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ON DOUGLAS TENSOR OF INFINITE SERIES FINSLER SPACE

M. K. Gupta[1], Abha Sahu[1] and Cenap Özel[2]

[1] **Department of Mathematics, Guru Ghasidas Vishwavidyalaya**
Bilaspur (C.G.), India
[2] **Faculty of Science, Department of Mathematics**
King Abdulaziz University, Joddah 21589, Saudi Arabia

ORCID IDs: M. K. Gupta https://orcid.org/0000-0002-1905-2068
Abha Sahu https://orcid.org/0000-0002-4292-7424
Cenap Özel https://orcid.org/0000-0001-9673-8875

Abstract. In this article, we consider the Finsler space F^n $(n > 2)$ with an infinite series (α, β)-metric and establish the necessary and sufficient conditions for it to be of Douglas type. Additionally, we demonstrate the criteria under which this metric in a Finsler space becomes a Berwald space. Furthermore, the space is shown to be projectively flat if it is a Berwald space.
Keywords: Finsler space, infinite series (α, β)-metric, Matsumoto metric, Randers metric, Douglas space, Projectively flat.

1. Introduction

A smooth manifold can encompass various Riemannian and non-Riemannian Finsler metrics. Among these, Finsler metrics are categorized based on their geometric properties, with one notable class being the Douglas metrics. In 1927, J. Douglas introduced the concept of the Douglas tensor, a fundamental projective invariant in Finsler geometry. This tensor is non-Riemannian since the Douglas curvature of a Riemannian metric is inherently zero. Building on the concept of Berwald spaces, Basco and Matsumoto [1] introduced the idea of a Finsler space of Douglas type. A Finsler space is termed a Douglas space if the Douglas tensor

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Corresponding Author: C. Özel
E-mail addresses: mkgiaps@gmail.com (M. K. Gupta), abhasahu118@gmail.com (A. Sahu), cenap.ozel@gmail.com (C. Özel)
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vanishes identically. The Randers metric, for instance, is of Douglas type if and only if the associated one-form β is closed.

By extending the Randers metric, M. Matsumoto [5] introduced the concept of (α, β)-metrics in 1972. Following this, various researchers [6, 7, 8, 9] have explored the theory of Finsler spaces with (α, β)-metrics, significantly influencing the development of Finsler geometry. On an n-dimensional manifold M, a Finsler metric $L(x, y)$ is called an (α, β)-metric if it is a positive homogeneous function of degree one in the Riemannian metric $\alpha(x, y) = a_{ij}y^i y^j$ and the one-form $\beta(x, y) = b_i y^i$. This can be expressed as $F = \alpha\phi(s)$, where $\phi(s)$ is a smooth function of $s = \frac{\beta}{\alpha}$.

Park and Lee [10] studied the Douglas space for the approximate Matsumoto metric up to the second approximation, while Shanker and Yadav [11] extended this study to the third approximation. In both cases, the requirement for the metric to be of Douglas type was that the one-form must be parallel to the Riemannian metric. However, Zhu [12] considered a class of general (α, β)-metrics and demonstrated that β does not necessarily need to be parallel to α for the metric to be of Douglas type. The study and characterization of all (α, β)-metrics of Douglas type remains a natural problem.

Consider the r-series (α, β)-metric:

$$L(\alpha, \beta) = \beta \sum_{r=0}^{r=\infty} \left(\frac{\alpha}{\beta}\right)^r$$

If $r = 1$, then it is a Randers metric. If $r = \infty$, then

$$L = \frac{\beta^2}{\beta - \alpha}.$$

This metric is called an infinite series (α, β)-metric. An interesting fact about this metric is that it can be expressed as the difference between a Randers metric and a Matsumoto metric. In this paper, we have proved following theorems:

Theorem 1.1. *A Finsler space F^n $(n > 2)$ with infinite series (α, β)-metric is Douglas space if and only if β is parallel to α, i.e, $b_{j;i} = 0$.*

L.Berwald defined a Berwald space as an affinely connected Finsler space whose connection coefficients depends solely on positional coordinates. In the second section of the present paper we have proved the following theorem.

