FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 31, No 1 (2016), 169–186

RIESZ LACUNARY ALMOST CONVERGENT DOUBLE SEQUENCE SPACES DEFINED BY ORLICZ FUNCTIONS

Şükran Konca and Metin Başarır

Abstract. The aim of this paper is to introduce a new concept for strong almost Pringsheim convergence with respect to an Orlicz function, combining with Riesz mean for double sequences and a double lacunary sequence. In addition, we study almost weighted lacunary statistical convergence for double sequences and present some inclusion theorems.
Keywords: Orlicz function, Pringsheim convergence, double sequences, Riesz convergence.

1. Introduction

A double sequence $x = (x_{k,l})$ is said to be convergent in the Pringsheim's sense (or P-convergent) if for given $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $\left|x_{k,l} - L\right| < \varepsilon$ whenever $k, l > N$ [23]. We shall write this as $\lim_{k,l\to\infty} x_{k,l} = L$, where k and l tending to infinity independent of each other. Let w_2 and c_2 be the spaces of all real or complex double sequences and P-convergent sequences, respectively. Throughout this paper limit of a double sequence means limit in the Pringsheim's sense. A double sequence x is bounded if $\|x\| = sup_{k,l\geq 0}|x_{k,l}| < \infty$. Note that, in contrast to the case for single sequences, a convergent double sequence need not be bounded. By ${c_2}^{\infty}$, we denote the space of double sequences which are bounded convergent and by ${l_2}^{\infty}$, the space of bounded double sequences.

We may refer to [1]-[4], [8], [14]-[22], [24], [26]-[29] for further results related with the concept of double sequence.

Mursaleen and Edely [21] defined the statistical convergence for double sequences $x = (x_{k,l})$ as follows: A real double sequence $x = (x_{k,l})$ is said to be P-statistically convergent to L provided that for each $\varepsilon > 0$

$$P - \lim_{m,n\to\infty} \frac{1}{mn} \left|\left\{j \leqslant m \; and \; k \leqslant n : \left|x_{j,k} - L\right| \geqslant \varepsilon\right\}\right| = 0,$$

Received August 17, 2015; Accepted December 05, 2015
2010 *Mathematics Subject Classification.* Primary 46A40; Secondary 40A35; 40G15; 46A45; 46A35

where the vertical bars denote the numbers of (j,k). Following Freedman et al. [9] and Fridy and Orhan [10], the ideas of lacunary sequence and lacunary statistical convergence were extended to double sequences by Savaş and Patterson in [22], [26]. But the concept of double lacunary density has been recently introduced by Çakan et al. in [8]. The concept of lacunary statistical convergence for double sequences has also been studied in [14]-[18], [20], [27]-[29]. For further results we recommend to the reader to see [11], [16], [17].

Recently, Başarır and Konca [5] have obtained a new lacunary sequence and a new concept of statistical convergence for single sequences which is called weighted lacunary statistical convergence by combining both of the definitions of lacunary sequence and Riesz mean, and have extended this new concept to locally solid Riesz spaces in [6] (see also [7], [12]).

The notion of almost convergence for double sequences had been introduced by Moricz and Rhoades [19], later the notion of strong almost convergence for double sequences was introduced by Başarır [3]. In [28] Savaş and Patterson gave the definition of lacunary sequence for double sequences and introduced a new concept for almost lacunary strong P-convergence. Recently, Alotaibi and Çakan [1] have introduced the Riesz convergence of double sequences (see also [15]).

In this paper, we introduce a new concept for strong almost Pringsheim convergence with respect to an Orlicz function, combining with Riesz mean and a lacunary sequence for double sequences. Further, we study almost weighted lacunary statistical convergence for double sequences and present some inclusion theorems.

2. Definitions and Preliminaries

Before the beginning of the presentation of the main results, we give some definitions and preliminaries.

Let $A = \left(a_{jk}^{mn}\right), \quad j,k = 0,1,...$ be a doubly infinite matrix of real numbers for all $m,n = 0,1,....$ Forming the sums

$$y_{mn} = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{jk}^{mn} x_{jk}$$

called the A-means of the sequence x, yields a method of summability. More exactly, we say that a sequence x is A-summable to the limit L if the A-means exist for all $m,n = 0,1,...$ in the sense of Pringsheim's convergence:

$$\lim_{p,q\to\infty} \sum_{j=0}^{p}\sum_{k=0}^{q} a_{jk}^{mn} x_{jk} = y_{mn}$$

and

$$\lim_{m,n\to\infty} y_{mn} = L.$$

We say that a matrix A is bounded-regular or RH-regular if every bounded and convergent sequence x is A-summable to the same limit and the A-means are also bounded. Necessary and sufficient conditions for A to be bounded-regular are

1. $\lim\limits_{m,n\to\infty} a_{jk}^{mn} = 0 \quad (j,k = 0,1,...)$

2. $\lim\limits_{m,n\to\infty} \sum\limits_{j=0}^{\infty}\sum\limits_{k=0}^{\infty} a_{jk}^{mn} = 1$

3. $\lim\limits_{m,n\to\infty} \sum\limits_{j=0}^{\infty} \left|a_{jk}^{mn}\right| = 0 \quad (k = 0,1,...)$

4. $\lim\limits_{m,n\to\infty} \sum\limits_{k=0}^{\infty} \left|a_{jk}^{mn}\right| = 0 \quad (j = 0,1,...)$

5. $\sum\limits_{j=0}^{\infty}\sum\limits_{k=0}^{\infty} \left|a_{jk}^{mn}\right| \leqslant C < \infty \quad (m,n = 0,1,...).$

These conditions were first established by Robison [24]. Actually (1) is a consequence of each of (3) and (4). We say that a matrix A is strongly regular if every almost convergent sequence x is A-summable to the same limit, and the A-means are also bounded.

Let $n,m \geqslant 1$. A double sequence $x = (x_{k,l})$ of real numbers is called almost P-convergent to a limit L if

$$\mathrm{P}-\lim_{n,m\to\infty}\sup_{\mu,\eta\geqslant 0}\left|\frac{1}{nm}\sum_{k=\mu}^{\mu+n-1}\sum_{l=\eta}^{\eta+m-1} x_{k,l} - L\right| = 0,$$

that is; the average value of $(x_{k,l})$ taken over any rectangle

$$\{(k,l) : \mu \leqslant k \leqslant \mu+n-1, \quad \eta \leqslant l \leqslant \eta+m-1\}$$

tends to L as both n and m tend to ∞, and this convergence is uniform in μ and η. A double sequence x is called strongly almost P-convergent to a number L if

$$\mathrm{P}-\lim_{n,m\to\infty}\sup_{\mu,\eta\geqslant 0}\frac{1}{nm}\sum_{k=\mu}^{\mu+n-1}\sum_{l=\eta}^{\eta+m-1}\left|x_{k,l} - L\right| = 0.$$

Let denote the set of sequences with this property as $[\hat{c}^2]$. By $\hat{c}^2$, we denote the space of all almost convergent double sequences. It is easy to see that the inclusions $c_2^\infty \subset [\hat{c}^2] \subset \hat{c}^2 \subset l_2^\infty$ strictly hold. As in the case of single sequences, every almost convergent double sequence is bounded. But a convergent double sequence need not be bounded. Thus a convergent double sequence need not be almost convergent. However, every bounded convergent double sequence is almost convergent.

We will use the following definition which may be called convergence in Pringsheim's sense as follows:

$$(x_{k,l} - \lambda) = o(1), \quad (k, l \to \infty).$$

Definition 2.1. [1] Let (p_n), $(\bar{p}_m)$ be sequences of positive numbers and $P_n = p_1 + p_2 + ... + p_n$, $\bar{P}_m = \bar{p}_1 + \bar{p}_2 + ... + \bar{p}_m$. Then the transformation given by

$$T_{n,m}(x) = \frac{1}{P_n \bar{P}_m} \sum_{k=1}^{n} \sum_{l=1}^{m} p_k \bar{p}_l x_{k,l}$$

is called the Riesz mean of double sequence $x = (x_{k,l})$. If P - $\lim_{\mathrm{n,m}} T_{n,m}(x) = L$, $L \in \mathbb{R}$, then the sequence $x = (x_{k,l})$ is said to be Riesz convergent to L. If $x = (x_{k,l})$ is Riesz convergent to L, then we write $\mathrm{P_R}$ - $\lim x = L$.

The double sequence $\theta_{r,s} = \{(k_r, l_s)\}$ is called double lacunary if there exist two increasing sequences of integers such that $k_0 = 0$, $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$ and $l_0 = 0$, $\bar{h}_s = l_s - l_{s-1} \to \infty$ as $s \to \infty$. Let $k_{r,s} = k_r l_s$, $h_{r,s} = h_r \bar{h}_s$ and $\theta_{r,s}$ is determined by $I_{r,s} = \{(k,l) : k_{r-1} < k \leqslant k_r$ and $\mathrm{l}_{s-1} < l \leqslant l_s\}$, $q_r = \frac{k_r}{k_{r-1}}$, $\bar{q}_s = \frac{l_s}{l_{s-1}}$ and $q_{r,s} = q_r \bar{q}_s$ [28].

