, one can conclude the following.

Corollary 3.1. *The conformal-Matsumoto change of m-th root Finsler metric is locally dually flat if and only if it is projectively flat.*

4. Finsler Metrics of Constant Flag Curvature

In this section, we have find the condition for transformed Finsler metric to be projectively flat with constant flag curvature. The scalar flag curvature $\mathbf{K}$ of projectively flat Finsler metric F is defined as $\mathbf{K} = F^{-2}\left(P^2 - P_{x^i}y^i\right)$, where the projective factor is given by [4]

$$P = \frac{F_{x^i}y^i}{2F}. \tag{4.1}$$

Proof of Theorem 1.3: Multiplying 2.1 with y^i implies that

$$[\overline{F}]_{x^i}y^i = \frac{e^{\sigma(x)}A^{\frac{2}{m}}}{(A^{\frac{1}{m}} - \beta)^2}\left\{\sigma_0 A^{\frac{1}{m}} + \frac{1}{m}A_0 A^{\frac{1}{m}-1} - \frac{2}{mA}\beta A_0 + (\beta_0 - \beta\sigma_0)\right\}.$$

Irreducibility of A and $deg(A_{x^i}) = m-1$ gives us that there exist a one-form $\theta = \theta_i(x)y^i$ such that $A_0 = m\theta A$. Then the above equation can be written as

$$[\overline{F}]_{x^i}y^i = \frac{e^{\sigma(x)}A^{\frac{2}{m}}}{(A^{\frac{1}{m}} - \beta)^2}\left\{(\sigma_0 + \theta)A^{\frac{1}{m}} + (\beta_0 - \beta\sigma_0 - 2\theta\beta)\right\}. \tag{4.2}$$

By considering 4.1, the projective factor of the transformed Finsler metric $\overline{F}$ is given by

$$\overline{P} = \frac{1}{2(A^{\frac{1}{m}} - \beta)}\left\{(\sigma_0 + \theta)A^{\frac{1}{m}} + (\beta_0 - \beta\sigma_0 - 2\theta\beta)\right\}. \tag{4.3}$$

Now, differentiating 4.3 with respect to x^i, we obtain

$$\overline{P}_{x^i} = \frac{1}{2(A^{\frac{1}{m}}-\beta)^2}\Bigg[(A^{\frac{1}{m}} - \beta)\Big\{(\sigma_{0x^i} + \theta_{x^i})A^{\frac{1}{m}} + \tfrac{1}{m}(\sigma_0 + \theta)A_{x^i}A^{\frac{1}{m}-1} + (\beta_{0x^i} - \beta_{x^i}\sigma_0 - \beta\sigma_{0x^i} - 2\theta\beta_{x^i})\Big\} - \Big\{(\sigma_0 + \theta)A^{\frac{1}{m}} + (\beta_0 - \beta\sigma_0 - 2\theta\beta)\Big\}\big(\tfrac{1}{m}A_{x^i}A^{\frac{1}{m}-1} - \beta_{x^i}\big)\Bigg],$$

which by multiplying it with y^i, we get

$$\overline{P}_0 = \frac{1}{2(A^{\frac{1}{m}} - \beta)^2}\Big\{(\sigma_{00} + \theta_0)A^{\frac{2}{m}} + (\beta_{00} + \theta^2\beta - 2\sigma_{00}\beta - 3\beta\theta_0 - 2\theta\beta_0)A^{\frac{1}{m}} + ({\beta_0}^2 - \beta\beta_{00} + \beta^2\sigma_{00} + 2\beta^2\theta_0)\Big\}.$$

Suppose $\overline{F}$ is of constant flag curvature, then $\mathbf{K}\overline{F}^2 + \overline{P}_0 - \overline{P}^2 = 0$. Using equation 4.3 and 4.4, we obtain

$$\mathbf{K}e^{\sigma(x)}\frac{A^{\frac{4}{m}}}{(A^{\frac{1}{m}}-\beta)^2} + \frac{1}{2(A^{\frac{1}{m}}-\beta)^2}\Big\{(\sigma_{00} + \theta_0)A^{\frac{2}{m}} + (\beta_{00} + \theta^2\beta - 2\beta\sigma_{00} - 3\beta\theta_0 - 2\beta_0\theta)A^{\frac{1}{m}} + (\beta_0^2 - \beta\beta_{00} + \beta^2\sigma_{00} + 2\beta^2\theta_0)\Big\} - \frac{1}{4(A^{\frac{1}{m}}-\beta)^2}\Big\{(\sigma_0 + \theta)^2 A^{\frac{2}{m}} + (\beta_0 - \beta\sigma_0 - 2\theta\beta)^2 + 2(\sigma_0 + \theta)(\beta_0 - \beta\sigma_0 - 2\theta\beta)A^{\frac{1}{m}}\Big\} = 0.$$

To calculate the above equation, we recall the following Lemma.