Theorem 1.2. *A Finsler space F^n $(n > 2)$ with infinite series (α, β)-metric is Berwald space if and only if β is parallel to α.*

In projective Finsler geometry, we examine projectively equivalent Finsler metrics on a manifold. On an open subset $U \subset \mathbb{R}^n$, a Finsler metric is referred to be projectively flat if all geodesics are straight. In 2007, Benling[2] have studied the projective flatness of the Matsumoto metric. Here, we have demonstrated the condition of projectively flatness for infinite series (α, β)-metric as

Theorem 1.3. *A Finsler space F^n $(n > 2)$ with infinite series (α, β)-metric is projectively flat if and only if α is projectively flat and β is parallel to α.*

2. Preliminaries

The spray coefficient of a Finsler metric L is denoted as G^i and defined by

$$G^i = \frac{g^{il}}{4}\left\{[L^2]_{x^k y^l} y^k - [L^2]_{x^l}\right\},$$

where metric tensor g_{ij} is $\frac{1}{2}[L^2]_{y^i y^j}$ and the spray coefficient of Riemannian metric is denoted by $\overline{G}^i$. Shen and Chern gave the following lemma

Lemma 2.1. *[3] The spray coefficients G^i are related to $\overline{G}^i$ by*

$$G^i = \overline{G}^i + K\left(-2I\alpha s_0 + r_{00}\right)\left(b^i - \frac{y^i}{\alpha}s\right) + J\left(-2Q\alpha s_0 + r_{00}\right)\frac{y^i}{\alpha}, \tag{2.1}$$

where

$$I := \frac{\phi'}{\phi - s\phi'}, \quad J := \frac{\phi'(\phi - s\phi')}{2\phi[(b^2 - s^2)\phi'' + (\phi - s\phi')]}, \quad K := \frac{\phi''}{2[(b^2 - s^2)\phi'' + (\phi - s\phi')]}, \tag{2.2}$$

and $b := ||\beta||_x$, $r_{ij} = \frac{1}{2}\left(b_{i;j} + b_{j;i}\right)$, $s_{ij} = \frac{1}{2}\left(b_{i;j} - b_{j;i}\right)$, $s_{l0} = s_{li}y^i$, $s_i = b_k s_i^k$, $s_0 = s_{l0}b^l$, $s_i b^i = 0$, $r_{00} = r_{ij}y^i y^j$, $b^2 = a^{ij}b_i b_j$.

Matsumoto [6] defined the spray coefficients $G^i(x, y)$ of a Finsler space with (α, β)-metric as

$$2G^i = \gamma^i_{00} + 2D^i,$$

where

$$D^i = \frac{\alpha L_\beta}{L_\alpha} s^i_0 + \overline{C}\left\{\frac{\beta L_\beta}{\alpha L} y^i - \frac{L_{\alpha\alpha}}{\beta L_\alpha}(\beta y^i - \alpha b^i)\right\}, \tag{2.3}$$

Lemma 2.2. *[3] An (α, β)-metric $F = \alpha\phi(s)$ where $s = \frac{\beta}{\alpha}$, is projectively flat on an open subset $U \subset \mathbb{R}^n$ if and only if*

$$(a_{kl}\alpha^2 - y_k y_l)\overline{G}^k + \alpha^3 I\, s_{l0} + K\alpha(-2\alpha\, Q\, s_0 + r_{00})(b_l\alpha - s y_l) = 0, \tag{2.4}$$

and $L_\alpha = \frac{\partial L}{\partial \alpha}$, $L_\beta = \frac{\partial L}{\partial \beta}$, $L_{\alpha\alpha} = \frac{\partial^2 L}{\partial\alpha\partial\alpha}$. The value of $\overline{C}$ is taken as

$$\overline{C} = \frac{\alpha\beta\left(r_{00}L\alpha - 2\alpha s_0 L_\beta\right)}{2\left(\beta^2 L\alpha + \alpha\gamma^2 L_{\alpha\alpha}\right)},$$

where $\gamma^2 = b^2\alpha^2 - \beta^2$, $b^2 = a_{ij}b^i b^j$.
Throughout the paper, *hp(r)* denotes the homogeneous polynomial in y^i of degree r. The Finsler space F^n with (α, β)-metric is a Douglas space, if and only if $D^{ij} = D^i y^j - D^j y^i$ are $hp(3)$ [1]. From equation 2.3, we have

$$D^{ij} = \frac{\alpha L_\beta}{L_\alpha}\left(s^i_0 y^j - s^j_0 y^i\right) + \frac{\alpha^2 L_{\alpha\alpha}}{\beta L_\alpha}\,\overline{C}\left(b^i y^j - b^j y^i\right). \tag{2.5}$$