Using the notations of lacunary sequence and Riesz mean for double sequences, we now present some new notations which will be used in the next section:

Let $\theta_{r,s} = \{(k_r, l_s)\}$ be a double lacunary sequence and (p_k), $(\bar{p}_l)$ be sequences of positive real numbers such that $P_{k_r} := \sum_{k \in (0, k_r]} p_k$, $\bar{P}_{l_s} := \sum_{l \in (0, l_s]} \bar{p}_l$ and $H_r := \sum_{k \in (k_{r-1}, k_r]} p_k$, $\bar{H}_s := \sum_{l \in (l_{s-1}, l_s]} \bar{p}_l$. Clearly, $H_r := P_{k_r} - P_{k_{r-1}}$, $\bar{H}_s := \bar{P}_{l_s} - \bar{P}_{l_{s-1}}$. If the Riesz transformation of double sequences is RH-regular, and $H_r := P_{k_r} - P_{k_{r-1}} \to \infty$ as $r \to \infty$, $\bar{H}_s := \bar{P}_{l_s} - \bar{P}_{l_{s-1}} \to \infty$ as $s \to \infty$, then $\theta'_{r,s} = \{(P_{k_r}, \bar{P}_{l_s})\}$ is a double lacunary sequence. Our obligation to add such provisions is the assumptions "$P_n \to \infty$ as $n \to \infty$" and "$\bar{P}_m \to \infty$ as $m \to \infty$" may be not enough to obtain the conditions "$H_r \to \infty$ as $r \to \infty$" and "$\bar{H}_s \to \infty$ as $s \to \infty$", respectively. To show these clearly; for any lacunary sequences (k_r) and (l_s) of integers, one can find sequences of positive real numbers (p_k) and $(\bar{p}_l)$ such that $P_n = p_1 + ... + p_n \to \infty$ $(n \to \infty)$ and $\bar{P}_m = \bar{p}_1 + ... + \bar{p}_m \to \infty$ $(m \to \infty)$, yet $H_r = P_{k_r} - P_{k_{r-1}}$ and $\bar{H}_s = \bar{P}_{l_s} - \bar{P}_{l_{s-1}}$ are bounded and strictly positive. For example, let

$$p_k = \begin{cases} 1, & \text{if } k = k_r \text{ for some } r \in \mathbb{N} \\ \frac{2}{3^k}, & \text{for all other } k \in \mathbb{N} \end{cases}$$

and

$$\bar{p}_l = \begin{cases} 1, & \text{if } l = l_s \text{ for some } s \in \mathbb{N} \\ \frac{1}{2^l}, & \text{for all other } l \in \mathbb{N}. \end{cases}$$

Then $P_n \to \infty$ $(n \to \infty)$ and $r < P_{k_r} < r+1$ as $P_{k_r} > p_{k_1} + p_{k_2} + ... + p_{k_r} = r$, $P_{k_r} < p_{k_1} + p_{k_2} + ... + p_{k_r} + \sum_{k=1}^{\infty} \frac{2}{3^k} = r+1$, and $\bar{P}_m \to \infty$ $(m \to \infty)$ and $s < \bar{P}_{l_s} < s+1$ as $\bar{P}_{l_s} > \bar{p}_{l_1} + \bar{p}_{l_2} + ... + \bar{p}_{l_s} = s$, $\bar{P}_{l_s} < \bar{p}_{l_1} + \bar{p}_{l_2} + ... + \bar{p}_{l_s} + \sum_{k=1}^{\infty} \frac{1}{2^l} = s+1$. Then we obtain $H_r = P_{k_r} - P_{k_{r-1}} < r+1-(r-1) < 2$ and $\bar{H}_s = \bar{P}_{l_s} - \bar{P}_{l_{s-1}} < s+1-(s-1) < 2$ are bounded from above and cannot diverge to infinity.

Throughout the paper, we assume that $P_n = p_1 + ... + p_n \to \infty$ $(n \to \infty)$, $\bar{P}_m = \bar{p}_1 + ... + \bar{p}_m \to \infty$ $(m \to \infty)$, such that $H_r = P_{k_r} - P_{k_{r-1}} \to \infty$ as $r \to \infty$ and $\bar{H}_s = \bar{P}_{l_s} - \bar{P}_{l_{s-1}} \to \infty$ as $s \to \infty$.

Let $P_{k_{r,s}} = P_{k_r}\bar{P}_{l_s}$, $H_{r,s} = H_r\bar{H}_s$, $I'_{r,s} = \{(k,l) : P_{k_{r-1}} < k \leqslant P_{k_r}$ and $\bar{P}_{l_{s-1}} < l \leqslant \bar{P}_{l_s}\}$, $Q_r = \frac{P_{k_r}}{P_{k_{r-1}}}$, $\bar{Q}_s = \frac{\bar{P}_{l_s}}{\bar{P}_{l_{s-1}}}$ and $Q_{r,s} = Q_r\bar{Q}_s$.

If we take $p_k = 1$, $\bar{p}_l = 1$ for all k and l, then $H_{r,s}$, $P_{k_{r,s}}$, $Q_{r,s}$ and $I'_{r,s}$ reduce to $h_{r,s}$, $k_{r,s}$, $q_{r,s}$ and $I_{r,s}$.

Recall in [13] that an Orlicz function M is continuous, convex, nondecreasing function define for $x > 0$ such that $M(0) = 0$ and $M(x) > 0$ for $x > 0$. If convexity of Orlicz function is replaced by $M(x+y) \leqslant M(x) + M(y)$ then this function is called the modulus function which is defined and characterized by Ruckle [25].

Without loss of generality, we will use the limit notation in Pringsheim's sense $\lim_{r,s}$ instead of $\lim_{r,s\to\infty}$, for brevity.

3. Main Results

Let M be an Orlicz function and $t = (t_{k,l})$ be any factorable double sequence of strictly positive real numbers, we define the following sequence spaces:

$$\left[\tilde{R}^2, \theta_{r,s}, p, M, t\right] = \left\{ \begin{array}{l} x = (x_{k,l}) : \mathrm{P}-\lim_{r,s\to\infty} \frac{1}{H_{r,s}} \sum_{(k,l)\in I_{r,s}} p_k\bar{p}_l \left[M\left(\frac{|x_{k+\mu,l+\eta} - L|}{\rho}\right)\right]^{t_{k,l}} = 0 \\ \text{uniformly in } \mu \text{ and } \eta, \ \text{ for some } \rho > 0 \end{array} \right\},$$

$$\left[\tilde{R}^2, \theta_{r,s}, p, M, t\right]_0 = \left\{ \begin{array}{l} x = (x_{k,l}) : \mathrm{P}-\lim_{r,s\to\infty} \frac{1}{H_{r,s}} \sum_{(k,l)\in I_{r,s}} p_k\bar{p}_l \left[M\left(\frac{|x_{k+\mu,l+\eta}|}{\rho}\right)\right]^{t_{k,l}} = 0 \\ \text{uniformly in } \mu \text{ and } \eta, \ \text{for some } \rho > 0 \end{array} \right\}.$$

Clearly, the proper inclusion $\left[\tilde{R}^2, \theta_{r,s}, p, M, t\right]_0 \subset \left[\tilde{R}^2, \theta_{r,s}, p, M, t\right]$ holds. We shall denote $\left[\tilde{R}^2, \theta_{r,s}, p, M, t\right]$ and $\left[\tilde{R}^2, \theta_{r,s}, p, M, t\right]_0$, as $\left[\tilde{R}^2, \theta_{r,s}, p, M\right]$ and $\left[\tilde{R}^2, \theta_{r,s}, p, M\right]_0$, respectively when $t_{k,l} = 1$ for all k and l. If a double sequence $x = (x_{k,l})$ is in $\left[\tilde{R}^2, \theta_{r,s}, p, M\right]$, we shall say that the double sequence $x = (x_{k,l})$ is Riesz lacunary strongly almost convergent with respect to the Orlicz function M. Note that $\left[\tilde{R}^2, \theta_{r,s}, p, M, t\right] = \left[\tilde{R}^2, \theta_{r,s}, p\right]$ and $\left[\tilde{R}^2, \theta_{r,s}, p, M, t\right]_0 = \left[\tilde{R}^2, \theta_{r,s}, p\right]_0$ when $\mathrm{M}(x) = x$ and $t_{k,l} = 1$ for all k and l.