Lemma 4.1. ([12]) *Suppose that the equation* $\phi A^{\frac{4}{m}} + \chi A^{\frac{2}{m}} + \Xi A^{\frac{1}{m}} + \Omega = 0$ *holds, where* ϕ, χ, Ξ *are homogeneous polynomials in* y *and* $m > 4$. *Then* $\phi = \chi = \Xi = \Omega = 0$.

Using Lemma 4.1, we have $\mathbf{K} = 0$ and the following conditions

$$2(\sigma_{00} + \theta_0) - (\sigma_0 + \theta)^2 = 0, \tag{4.4}$$

$$2(\beta_{00} + \theta^2\beta - 2\beta\sigma_{00} - 3\beta\theta_0 - 2\beta_0\theta) - 2(\sigma_0 + \theta)(\beta_0 - \beta\sigma_0 - 2\theta\beta) = 0, \tag{4.5}$$

$$2(\beta_0^2 - \beta\beta_{00} + \beta^2\sigma_{00} + 2\beta^2\theta_0) - (\beta_0 - \beta\sigma_0 - 2\theta\beta)^2 = 0. \tag{4.6}$$

Thus, we conclude the proof of theorem 1.3.
Theorem 1.3 can be reiterated as follows:

Corollary 4.1. *If* $\overline{F} = \overline{F}(x, y)$ *is the conformal-Matsumoto change of* m*-th root* ($m > 4$) *metric* $F = \sqrt[m]{A}$ *such that* A *is irreducible. Then* $\overline{F}$ *can not be projectively flat with non-zero constant flag curvature.*

5. Conclusion

For a Finsler metric, if the geodesic are straight lines then it must be of constant curvature. The Funk metric and Klein metric are projectively flat with negative constant flag curvature $\mathbf{K} = -1/4$ and $\mathbf{K} = -1$, respectively [9]. While, Bryant metric is projectively flat with positive constant flag curvature $\mathbf{K} = 1$ [3]. Although, Bao and Shen [2] have constructed a family of Randers metric on $\mathbb{S}^3$ of constant curvature $\mathbf{K} = 1$, which is not locally projectively flat. In the present paper, we have shown that it doesn't exist a Finsler metric in the form 1.1 such that it is projectively flat with non-zero constant flag curvature.
For future approach, *one can classify the family of Finsler metrics which is projectively flat either with non-zero constant flag curvature or with zero constant flag curvature.*

REFERENCES

1. S. I. Amari and H. Nagaoka: *Methods of Information Geometry*, AMS Translations Mathematical Monographs, Oxford University Press, 2000.
2. D. Bao and Z. Shen. *Finsler metrics of constant positive curvature on the Lie group* $\mathbb{S}^3$, Journal of London Mathematical society, **66(2)** (2002), 453–467.
3. R. Bryant. *Projectively flat Finsler* 2*-spheres of constant curvature*, Selecta Mathematica **3(2)** (1997), 161–203.
4. S. S. Chern and Z. Shen. *Riemannian-Finsler geometry*, Singapore, World Scientic, (2005).
5. G. Hamel. *Über die Geometrien in denen die Geraden die Kürzesten sind.,* Math Ann, **57** (1903), 231–264.

6. M. Kumar and C. K. Mishra. *On conformal Kropina transformation of m-th root metrics,* Journal of the Indian Math. Soc., **88(1-2)** (2021), 97–104.
7. M. Kumar. *Locally dually flatness and locally projectively flatness of Matsumoto change with m-th root Finsler metrics,* Journal of Finsler Geometry and its Applications, **2(1)** (2021), 31–39.
8. Z. Shen and Q. Xia. *A class of Randers metrics of scalar flag curvature*, Int. J. Math., **24(7)** (2013), 1350055.
9. Z. Shen. *Differential geometry of spray and Finsler spaces*, Springer, 2013.
10. H.Shimada. *On Finsler space with the metric $L = \sqrt[m]{a_{i_1 \ldots i_m} y^{i_1} \ldots y^{i_m}}$,* Tensor N.S., **33(3)** (1979), 365–372.
11. A. Tayebi, M. Shahbazi Nia, and E. Peyghan. *On generalized m-th root Finsler metrics,* Linear Algebra and its Applications, **437** (2012), 675–683.
12. A. Tayebi and B. Najafi. *On m-th root Finsler metrics*, Journal of Geometry and Physics, **8(1)** (2015), 14–20.
13. B. Tiwari, M. Kumar and A. Tayebi. *On generalized Kropina change of generalized m-th root Finsler metrics,* Proc. Natl. Acad. Sci., India, Sect. A Phys. Sci., **91(3)** (2021), 443–450.