The following lemma is stated for subsequent use [4]:

Lemma 2.3. *If $\alpha^2 \equiv 0$ (mod β), that is $a_{ij}y^i y^j$ contains $b_i(x)y^i$ as a factor, then the dimension is equal to two and b^2 vanishes. In this case, we have $\delta = d_i(x)y^i$ satisfying $\alpha^2 = \beta\delta$ and $d_i b^i = 2$.*

3. Douglas Space

The purpose of the current section is to determine the conditions under which a Finsler space with the infinite series (α, β)-metric is a Douglas space.

Proof of Theorem 1.1: For infinite series (α, β)-metric

$$L_\alpha = \frac{\beta^2}{(\beta-\alpha)^2},\ L_\beta = \frac{\beta(\beta-2\alpha)}{(\beta-\alpha)^2},\ L_{\alpha\alpha} = \frac{2\beta^2}{(\beta-\alpha)^3} \tag{3.1}$$

Substituting the value from equation 3.1 in equation 2.5, we obtain

$$D^{ij} = \frac{\alpha(\beta-2\alpha)}{\beta}(s_0^i y^j - s_0^j y^i) + \frac{\alpha^3(\beta r_{00} - 2\alpha\beta s_0 + 4\alpha^2 s_0)}{\beta(\beta^3 - 2\alpha\beta^2 + 2b^2\alpha^3)}(b^i y^j - b^j y^i) \tag{3.2}$$

Since α is irrational function in y^i, therefore separating the rational and irrational part, we get

$$\beta^4 D^{ij} = (-2\alpha^2\beta^3 + 2b^2\alpha^4\beta - 2\alpha^2\beta^3)(s_0^i y^j - s_0^j y^i) - 2\alpha^4\beta s_0(b^i y^j - b^j y^i) \tag{3.3}$$

and

$$\begin{aligned}(-2\alpha\beta^3 + 2b^2\alpha^4)D^{ij} &= (\alpha\beta^4 + 4b^2\alpha^5 - 4\alpha^3\beta^2)(s_0^i y^j - s_0^j y^i) \\ &\quad + (\alpha^3\beta r_{00} + 4\alpha^5 s_0)(b^i y^j - b^j y^i)\end{aligned} \tag{3.4}$$

After eliminating D^{ij} from above equations, we have

$$A(s_0^i y^j - s_0^j y^i) + B(b^i y^j - b^j y^i) = 0 \tag{3.5}$$

where

$$\begin{aligned} A &= \left(-8b^2\alpha^5\beta^2 + 4b^4\alpha^7 - \beta^7 + 4\alpha^2\beta^5\right) \\ B &= -(4b^2\alpha^7 s_0 + \alpha^2\beta^4 r_{00}) \end{aligned}$$

Transvecting above equation with $b_i y_j$ gives us

$$A s_0 \alpha^2 + B(b^2\alpha^2 - \beta^2) = 0. \tag{3.6}$$

Separating the rational and irrational terms, we get

$$(b^2\alpha^2 - 2\beta^2)s_0\alpha^2 - \alpha^2 s_0(b^2\alpha^2 - \beta^2) = 0 \tag{3.7}$$

and

$$(4\alpha^2\beta - \beta^3)s_0\alpha^2 - \alpha^2 r_{00}(b^2\alpha^2 - \beta^2) = 0 \tag{3.8}$$