If we choose $p_k = 1, \bar{p}_l = 1$ for all k and l, then we obtain the following sequence spaces which can be seen in [28].

$$\left[AC_{\theta_{r,s}}, M, t\right] = \left\{ \begin{array}{l} x = (x_{k,l}) : \mathrm{P} - \lim\limits_{r,s\to\infty} \frac{1}{h_{r,s}} \sum\limits_{(k,l)\in I_{r,s}} \left[M\left(\frac{|x_{k+\mu,l+\eta} - L|}{\rho}\right)\right]^{t_{k,l}} = 0, \\ \text{uniformly in } \mu \text{ and } \eta, \text{ for some } \rho > 0 \end{array} \right\},$$

$$\left[AC_{\theta_{r,s}}, M, t\right]_0 = \left\{ \begin{array}{l} x = (x_{k,l}) : \mathrm{P} - \lim\limits_{r,s\to\infty} \frac{1}{h_{r,s}} \sum\limits_{(k,l)\in I_{r,s}} \left[M\left(\frac{|x_{k+\mu,l+\eta}|}{\rho}\right)\right]^{t_{k,l}} = 0, \\ \text{uniformly in } \mu \text{ and } \eta, \text{ for some } \rho > 0 \end{array} \right\}.$$

Let M be an Orlicz function, $t = (t_{k,l})$ be any factorable double sequence of strictly positive real numbers and (p_n), $(\bar{p}_m)$ be sequences of positive numbers and $P_n = p_1 + p_2 + ... + p_n$, $\bar{P}_m = \bar{p}_1 + \bar{p}_2 + ... + \bar{p}_m$. We define the following sequence spaces:

$$\left[\tilde{R}^2, p, M, t\right] = \left\{ \begin{array}{l} x = (x_{k,l}) : \mathrm{P} - \lim\limits_{n,m\to\infty} \frac{1}{P_n \bar{P}_m} \sum\limits_{k=1}^{n} \sum\limits_{l=1}^{m} p_k \bar{p}_l \left[M\left(\frac{|x_{k+\mu,l+\eta} - L|}{\rho}\right)\right]^{t_{k,l}} = 0 \\ \text{uniformly in } \mu \text{ and } \eta, \text{ for some } \rho > 0 \end{array} \right\},$$

$$\left[\tilde{R}^2, p, M, t\right]_0 = \left\{ \begin{array}{l} x = (x_{k,l}) : \mathrm{P} - \lim\limits_{n,m\to\infty} \frac{1}{P_n \bar{P}_m} \sum\limits_{k=1}^{n} \sum\limits_{l=1}^{m} p_k \bar{p}_l \left[M\left(\frac{|x_{k+\mu,l+\eta}|}{\rho}\right)\right]^{t_{k,l}} = 0 \\ \text{uniformly in } \mu \text{ and } \eta, \text{ for some } \rho > 0 \end{array} \right\}.$$

We will investigate the inclusion relations between these sequence spaces given above, later. We have the following theorem whose proof is left to the reader.

Theorem 3.1. *For any Orlicz function M and a bounded factorable positive double number sequence* $t_{k,l}$, $\left[\tilde{R}^2, \theta_{r,s}, p, M, t\right]$ *and* $\left[\tilde{R}^2, \theta_{r,s}, p, M, t\right]_0$ *are linear spaces.*

Theorem 3.2. *Let* $\theta_{r,s} = \{(k_r, l_s)\}$ *be a double lacunary sequence and* p_k, $\bar{p}_l$ *be sequences of positive numbers. If* $\liminf\limits_r Q_r > 1$ *and* $\liminf\limits_s \bar{Q}_s > 1$, *then for any Orlicz function M,* $\left[\tilde{R}^2, p, M, t\right] \subseteq \left[\tilde{R}^2, \theta_{r,s}, p, M, t\right]$.

Proof. Assume that $\liminf\limits_r Q_r > 1$ and $\liminf\limits_s \bar{Q}_s > 1$, then there exists $\delta > 0$ such that $Q_r > 1 + \delta$ and $\bar{Q}_s > 1 + \delta$. This implies $\frac{H_r}{P_{k_r}} \geqslant \frac{\delta}{1+\delta}$ and $\frac{\bar{H}_s}{\bar{P}_{l_s}} \geqslant \frac{\delta}{1+\delta}$. Then for

$x \in \left[\tilde{R}^2, p, M, t\right]$, we can write for each μ and η

$$\begin{aligned}
A_{r,s} &= \frac{1}{H_{r,s}} \sum_{(k,l)\in I_{r,s}} p_k \bar{p}_l \left[M\left(\frac{\left|x_{k+\mu,l+\eta}-L\right|}{\rho}\right)\right]^{t_{k,l}} \\
&= \frac{1}{H_{r,s}} \sum_{k=1}^{k_r} \sum_{l=1}^{l_s} p_k \bar{p}_l \left[M\left(\frac{\left|x_{k+\mu,l+\eta}-L\right|}{\rho}\right)\right]^{t_{k,l}} \\
&- \frac{1}{H_{r,s}} \sum_{k=1}^{k_{r-1}} \sum_{l=1}^{l_{s-1}} p_k \bar{p}_l \left[M\left(\frac{\left|x_{k+\mu,l+\eta}-L\right|}{\rho}\right)\right]^{t_{k,l}} \\
&- \frac{1}{H_{r,s}} \sum_{k=k_{r-1}+1}^{k_r} \sum_{l=1}^{l_{s-1}} p_k \bar{p}_l \left[M\left(\frac{\left|x_{k+\mu,l+\eta}-L\right|}{\rho}\right)\right]^{t_{k,l}} \\
&- \frac{1}{H_{r,s}} \sum_{l=l_{s-1}+1}^{l_s} \sum_{k=1}^{k_{r-1}} p_k \bar{p}_l \left[M\left(\frac{\left|x_{k+\mu,l+\eta}-L\right|}{\rho}\right)\right]^{t_{k,l}} \\
&= \frac{P_{k_r}\bar{P}_{l_s}}{H_{r,s}} \left(\frac{1}{P_{k_r}\bar{P}_{l_s}} \sum_{k=1}^{k_r} \sum_{l=1}^{l_s} p_k \bar{p}_l \left[M\left(\frac{\left|x_{k+\mu,l+\eta}-L\right|}{\rho}\right)\right]^{t_{k,l}}\right) \\
&- \frac{P_{k_{r-1}}\bar{P}_{l_{s-1}}}{H_{r,s}} \left(\frac{1}{P_{k_{r-1}}\bar{P}_{l_{s-1}}} \sum_{k=1}^{k_{r-1}} \sum_{l=1}^{l_{s-1}} p_k \bar{p}_l \left[M\left(\frac{\left|x_{k+\mu,l+\eta}-L\right|}{\rho}\right)\right]^{t_{k,l}}\right) \\
&- \frac{1}{H_r} \sum_{k=k_{r-1}+1}^{k_r} \frac{\bar{P}_{l_{s-1}}}{\bar{H}_s} \frac{1}{\bar{P}_{l_{s-1}}} \sum_{l=1}^{l_{s-1}} p_k \bar{p}_l \left[M\left(\frac{\left|x_{k+\mu,l+\eta}-L\right|}{\rho}\right)\right]^{t_{k,l}} \\
&- \frac{1}{\bar{H}_s} \sum_{l=l_{s-1}+1}^{l_s} \frac{P_{k_{r-1}}}{H_r} \frac{1}{P_{k_{r-1}}} \sum_{k=1}^{k_{r-1}} p_k \bar{p}_l \left[M\left(\frac{\left|x_{k+\mu,l+\eta}-L\right|}{\rho}\right)\right]^{t_{k,l}}.
\end{aligned}$$

Since $x \in \left[\tilde{R}^2, p, M, t\right]$ the last two terms tend to zero uniformly in μ, η in the Pringsheim sense, thus for each μ and η

$$A_{r,s} = \frac{P_{k_r}\bar{P}_{l_s}}{H_{r,s}} \left(\frac{1}{P_{k_r}\bar{P}_{l_s}} \sum_{k=1}^{k_r} \sum_{l=1}^{l_s} p_k \bar{p}_l \left[M\left(\frac{\left|x_{k+\mu,l+\eta}-L\right|}{\rho}\right)\right]^{t_{k,l}}\right)$$

$$-\frac{P_{k_{r-1}}\bar{P}_{l_{s-1}}}{H_{r,s}} \left(\frac{1}{P_{k_{r-1}}\bar{P}_{l_{s-1}}} \sum_{k=1}^{k_{r-1}} \sum_{l=1}^{l_{s-1}} p_k \bar{p}_l \left[M\left(\frac{\left|x_{k+\mu,l+\eta}-L\right|}{\rho}\right)\right]^{t_{k,l}}\right) + o(1).$$

Since $H_{r,s} = P_{k_r}P_{l_s} - P_{k_{r-1}}P_{l_{s-1}}$, for each μ and η we have the following:

$$\frac{P_{k_r}P_{l_s}}{H_{r,s}} \leqslant \frac{1+\delta}{\delta} \text{ and } \frac{P_{k_{r-1}}P_{l_{s-1}}}{H_{r,s}} \leqslant \frac{1}{\delta}.$$

The terms

$$\frac{1}{P_{k_r}\bar{P}_{l_s}} \sum_{k=1}^{k_r} \sum_{l=1}^{l_s} p_k \bar{p}_l \left[M\left(\frac{\left|x_{k+\mu,l+\eta}-L\right|}{\rho}\right)\right]^{t_{k,l}}$$

and

$$\frac{1}{P_{k_{r-1}}\bar{P}_{l_{s-1}}} \sum_{k=1}^{k_{r-1}} \sum_{l=1}^{l_{s-1}} p_k \bar{p}_l \left[M\left(\frac{\left|x_{k+\mu,l+\eta}-L\right|}{\rho}\right)\right]^{t_{k,l}}$$

are both Pringsheim null sequences for all μ and η. Thus $A_{r,s}$ is a Pringsheim null sequence for each μ and η. Therefore x is in $\left[\tilde{R}^2, \theta_{r,s}, p, M, t\right]$. This completes the proof. □

Theorem 3.3. *Let $\theta_{r,s} = \{(k_r, l_s)\}$ be a double lacunary sequence and p_k, $\bar{p}_l$ be sequences of positive numbers. If $\limsup_r Q_r < \infty$ and $\limsup_s \bar{Q}_s < \infty$, then for any Orlicz function M, $\left[\tilde{R}^2, \theta_{r,s}, p, M, t\right] \subseteq \left[\tilde{R}^2, p, M, t\right]$.*

Proof. Since $\limsup_r Q_r < \infty$ and $\limsup_s \bar{Q}_s < \infty$, there exists $H > 0$ such that $Q_r < H$ and $\bar{Q}_s < H$ for all r and s. Let $x \in \left[\tilde{R}^2, \theta_{r,s}, p, M, t\right]$ and $\varepsilon > 0$. Then there exist $r_0 > 0$ and $s_0 > 0$ such that for every $i \geqslant r_0$ and $j \geqslant s_0$ and for all μ and η,

$$A'_{i,j} = \frac{1}{H_{i,j}} \sum_{(k,l)\in I_{i,j}} p_k \bar{p}_l \left[M\left(\frac{\left| x_{k+\mu,l+\eta} \right|}{\rho} \right) \right]^{t_{k,l}} < \varepsilon.$$

Let $M' = \max\left\{A'_{i,j} : 1 \leqslant i \leqslant r_0 \text{ and } 1 \leqslant j \leqslant s_0\right\}$, and n and m be such that $k_{r-1} < n \leqslant k_r$ and $l_{s-1} < m \leqslant l_s$. Thus we obtain the following:

$$\begin{aligned}
&\frac{1}{P_n \bar{P}_m} \sum_{k=1}^{n} \sum_{l=1}^{m} p_k \bar{p}_l \left[M\left(\frac{\left| x_{k+\mu,l+\eta} - L \right|}{\rho} \right) \right]^{t_{k,l}} \\
&\leqslant \frac{1}{P_{k_{r-1}} \bar{P}_{l_{s-1}}} \sum_{k=1}^{k_r} \sum_{l=1}^{l_s} p_k \bar{p}_l \left[M\left(\frac{\left| x_{k+\mu,l+\eta} - L \right|}{\rho} \right) \right]^{t_{k,l}} \\
&= \frac{1}{P_{k_{r-1}} \bar{P}_{l_{s-1}}} \sum_{t,u=1,1}^{r,s} \left(\sum_{(k,l)\in I_{t,u}} p_k \bar{p}_l \left[M\left(\frac{\left| x_{k+\mu,l+\eta} - L \right|}{\rho} \right) \right]^{t_{k,l}} \right) \\
&= \frac{1}{P_{k_{r-1}} \bar{P}_{l_{s-1}}} \sum_{t,u=1,1}^{r_0,s_0} H_{t,u} A'_{t,u} + \frac{1}{P_{k_{r-1}} \bar{P}_{l_{s-1}}} \sum_{(r_0<t\leqslant r)\cup(s_0<u\leqslant s)} H_{t,u} A'_{t,u} \\
&\leqslant \frac{M' P_{k_{r_0}} \bar{P}_{l_{s_0}}}{P_{k_{r-1}} \bar{P}_{l_{s-1}}} + \left(\sup_{t\geqslant r_0 \cup u \geqslant s_0} A'_{t,u} \right) \frac{1}{P_{k_{r-1}} \bar{P}_{l_{s-1}}} \sum_{(r_0<t\leqslant r)\cup(s_0<u\leqslant s)} H_{t,u} \\
&\leqslant \frac{M' P_{k_{r_0}} \bar{P}_{l_{s_0}}}{P_{k_{r-1}} \bar{P}_{l_{s-1}}} + \frac{\varepsilon}{P_{k_{r-1}} \bar{P}_{l_{s-1}}} \sum_{(r_0<t\leqslant r)\cup(s_0<u\leqslant s)} H_{t,u} \\
&\leq \frac{M' P_{k_{r_0}} \bar{P}_{l_{s_0}}}{P_{k_{r-1}} \bar{P}_{l_{s-1}}} + \frac{P_{k_r}}{P_{k_{r-1}}} \frac{\bar{P}_{l_s}}{\bar{P}_{l_{s-1}}} \varepsilon = \frac{M' P_{k_{r_0}} \bar{P}_{l_{s_0}}}{P_{k_{r-1}} \bar{P}_{l_{s-1}}} + Q_r \bar{Q}_s \varepsilon \\
&\leq \frac{M' P_{k_{r_0}} \bar{P}_{l_{s_0}}}{P_{k_{r-1}} \bar{P}_{l_{s-1}}} + \varepsilon H^2.
\end{aligned}$$

Since $P_{k_{r-1}} \to \infty$ and $\bar{P}_{l_{s-1}} \to \infty$ as $r, s \to \infty$, it follows that

$$\frac{1}{P_n \bar{P}_m} \sum_{k=1}^{n} \sum_{l=1}^{m} p_k \bar{p}_l \left[M\left(\frac{\left| x_{k+\mu,l+\eta} - L \right|}{\rho} \right) \right]^{t_{k,l}} \to 0,$$

uniformly in μ and η. Therefore $x \in \left[\tilde{R}^2, p, M, t\right]$. □

We observe the following corollary as an outcome of the Theorem 3.2 and Theorem 3.3.

Corollary 3.1. *Let $\theta_{r,s} = \{(k_r, l_s)\}$ be a double lacunary sequence and p_k, $\bar{p}_l$ be sequences of positive numbers. If $1 < \liminf_{r,s} Q_{r,s} \leqslant \limsup_{r,s} Q_{r,s} < \infty$, then for any Orlicz function M, $\left[\tilde{R}^2, \theta_{r,s}, p, M, t\right] = \left[\tilde{R}^2, p, M, t\right]$.*

Theorem 3.4. *The following statements are true:*

1. *If $p_k < 1$ for all $k \in \mathbb{N}$ and $\bar{p}_l < 1$ for all $l \in \mathbb{N}$, then $\left[AC_{\theta_{r,s}}, M, t\right] \subset \left[\tilde{R}^2, \theta_{r,s}, p, M, t\right]$ with $\left[AC_{\theta_{r,s}}, M, t\right]$-$P$-$\lim x = \left[\tilde{R}^2, \theta_{r,s}, p, M, t\right]$-$P$-$\lim x = L$.*

2. *If $p_k > 1$ for all $k \in \mathbb{N}$ and $\bar{p}_l > 1$ for all $l \in \mathbb{N}$, and $\frac{H_r}{h_r}$ and $\frac{\bar{H}_s}{\bar{h}_s}$ are upper bounded, then $\left[\tilde{R}^2, \theta_{r,s}, p, M, t\right] \subset \left[AC_{\theta_{r,s}}, M, t\right]$ with $\left[\tilde{R}^2, \theta_{r,s}, p, M, t\right]$-$P$-$\lim x = \left[AC_{\theta_{r,s}}, M, t\right]$-$P$-$\lim x = L$.*

Proof. 1. If $p_k < 1$ for all $k \in \mathbb{N}$ and $\bar{p}_l < 1$ for all $l \in \mathbb{N}$, then $H_r < h_r$ for all $r \in \mathbb{N}$ and $\bar{H}_s < \bar{h}_s$ for all $s \in \mathbb{N}$, respectively. So, there exist M_1 and M_2 constants such that $0 < M_1 \leqslant \frac{H_r}{h_r} < 1$ for all $r \in \mathbb{N}$ and $0 < M_2 \leqslant \frac{\bar{H}_s}{\bar{h}_s} < 1$ for all $s \in \mathbb{N}$. Let $x = (x_{k,l})$ be a double sequence which converges to the P-limit L in $\left[AC_{\theta_{r,s}}, M, t\right]$, then for each μ and η

$$
\begin{aligned}
&\frac{1}{H_{r,s}} \sum_{(k,l)\in I_{r,s}} p_k \bar{p}_l \left[M\left(\frac{\left|x_{k+\mu,l+\eta} - L\right|}{\rho}\right)\right]^{t_{k,l}} \\
&= \frac{1}{H_r \bar{H}_s} \sum_{(k,l)\in I_{r,s}} p_k \bar{p}_l \left[M\left(\frac{\left|x_{k+\mu,l+\eta} - L\right|}{\rho}\right)\right]^{t_{k,l}} \\
&< \frac{1}{M_1.h_r} \frac{1}{M_2.\bar{h}_s} \sum_{(k,l)\in I_{r,s}} \left[M\left(\frac{\left|x_{k+\mu,l+\eta} - L\right|}{\rho}\right)\right]^{t_{k,l}} \\
&= \frac{1}{M_{1,2}} \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \left[M\left(\frac{\left|x_{k+\mu,l+\eta} - L\right|}{\rho}\right)\right]^{t_{k,l}}
\end{aligned}
$$

where $M_{1,2} := M_1 M_2$. Hence, we obtain the result by taking the P-limit as $r, s \to \infty$.