On solving, we get $\beta^2\alpha^2 s_0 = 0$ which implies $s_0 = 0$. From the equation 3.8, we get $r_{00} = 0$. Substituting the value of $s_0 = 0$ and $r_{00} = 0$ in equation 3.5, we get

$$A\left(s_0^i y^j - s_0^j y^i\right) = 0. \tag{3.9}$$

From above equation, we have either $A = 0$ or $\left(s^i_0 y^j - s^j_0 y^i\right) = 0$. Let if possible $A = 0$, then, we have

$$-8b^2\alpha^5\beta^2 + 4b^4\alpha^7 - \beta^7 + 4\alpha^2\beta^5 = 0$$

On rewriting above equation

$$-8b^2\alpha^5\beta^2 + 4b^4\alpha^7 + 4\alpha^2\beta^5 = \beta^7$$

The terms on the left hand side are multiple of α^2, thus their exists w_2 of $hp(2)$ such that

$$\beta^7 = \alpha^2 w_2.$$

Since, α^2 is not contained in β^4, thus $w_2 = 0$ which leads to contradiction. Therefore $A \neq 0$, then equation 3.9 implies $\left(s^i_0 y^j - s^j_0 y^i\right) = 0$, which on transvecting with y_j gives $s^i_0 = 0$. Therefore, we obtain $r_{ij} = s_{ij} = 0$ *i.e.* $b_{i;j} = 0$.
Conversely, Let $b_{i;j} = 0$. then from equation 2.5, $D^{ij} = 0$. Therefore, the Finsler space with infinite series (α, β)-metric is a Douglas space. This concludes our proof.

4. Berwald Space

Matsumoto [6] gave the condition for a Finsler space to be Berwald space using Berwald connection $B\Gamma = (G^i_{jk}, G^i_j, 0)$ of F^n with (α, β)-metric which is given as follows:

(4.1) $$L_\alpha B^k_{ji} y^j y_k + \alpha L_\beta (B^k_{ji} b_k - b_{j;i}) y^j = 0$$

Equation 4.1 can be rewritten as

(4.2) $$L_\alpha B_{jki} y^j y^k + \alpha L_\beta (B_{jki} b^k - b_{j;i}) y^j = 0$$

Proof of Theorem 1.2: Substituting the values from equation 3.1 in above eqution we get

(4.3) $$\frac{\beta^2}{(\beta - \alpha)^2} B_{jki} y^j y^k + \frac{\beta(\beta - 2\alpha)}{(\beta - \alpha)^2} (B_{jki} b^k - b_{j;i}) y^j = 0$$

On simplfying above equation, we have

(4.4) $$\beta^2 B_{jki} y^j y^k + \beta(\beta - 2\alpha)(B_{jki} b^k - b_{j;i}) y^j = 0$$

Separating rational and irrational terms in y^i, we obtain

(4.5) $$\alpha\beta^2 (B_{jki} b^k - b_{j;i}) y^j = 0,$$

and

(4.6) $$\beta^2 B_{jki} y^j y^k - 2\alpha^2\beta (B_{jki} b^k - b_{j;i}) y^j = 0$$

From the equation 4.6, we have $B_{jki} y^j y^k = 0$ which on differentiating twice gives us $B_{jki} + B_{kji} = 0$. Since, B_{jki} is symmetric in first two indices, we obtain $B_{jki} = 0$. In view of equation 4.5, we get $b_{j;i} y^j = 0$ which on differentiating gives $b_{j;i} = 0$. In contrast, Hashiguchi has demonstrated that if $b_{j;i} = 0$ holds, then the Finsler space is Berwald space. This completes the proof of theorem 1.2.

5. Projective Flat

Proof of Theorem 1.3: For infinite series (α,β)-metric the values of I, J, K in equation 2.2 take the forms

$$I=\frac{\beta-2\alpha}{\beta},\ J=\frac{(\beta-\alpha)(\beta^2-2\alpha\beta)}{2(2b^2\alpha^3-3\alpha\beta^2-3\beta^3)},\ K=\frac{\alpha^3}{2(2b^2\alpha^3-3\alpha\beta^2-3\beta^3)} \tag{5.1}$$

Substitutuing above values in equation 2.4, we have

$$\begin{aligned}\beta(2b^2\alpha^3-3\alpha\beta^3)(a_{kl}\alpha^2-y_ky_l)\overline{G}^k+\alpha^3(\beta-2\alpha)(2b^2\alpha^3\\ -3\alpha\beta^2+\beta^3)s_{l0}+\alpha^4(-2\alpha\beta s0+4\alpha^2s_0+\beta r_{00})(b_l\alpha-sy_l)=0\end{aligned} \tag{5.2}$$