2. Let $\frac{H_r}{h_r}$ and $\frac{\bar{H}_s}{\bar{h}_s}$ be upper bounded and $p_k > 1$ for all $k \in \mathbb{N}$ and $\bar{p}_l > 1$ for all $l \in \mathbb{N}$. Then $H_r > h_r$ for all $r \in \mathbb{N}$ and $\bar{H}_s > \bar{h}_s$ for all $s \in \mathbb{N}$. So, there exist N_1 and N_2 constants such that $1 < \frac{H_r}{h_r} \leqslant N_1 < \infty$ for all $r \in \mathbb{N}$ and $1 < \frac{\bar{H}_s}{\bar{h}_s} \leqslant N_2 < \infty$ for all $s \in \mathbb{N}$. Assume that the double sequence $x = (x_{k,l})$ converges to the P-limit L in $\left[\tilde{R}^2, \theta_{r,s}, p, M, t\right]$, with $\left[\tilde{R}^2, \theta_{r,s}, p, M, t\right]$- P-$\lim x = L$, then for each μ

and η we have

$$\begin{aligned}
&\frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \left[M\left(\frac{\left| x_{k+\mu,l+\eta} - L \right|}{\rho} \right) \right]^{t_{k,l}} \\
&= \frac{1}{h_r \bar{h}_s} \sum_{(k,l)\in I_{r,s}} \left[M\left(\frac{\left| x_{k+\mu,l+\eta} - L \right|}{\rho} \right) \right]^{t_{k,l}} \\
&< \frac{N_1}{H_r} \frac{N_2}{\bar{H}_s} \sum_{(k,l)\in I_{r,s}} p_k \bar{p}_l \left[M\left(\frac{\left| x_{k+\mu,l+\eta} - L \right|}{\rho} \right) \right]^{t_{k,l}} \\
&= \frac{1}{N_{1,2}} \frac{1}{H_{r,s}} \sum_{(k,l)\in I_{r,s}} p_k \bar{p}_l \left[M\left(\frac{\left| x_{k+\mu,l+\eta} - L \right|}{\rho} \right) \right]^{t_{k,l}}
\end{aligned}$$

where $N_{1,2} := N_1 N_2$. Hence, the result is obtained by taking the P-limit as $r, s \to \infty$.

□

Definition 3.1. Let $\theta_{r,s} = \{(k_r, l_s)\}$ be a double lacunary sequence. The double number sequence x is said to be $S_{(\bar{R}^2, \theta_{r,s})}$-$P$-convergent to L provided that for every $\varepsilon > 0$,

$$\mathrm{P} - \lim_{r,s} \frac{1}{H_{r,s}} \sup_{\mu,\eta} \left| \left\{ (k,l) \in I'_{r,s} : p_k \bar{p}_l \left| x_{k+\mu,l+\eta} - L \right| \geqslant \varepsilon \right\} \right| = 0.$$

In this case we write $S_{(\bar{R}^2, \theta_{r,s})}$-$P$-$\lim x = L$.

Theorem 3.5. *Let $\theta_{r,s} = \{(k_r, l_s)\}$ be a double lacunary sequence. If $I'_{r,s} \subseteq I_{r,s}$, then the inclusion $\left[\tilde{R}^2, \theta_{r,s}, p\right] \subset S_{(\bar{R}^2, \theta_{r,s})}$ is strict and $\left[\tilde{R}^2, \theta_{r,s}, p\right]$ - P-$\lim x = S_{(\bar{R}^2, \theta_{r,s})}$-$P$-$\lim x = L$.*

Proof. Let

$$K_{P_{r,s}}(\varepsilon) = \left\{ (k,l) \in I'_{r,s} : p_k \bar{p}_l \left| x_{k+\mu,l+\eta} - L \right| \geqslant \varepsilon \right\}. \tag{3.1}$$

Suppose that $x \in \left[\tilde{R}^2, \theta_{r,s}, p\right]$. Then for each μ and η

$$\mathrm{P} - \lim_{r,s} \frac{1}{H_{r,s}} \sum_{(k,l)\in I_{r,s}} p_k \bar{p}_l \left| x_{k+\mu,l+\eta} - L \right| = 0.$$

Since

$$\begin{aligned}
&\frac{1}{H_{r,s}} \sum_{(k,l)\in I_{r,s}} p_k \bar{p}_l \left| x_{k+\mu,l+\eta} - L \right| \geq \frac{1}{H_{r,s}} \sum_{(k,l)\in I'_{r,s}} p_k \bar{p}_l \left| x_{k+\mu,l+\eta} - L \right| \\
&= \frac{1}{H_{r,s}} \sum_{\substack{(k,l)\in I'_{r,s} \\ (k,l)\in K_{P_{r,s}}(\varepsilon)}} p_k \bar{p}_l \left| x_{k+\mu,l+\eta} - L \right| + \frac{1}{H_{r,s}} \sum_{\substack{(k,l)\in I'_{r,s} \\ (k,l)\notin K_{P_{r,s}}(\varepsilon)}} p_k \bar{p}_l \left| x_{k+\mu,l+\eta} - L \right| \\
&\geqslant \frac{1}{H_{r,s}} \sum_{\substack{(k,l)\in I'_{r,s} \\ (k,l)\in K_{P_{r,s}}(\varepsilon)}} p_k \bar{p}_l \left| x_{k+\mu,l+\eta} - L \right| = \frac{\left| K_{P_{r,s}}(\varepsilon) \right|}{H_{r,s}},
\end{aligned}$$

for all μ and η, we get P-$\lim_{r,s} \frac{|K_{P_{r,s}}(\varepsilon)|}{H_{r,s}} = 0$ for each μ and η. This implies that $x \in S_{(\tilde{R}^2,\theta_{r,s})}$.
To show that this inclusion is strict, let $x = (x_{k,l})$ be defined as

$$x_{k,l} := \begin{pmatrix} 1 & 2 & 3 & \cdots & \sqrt[3]{H_{r,s}}-1 & \sqrt[3]{H_{r,s}} & 0 & \cdots \\ 2 & 2 & 3 & \cdots & \sqrt[3]{H_{r,s}}-1 & \sqrt[3]{H_{r,s}} & 0 & \cdots \\ 3 & 2 & 3 & \cdots & \sqrt[3]{H_{r,s}}-1 & \sqrt[3]{H_{r,s}} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \sqrt[3]{H_{r,s}}-1 & 2 & 3 & \cdots & \sqrt[3]{H_{r,s}}-1 & \sqrt[3]{H_{r,s}} & 0 & \cdots \\ \sqrt[3]{H_{r,s}} & \sqrt[3]{H_{r,s}} & \sqrt[3]{H_{r,s}} & \cdots & \sqrt[3]{H_{r,s}} & \sqrt[3]{H_{r,s}} & 0 & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

and $p_k := 1$, $p_l := 1$ for all k and l. Clearly, x is an unbounded sequence. For $\varepsilon > 0$ and for all μ and η we have

$$\mathrm{P}-\lim_{r,s} \frac{1}{H_{r,s}} \left|\left\{(k,l) \in I'_{r,s} : p_k \bar{p}_l \left|x_{k+\mu,l+\eta} - L\right| \geqslant \varepsilon\right\}\right| = \mathrm{P}-\lim_{r,s} \frac{\sqrt[3]{H_{r,s}}\sqrt[3]{H_{r,s}}}{H_{r,s}} = 0.$$

Therefore $x \in S_{(\tilde{R}^2,\theta_{r,s})}$ with the P-limit $L = 0$. Also note that

$$\mathrm{P}-\lim_{r,s} \frac{1}{H_{r,s}} \sum_{(k,l)\in I_{r,s}} p_k \bar{p}_l \left|x_{k+\mu,l+\eta} - 0\right| = \mathrm{P}-\lim_{r,s} \frac{H_{r,s} + 3\sqrt[3]{H_{r,s}}^2 - 4\sqrt[3]{H_{r,s}} + 2}{2H_{r,s}} = \frac{1}{2}.$$

Hence $x_{k,l} \notin \left[\tilde{R}^2, \theta_{r,s}, p\right]$. This completes the proof. □

Theorem 3.6. *Let M be a constant such that $p_k \bar{p}_l \left|x_{k+\mu,l+\eta} - L\right| \leqslant M$, for all $k,l \in \mathbb{N}$ and for all μ and η. If $I_{r,s} \subseteq I'_{r,s}$, then $S_{(\tilde{R}^2,\theta_{r,s})} \subset \left[\tilde{R}^2, \theta_{r,s}, p\right]$ with $\left[\tilde{R}^2, \theta_{r,s}, p\right]$-$P$-$\lim x = S_{(\tilde{R}^2,\theta_{r,s})}$-$P$-$\lim x = L$.*

Proof. Assume that $\theta_{r,s} = \{(k_r, l_s)\}$ be a double lacunary sequence, $p_k \bar{p}_l \left|x_{k+\mu,l+\eta} - L\right| \leqslant M$, for all $k,l \in \mathbb{N}$ and for all μ and η. Let $I_{r,s} \subseteq I'_{r,s}$ and $K_{P_{r,s}}(\varepsilon)$ be as defined in the previous theorem. Since $x \in S_{(\tilde{R}^2,\theta_{r,s})}$ with $S_{(\tilde{R}^2,\theta_{r,s})}$-$P$-$\lim x = L$, then P-$\lim_{r,s} \frac{|K_{P_{r,s}}(\varepsilon)|}{H_{r,s}} = 0$. For a given $\varepsilon > 0$ and for all μ and η we have the following.

$$\begin{aligned} &\tfrac{1}{H_{r,s}} \sum_{(k,l)\in I_{r,s}} p_k \bar{p}_l \left|x_{k+\mu,l+\eta} - L\right| \leq \tfrac{1}{H_{r,s}} \sum_{(k,l)\in I'_{r,s}} p_k \bar{p}_l \left|x_{k+\mu,l+\eta} - L\right| \\ &= \tfrac{1}{H_{r,s}} \sum_{\substack{(k,l)\in I'_{r,s} \\ (k,l)\in K_{P_{r,s}}(\varepsilon)}} p_k \bar{p}_l \left|x_{k+\mu,l+\eta} - L\right| + \tfrac{1}{H_{r,s}} \sum_{\substack{(k,l)\in I'_{r,s} \\ (k,l)\notin K_{P_{r,s}}(\varepsilon)}} p_k \bar{p}_l \left|x_{k+\mu,l+\eta} - L\right| \\ &\leqslant M \tfrac{|K_{P_{r,s}}(\varepsilon)|}{H_{r,s}} + \varepsilon. \end{aligned}$$

Since ε is arbitrary, we get $x \in \left[\tilde{R}^2, \theta_{r,s}, p\right]$ with the same P-limit L. □

Theorem 3.7. *The following statements are true:*

1. *If $p_k \leqslant 1$ for all $k \in \mathbb{N}$ and $\bar{p}_l \leqslant 1$ for all $l \in \mathbb{N}$, then $S_{\theta_{r,s}} \subseteq S_{(\bar{R}^2,\theta_{r,s})}$ with $S_{\theta_{r,s}}$-P-$\lim x = S_{(\bar{R}^2,\theta_{r,s})}$-$P$-$\lim x = L$.*

2. *If $p_k \geqslant 1$ for all $k \in \mathbb{N}$ and $\bar{p}_l \geqslant 1$ for all $l \in \mathbb{N}$, and $\frac{H_r}{h_r}$ and $\frac{\bar{H}_s}{\bar{h}_s}$ are upper bounded, then $S_{(\bar{R}^2,\theta_{r,s})} \subseteq S_{\theta_{r,s}}$ with $S_{(\bar{R}^2,\theta_{r,s})}$-$P$-$\lim x = S_{\theta_{r,s}}$-$P$-$\lim x = L$.*

Proof. 1. If $p_k \leqslant 1$ for all $k \in \mathbb{N}$ and $\bar{p}_l \leqslant 1$ for all $l \in \mathbb{N}$, then $H_r \leqslant h_r$ for all $r \in \mathbb{N}$ and $\bar{H}_s \leqslant \bar{h}_s$ for all $s \in \mathbb{N}$. So, there exist $\bar{M}_1$ and $\bar{M}_2$ constants such that $0 < \bar{M}_1 \leqslant \frac{H_r}{h_r} \leqslant 1$ for all $r \in \mathbb{N}$ and $0 < \bar{M}_2 \leqslant \frac{\bar{H}_s}{\bar{h}_s} \leqslant 1$ for all $s \in \mathbb{N}$. Let $x = (x_{k,l})$ be a double sequence which converges to the P-limit L in $S_{\theta_{r,s}}$, then for an arbitrary $\varepsilon > 0$ and, for all μ and η we have

$$
\begin{aligned}
&\tfrac{1}{H_{r,s}} \left|\left\{(k,l) \in I'_{r,s} : p_k\bar{p}_l \left|x_{k+\mu,l+\eta} - L\right| \geqslant \varepsilon\right\}\right| \\
&= \tfrac{1}{H_r\bar{H}_s} \left|\left\{P_{k_{r-1}} < k \leqslant P_{k_r} \text{ and } \bar{P}_{l_{s-1}} < l \leqslant \bar{P}_{l_s} : p_k\bar{p}_l \left|x_{k+\mu,l+\eta} - L\right| \geqslant \varepsilon\right\}\right| \\
&\leqslant \tfrac{1}{\bar{M}_1\bar{M}_2} \tfrac{1}{h_r\bar{h}_s} \Big|\{P_{k_{r-1}} \leqslant k_{r-1} < k \leqslant P_{k_r} \leqslant k_r \text{ and} \\
&\qquad \bar{P}_{l_{s-1}} \leqslant l_{s-1} < l \leqslant \bar{P}_{l_s} \leqslant l_s : \left|x_{k+\mu,l+\eta} - L\right| \geqslant \varepsilon\}\Big| \\
&= \tfrac{1}{\bar{M}_{1,2}} \tfrac{1}{h_{r,s}} \left|\left\{k_{r-1} < k \leqslant k_r \text{ and } l_{s-1} < l \leqslant l_s : \left|x_{k+\mu,l+\eta} - L\right| \geqslant \varepsilon\right\}\right| \\
&= \tfrac{1}{\bar{M}_{1,2}} \tfrac{1}{h_{r,s}} \left|\left\{(k,l) \in I_{r,s} : \left|x_{k+\mu,l+\eta} - L\right| \geqslant \varepsilon\right\}\right|
\end{aligned}
$$

where $\bar{M}_{1,2} := \bar{M}_1\bar{M}_2$. Hence, we obtain the result by taking the P-limit as $r, s \to \infty$.

2. Let $\frac{H_r}{h_r}$ and $\frac{\bar{H}_s}{\bar{h}_s}$ be upper bounded and $p_k \geqslant 1$ for all $k \in \mathbb{N}$ and $\bar{p}_l \geqslant 1$ for all $l \in \mathbb{N}$. Then $H_r \geqslant h_r$ for all $r \in \mathbb{N}$ and $\bar{H}_s \geqslant \bar{h}_s$ for all $s \in \mathbb{N}$. So, there exist $\bar{N}_1$ and $\bar{N}_2$ constants such that $1 \leqslant \frac{H_r}{h_r} \leqslant \bar{N}_1 < \infty$ for all $r \in \mathbb{N}$ and $1 \leqslant \frac{\bar{H}_s}{\bar{h}_s} \leqslant \bar{N}_2 < \infty$ for all $s \in \mathbb{N}$. Assume that the double sequence $x = (x_{k,l})$ converges to the P-limit L in $S_{(\bar{R}^2,\theta_{r,s})}$ with $S_{(\bar{R}^2,\theta_{r,s})}$ -P-$\lim x = L$, then for an arbitrary $\varepsilon > 0$ and, for all μ and η we have

$$\frac{1}{h_{r,s}}\left|\left\{(k,l)\in I_{r,s}:\left|x_{k+\mu,l+\eta}-L\right|\geqslant\varepsilon\right\}\right|$$

$$=\frac{1}{h_r\bar{h}_s}\left|\left\{k_{r-1}<k\leqslant k_r \text{ and } l_{s-1}<l\leqslant l_s:\left|x_{k+\mu,l+\eta}-L\right|\geqslant\varepsilon\right\}\right|$$

$$\leqslant\frac{\bar{N}_1}{H_r}\frac{\bar{N}_2}{\bar{H}_s}\Big|\{k_{r-1}\leqslant P_{k_{r-1}}<k\leqslant k_r\leqslant P_{k_r} \text{ and } l_{s-1}\leqslant\bar{P}_{l_{s-1}}<l\leqslant l_s\leqslant\bar{P}_{l_s}:p_k\bar{p}_l\left|x_{k+\mu,l+\eta}-L\right|\geqslant\varepsilon\Big\}\Big|$$

$$=\frac{\bar{N}_1\bar{N}_2}{H_r\bar{H}_s}\left|\left\{P_{k_{r-1}}<k\leqslant P_{k_r} \text{ and } \bar{P}_{l_{s-1}}<l\leqslant\bar{P}_{l_s}:p_k\bar{p}_l\left|x_{k+\mu,l+\eta}-L\right|\geqslant\varepsilon\right\}\right|$$

$$=\bar{N}_{1,2}.\frac{1}{H_{r,s}}\left|\left\{(k,l)\in I'_{r,s}:p_k\bar{p}_l\left|x_{k+\mu,l+\eta}-L\right|\geqslant\varepsilon\right\}\right|$$

where $\bar{N}_{1,2}=\bar{N}_1\bar{N}_2$. Hence, the result is obtained by taking the P-limit as $r,s\to\infty$.
□

Theorem 3.8. *Let $\theta_{r,s}=\{(k_r,l_s)\}$ be a double lacunary sequence. Then we have the followings.*

1. *If $\liminf_r Q_r>1$ and $\liminf_s \bar{Q}_s>1$, then $S_{(\bar{R}^2)}\subseteq S_{(\bar{R}^2,\theta_{r,s})}$.*
2. *If $\limsup_r Q_r<\infty$ and $\limsup_s \bar{Q}_s<\infty$, then $S_{(\bar{R}^2,\theta_{r,s})}\subseteq S_{(\bar{R}^2)}$.*
3. *If $1<\liminf_{r,s} Q_{r,s}\leqslant\limsup_{r,s} Q_{r,s}<\infty$, then $S_{(\bar{R}^2,\theta_{r,s})}=S_{(\bar{R}^2)}$*

Proof. The item (3) is a consequence of (1) and (2). The proof can be done in a similar manner as in Theorem 3.2, Theorem 3.3 and Corollary 3.1. For this purpose, we left the proof to the reader. □

Definition 3.2. A double sequence $x=(x_{k,l})$ is said to be Riesz lacunary almost P-convergent to L if $P-\lim_{r,s}\omega_{rs}^{\mu\eta}(x)=L$, uniformly in μ and η, where $\omega_{rs}^{\mu\eta}=\omega_{rs}^{\mu\eta}(x)=\frac{1}{H_{r,s}}\sum_{(k,l)\in I_{r,s}}p_k\bar{p}_l\,x_{k+\mu,l+\eta}$.