Separating rational and irrational terms, we obtain

$$\begin{aligned}\beta(2b^2\alpha^3-3\alpha\beta^3)(a_{kl}\alpha^2-y_ky_l)\overline{G}^k+(\alpha^3\beta^4-4b^2\alpha^7+6\alpha^5\beta^2)s_{l0}\\ +(-2\alpha^5\beta s_0)(b_l\alpha-sy_l)=0\end{aligned} \tag{5.3}$$

and

$$\begin{aligned}\beta^4(a_{kl}\alpha^2-y_ky_l)\overline{G}^k+(2b^2\alpha^6\beta-3\alpha^4\beta^3-2\alpha^4\beta^3)s_{l0}\\ (4\alpha^6s_0+\alpha^4\beta r_{00})(b_l\alpha-sy_l)=0\end{aligned} \tag{5.4}$$

transevecting above equations b^l, we get

$$\begin{aligned}\alpha\beta(2b^2\alpha^2-3\beta^3)(b_k\alpha^2-\beta y_k)\overline{G}^k\\ +(\alpha^3\beta^4-4b^2\alpha^7+6\alpha^5\beta^2)s_0-2\alpha^5\beta s_0(b^2\alpha-s\beta)=0\end{aligned} \tag{5.5}$$

and

$$\begin{aligned}\beta^4(b_k\alpha^2-\beta y_k)\overline{G}^k+(2b^2\alpha^6\beta-3\alpha^4\beta^3-2\alpha^4\beta^3)s_0\\ (4\alpha^6s_0+\alpha^4\beta r_{00})(b^2\alpha-s\beta)=0\end{aligned} \tag{5.6}$$

On solving above equation, we get

$$\begin{aligned}\beta^4s_0(\alpha^2\beta^4-4b^2\alpha^6+6\alpha^4\beta^2-2b^2\alpha^5\beta+2\alpha^3\beta^3)=s_0\beta(2b^2\alpha^2-3\beta^2)\\ (2b^2\alpha^6\beta-5\alpha^4\beta^3+4b^2\alpha^7-4\alpha^5\beta^2)+r_{00}(b^2\alpha^2-\beta^2)\alpha^3\beta^2(2b^2\alpha^2-3\beta^2)\end{aligned} \tag{5.7}$$

and after simplifying, we have

$$\begin{aligned}s_0(\beta^7-8b^4\alpha^7+2\alpha\beta^6-9\alpha^2\beta^5-14\alpha^3\beta^4+12b^2\alpha^4\beta^3\\ +20b^2\alpha^5\beta^2-4b^4\alpha^6\beta)=r_{00}(2b^4\alpha^5\beta-5b^2\alpha^3\beta^3+3\alpha\beta^5)\end{aligned} \tag{5.8}$$

Separating rational and irrational terms, we obtain

$$s_0(\beta^7-9\alpha^2\beta^5+12b^2\alpha^4\beta^3-4b^4\alpha^6\beta)=0 \tag{5.9}$$

$$(5.10) s_0(-8b^4\alpha^7 + 2\alpha\beta^6 - 14\alpha^3\beta^4 + 20b^2\alpha^5\beta^2) = r_{00}(2b^4\alpha^5\beta - 5b^2\alpha^3\beta^3 + 3\alpha\beta^5)$$

From equation 5.9, we have $s_0 = 0$, which on substituting in equation 5.10 gives $r_{00} = 0$. Then from equation 5.6, we get

$$(b_k\alpha^2 - \beta y_k)\overline{G}^k = 0$$

Transvecting above equation by a_{il}, we obtain $\alpha^2\overline{G}^i - y_k y^i \overline{G}^k = 0$. Assume $\eta(x, y) = y_k \frac{\overline{G}^k}{\alpha^2}$, then we have $\overline{G}^i = \eta\, y^i$. Thus, we have α is projectively flat. Hence, from lemma 2.2, we can say that the Finsler space with infinite series (α, β)-metric is projectively flat. This completes the proof of theorem 1.3.

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