Definition 3.3. A double sequence $x=(x_{k,l})$ is said to be Riesz lacunary almost statistically summable to L if for every $\varepsilon>0$ the set

$$K_\varepsilon:=\left\{(r,s)\in\mathbb{N}\times\mathbb{N}:\left|\omega_{rs}^{\mu\eta}-L\right|\geqslant\varepsilon\right\}$$

has double natural density zero, i.e., $\delta_2(K_\varepsilon)=0$. In this case, we write $\left(\tilde{R},\theta\right)_{st_2}$-$P$-$\lim x=L$. That is; for every $\varepsilon>0$, P - $\lim_{m,n}\frac{1}{mn}\left|\left\{r\leqslant m,\ s\leqslant n:\left|\omega_{rs}^{\mu\eta}-L\right|\geqslant\varepsilon\right\}\right|=0$, uniformly in μ and η. Hence, a double sequence $x = (x_{k,l})$ is Riesz lacunary almost statistically summable to L if and only if the double sequence $\left(\omega_{rs}^{\mu\eta}(x)\right)$ is almost statistically P-convergent to L.

Note that since a convergent double sequence is also statistically convergent to the same value, a Riesz lacunary almost convergent double sequence is also Riesz lacunary almost statistically summable with the same P-limit.

A double sequence $x = (x_{k,l})$ is said to be strongly $\left[\tilde{R}^2, \theta_{r,s}, p\right]_q$-almost convergent $(0 < q < \infty)$ to the number L if $P\text{-}\lim\limits_{r,s} \omega_{rs}^{\mu\eta}\left(|x - L|^q\right) = 0$, uniformly in μ and η. In this case, we write $x_{k,l} \to L\left(\left[\tilde{R}^2, \theta_{r,s}, p\right]_q\right)$ and L is called $\left[\tilde{R}^2, \theta_{r,s}, p\right]_q$-$P$-limit of x. Also, we denote the set of all strongly $\left[\tilde{R}^2, \theta_{r,s}, p\right]_q$-almost P-convergent double sequences by $\left[\tilde{R}^2, \theta_{r,s}, p\right]_q$.

Theorem 3.9. *Let $I_{r,s} \subseteq I'_{r,s}$ and $p_k\bar{p}_l \left|x_{k+\mu,l+\eta} - L\right| \leqslant M$ for all $k, l \in \mathbb{N}$ and for all μ and η. If the followings hold, then $S_{\left(\tilde{R}^2,\theta_{r,s}\right)} \subset \left[\tilde{R}^2, \theta_{r,s}, p\right]_q$ and $S_{\left(\tilde{R}^2,\theta_{r,s}\right)}$-$P$-$\lim x = \left[\tilde{R}^2, \theta_{r,s}, p\right]_q$-$P$-$\lim x = L$.*

1. $0 < q < 1$ *and* $1 \leqslant \left|x_{k+\mu,l+\eta} - L\right| < \infty$.
2. $1 \leqslant q < \infty$ *and* $0 \leqslant \left|x_{k+\mu,l+\eta} - L\right| < 1$.

Proof. Assume that $x = (x_{k,l}) \in S_{\left(\tilde{R}^2,\theta_{r,s}\right)}$ with P - $\lim\limits_{r,s\to\infty} \frac{1}{H_{r,s}} \left|K_{P_{r,s}}(\varepsilon)\right| = 0$, where $K_{P_{r,s}}(\varepsilon)$ was given by (3.1).

Since $p_k\bar{p}_l \left|x_{k+\mu,l+\eta} - L\right| \leqslant M$ for all $k, l \in \mathbb{N}$ and for all μ and η, and $I_{r,s} \subseteq I'_{r,s}$, then for a given $\varepsilon > 0$ and for all μ and η, we have

$$\begin{aligned}
&\tfrac{1}{H_{r,s}} \sum_{(k,l)\in I_{r,s}} p_k\bar{p}_l\left|x_{k+\mu,l+\eta} - L\right|^q \\
&\leqslant \tfrac{1}{H_{r,s}} \sum_{(k,l)\in I'_{r,s}} p_k\bar{p}_l\left|x_{k+\mu,l+\eta} - L\right|^q \\
&= \tfrac{1}{H_{r,s}} \sum_{\substack{(k,l)\in I'_{r,s} \\ (k,l)\notin K_{P_{r,s}}(\varepsilon)}} p_k\bar{p}_l \left|x_{k+\mu,l+\eta} - L\right|^q + \tfrac{1}{H_{r,s}} \sum_{\substack{(k,l)\in I'_{r,s} \\ (k,l)\in K_{P_{r,s}}(\varepsilon)}} p_k\bar{p}_l\left|x_{k+\mu,l+\eta} - L\right|^q \\
&= T_{r,s} + T'_{r,s},
\end{aligned}$$

where

$$T_{r,s} = \frac{1}{H_{r,s}} \sum_{\substack{(k,l)\in I'_{r,s} \\ (k,l)\notin K_{P_{r,s}}(\varepsilon)}} p_k\bar{p}_l \left|x_{k+\mu,l+\eta} - L\right|^q$$

and

$$T'_{r,s} = \frac{1}{H_{r,s}} \sum_{\substack{(k,l)\in I'_{r,s} \\ (k,l)\in K_{P_{r,s}}(\varepsilon)}} p_k \bar{p}_l \left| x_{k+\mu,l+\eta} - L \right|^q.$$

For $(k,l) \notin K_{P_{r,s}}(\varepsilon)$, we have

$$T_{r,s} = \frac{1}{H_{r,s}} \sum_{\substack{(k,l)\in I'_{r,s} \\ (k,l)\notin K_{P_{r,s}}(\varepsilon)}} p_k \bar{p}_l \left| x_{k+\mu,l+\eta} - L \right|^q \leqslant \frac{1}{H_{r,s}} \sum_{\substack{(k,l)\in I'_{r,s} \\ (k,l)\notin K_{P_{r,s}}(\varepsilon)}} p_k \bar{p}_l \left| x_{k+\mu,l+\eta} - L \right| \leqslant \varepsilon.$$

If $(k,l) \in K_{P_{r,s}}(\varepsilon)$, then

$$\begin{aligned} T'_{r,s} &= \frac{1}{H_{r,s}} \sum_{\substack{(k,l)\in I'_{r,s} \\ (k,l)\in K_{P_{r,s}}(\varepsilon)}} p_k \bar{p}_l \left| x_{k+\mu,l+\eta} - L \right|^q \\ &\leq \frac{1}{H_{r,s}} \sum_{\substack{(k,l)\in I'_{r,s} \\ (k,l)\in K_{P_{r,s}}(\varepsilon)}} p_k \bar{p}_l \left| x_{k+\mu,l+\eta} - L \right| \leqslant \frac{M}{H_{r,s}} \left| K_{P_{r,s}}(\varepsilon) \right|. \end{aligned}$$

Hence, $\frac{1}{H_{r,s}} \sum_{(k,l)\in I_{r,s}} p_k \bar{p}_l \left| x_{k+\mu,l+\eta} - L \right|^q \to 0$ as $r,s \to \infty$, uniformly in μ and η. This completes the proof. □

Theorem 3.10. *Let $I'_{r,s} \subseteq I_{r,s}$. If the following conditions hold, then $\left[\tilde{R}^2, \theta_{r,s}, p\right]_q \subset S_{(\tilde{R}^2,\theta_{r,s})}$ and $\left[\tilde{R}^2, \theta_{r,s}, p\right]_q$-$P$-$\lim x = S_{(\tilde{R}^2,\theta_{r,s})}$-$P$-$\lim x = L$.*

1. *$0 < q < 1$ and $0 \leqslant \left| x_{k+\mu,l+\eta} - L \right| < 1$.*
2. *$1 \leqslant q < \infty$ and $1 \leqslant \left| x_{k+\mu,l+\eta} - L \right| < \infty$.*

Proof. Let $x = (x_{k,l})$ be strongly $\left[\tilde{R}^2, \theta_{r,s}, p\right]_q$-almost P-convergent to the limit L. Since $p_k \bar{p}_l \left| x_{k+\mu,l+\eta} - L \right|^q \geqslant p_k \bar{p}_l \left| x_{k+\mu,l+\eta} - L \right|$ for case (1) and (2), then for all μ and η, we have

$$\begin{aligned} &\frac{1}{H_{r,s}} \sum_{(k,l)\in I_{r,s}} p_k \bar{p}_l \left| x_{k+\mu,l+\eta} - L \right|^q \\ &\geqslant \frac{1}{H_{r,s}} \sum_{(k,l)\in I'_{r,s}} p_k \bar{p}_l \left| x_{k+\mu,l+\eta} - L \right| \\ &\geqslant \frac{1}{H_{r,s}} \sum_{\substack{(k,l)\in I'_{r,s} \\ (k,l)\in K_{P_{r,s}}(\varepsilon)}} p_k \bar{p}_l \left| x_{k+\mu,l+\eta} - L \right|^q \\ &\geqslant \varepsilon \frac{1}{H_{r,s}} \left| K_{P_{r,s}}(\varepsilon) \right| \end{aligned}$$

where $K_{P_{r,s}}(\varepsilon)$ is as in (3.1). Taking limit as $r,s \to \infty$ in both sides of the above inequality, we conclude that $S_{(\tilde{R}^2,\theta_{r,s})}$ - P-$\lim x = L$. □

Theorem 3.11. *Let $I_{r,s} \subseteq I'_{r,s}$ and $p_k \bar{p}_l \left| x_{k+\mu,l+\eta} - L \right| \leqslant M$ for all $k, l \in \mathbb{N}$ and for each μ and η. If a double sequence $x = (x_{k,l})$ is Riesz lacunary almost statistically P-convergent to L, then it is Riesz lacunary almost statistically summable to L but not conversely.*

Proof. Assume that $I_{r,s} \subseteq I'_{r,s}$ and $p_k \bar{p}_l \left| x_{k+\mu,l+\eta} - L \right| \leqslant M$ for all $k, l \in \mathbb{N}$ and for each μ and η. Let $x = (x_{k,l})$ be $S_{\left(\bar{R}^2, \theta_{r,s}\right)}$-$P$-convergent to L. Put $K_{P_{r,s}}(\varepsilon) = \left\{ (k,l) \in I'_{r,s} : p_k \bar{p}_l \left| x_{k+\mu,l+\eta} - L \right| \geqslant \varepsilon \right\}$. Then

$$
\begin{aligned}
\left| \omega_{rs}^{\mu\eta} - L \right| &= \left| \frac{1}{H_{r,s}} \sum_{(k,l) \in I_{r,s}} p_k \bar{p}_l \, x_{k+\mu,l+\eta} - L \right| \\
&= \left| \frac{1}{H_{r,s}} \sum_{(k,l) \in I_{r,s}} p_k \bar{p}_l \, \left(x_{k+\mu,l+\eta} - L \right) \right| \\
&\leqslant \frac{1}{H_{r,s}} \sum_{(k,l) \in I_{r,s}} p_k \bar{p}_l \, \left| x_{k+\mu,l+\eta} - L \right| \\
&\leqslant \frac{1}{H_{r,s}} \sum_{(k,l) \in I'_{r,s}} p_k \bar{p}_l \, \left| x_{k+\mu,l+\eta} - L \right| \\
&= \frac{1}{H_{r,s}} \sum_{\substack{(k,l) \in I'_{r,s} \\ (k,l) \in K_{P_{r,s}}(\varepsilon)}} p_k \bar{p}_l \, \left| x_{k+\mu,l+\eta} - L \right| \\
&\quad + \frac{1}{H_{r,s}} \sum_{\substack{(k,l) \in I'_{r,s} \\ (k,l) \notin K_{P_{r,s}}(\varepsilon)}} p_k \bar{p}_l \, \left| x_{k+\mu,l+\eta} - L \right| \\
&\leqslant \frac{M}{H_{r,s}} \left| K_{P_{r,s}}(\varepsilon) \right| + \varepsilon
\end{aligned}
$$

for each μ and η, which implies that $P\text{-}\lim\limits_{r,s} \omega_{rs}^{\mu\eta}(x) = L$ uniformly in μ and η. Hence, st_2-P-$\lim\limits_{r,s} \omega_{rs}^{\mu\eta}(x) = L$ uniformly in μ and η and so, $\left(\tilde{R}, \theta\right)_{st_2}$-$P$-$\lim x = L$.

To see that the converse is not true, consider the double lacunary sequence $\theta_{r,s} = \left\{ \left(2^{r-1}, 3^{s-1} \right) \right\}$, $p_k = 1$, $\bar{p}_l = 1$ for all k and l, and the double sequence $x = (x_{k,l})$ defined as $x_{k,l} = (-1)^k$ for all l. □

Acknowledgement. The authors are grateful to anonymous referees for their careful reading of the paper which improved it greatly.

REFERENCES

1. A. M. Alotaibi and C. Çakan: *The Riesz convergence and Riesz core of double sequences*. J. Ineq. Appl. **2012** (56) (2012).
2. A. Alotaibi, B. Hazarika and S. A. Mohiuddine: *On lacunary statistical convergence of double sequences in locally solid Riesz spaces*. J. Comput. Anal. Appl. **17** (1) (2014) 156-165.
3. M. Başarır: *On the strong almost convergence of double sequences*. Period. Math. Hungar. **30** (3) (1995) 177–181.

4. M. Başarir and Ş. Konca: *On some lacunary almost convergent double sequence spaces and Banach limits*. Hindawi Publ. Cor., Abst. Appl. Anal. (2012) Article ID 426357, 17 pages. doi:10.1155/2012/426357.

5. M. Başarir and Ş. Konca: *On some spaces of lacunary convergent sequences derived by Norlund-type mean and weighted lacunary statistical convergence*. Arab J. Math. Sci. **20** (2) (2014) 250-263. doi: 10.1016/j.ajmsc.2013.09.002.

6. M. Başarir and Ş. Konca: *Weighted lacunary statistical convergence in locally solid Riesz spaces*. Filomat. **28** (10) (2014) 2059-2067. DOI 10.2298/FIL1410059B

7. M. Başarir and Ş. Konca: *On some spaces of weighted lacunary B_m^η-convergent sequences defined by an Orlicz function and B_m^η-weighted lacunary statistical convergence*. Contemp. Anal. Appl. Math. **2** (2) (2014) 242–257.

8. C. Çakan, B. Altay and H. Coşkun: *Double lacunary density and lacunary statistical convergence of double sequences*. Stud. Sci. Math. Hungar. **47** (1) (2010) 35–45.

9. A. R. Freedman, J. J. Sember and M.Raphael: *Some Cesaro type summability spaces*. Proc. London Math. Soc. **37** (1978) 508–520.

10. J. A. Fridy and C. Orhan: *Lacunary statistical convergence*. Pacific J. Math. **160** (1993) 43–51.

11. B. Hazarika, S. A. Mohiuddine and M. Mursaleen: *Some inclusion results for lacunary statistical convergence in locally solid Riesz spaces*. Iranian J. Sci. Tech., Transaction A: Science. 38A1 (2014) 61-68.

12. Ş. Konca and M. Başarir: *On some spaces of almost lacunary convergent sequences derived by Riesz mean and weighted almost lacunary statistical convergence in a real n-normed space*. J. Ineq. Appl. **2014** (81) (2014) doi.10.1186/1029-242X-2014-81.

13. M. A. Krasnoselskii and Y. B. Rutisky: *Convex function and Orlicz spaces*. Groningen, Netherlands, 1961.

14. S. A. Mohiuddine and E. Savaş: *Lacunary statistically convergent double sequences in probabilistic normed spaces*. Annali dellUniversita di Ferrara. **58** (2) (2012) 331–339.

15. S. A. S.A. Mohiuddine: *An application of almost convergence in approximation theorems*. Appl. Math. Lett. **24** (2011) 18561860.

16. S. A. Mohiuddine and M. Aiyub: *Lacunary statistical convergence in random 2-normed spaces*. Appl. Math. Inf. Sci. **6** (3) (2012) 581-585.

17. S. A. Mohiuddine and M. A. Alghamdi: *Statistical summability through lacunary sequence in locally solid Riesz spaces*. J. Ineq. Appl. **2012**:225 (2012). doi:10.1186/1029-242X-2012-225

18. S. A. Mohiuddine, A. Alotaibi and M. Mursaleen: *Statistical convergence of double sequences in locally solid Riesz spaces*. Abst. Appl. Anal. 2012, Article ID 719729, 9 pages. doi:10.1155/2012/719729

19. F. Moricz and B.E.Rhoades: *Almost convergence of double sequences and strong regularity of summability matrices*. Math. Proc. Cambridge Philos. Soc. **104** (2), (1988), 283–294.

20. M. Mursaleen and C. Belen: *On statistical lacunary summability of double sequences*. Filomat. **28** (2) (2014) 231–239.

21. M. Mursaleen and O. H. Edely: *Statistical convergence of double sequences*. J. Math. Anal. Appl. **288** (2003) 223–231.

22. R. F. Patterson and E. Savaş: *Lacunary statistical convergence of double sequences*. Math. Commun. **10** (2005) 55–61.

23. A. Pringsheim: *Zur Theorie der zweifach unendlichen Zahlenfolgen*. Math. Ann. **53** (3), (1900), 289–321.

24. G. M. Robison: *Divergent double sequences and series*. Trans. Amer. Math. Soc. **28** (1926), 50–73.

25. W. H. Ruckle: *FK-spaces in which the sequence of coordinate vectors in bounded*. Canadian J. Math. **25** (1973) 973–978.

26. E. Savaş and R. Patterson: *Lacunary statistical convergence of multiple sequences*. Appl. Math. Lett. **19** (2006) 527–534.

27. E. Savaş and R. F. Patterson: *Double sequence spaces characterized by lacunary sequences*. Appl. Math. Lett. **20** (2007), 964–970.

28. E. Savaş and R. F. Patterson: *On some double almost lacunary sequence spaces defined by Orlicz functions*. Filomat. **19** (2005) 35–44.

29. E. Savaş: *On lacunary double statistical convergence in locally solid Riesz spaces*. J. Ineq. Appl. **2013** (99) (2013). doi:10.1186/1029-242X-2013-99.

Şükran Konca
Bitlis Eren University
Department of Mathematics
13000, Bitlis, Turkey
skonca@beu.edu.tr

Metin Başarır
Sakarya University
Department of Mathematics
13000, Bitlis, Turkey
basarir@sakarya.edu.tr