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COUPLED AND COMMON COUPLED FIXED POINT THEOREMS UNDER NEW COUPLED IMPLICIT RELATION IN PARTIAL METRIC SPACES

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Abstract. The purpose of this paper is to study existence and uniqueness of coupled and common coupled fixed point theorems for self-mappings satisfying a new coupled implicit relation in the setting of partial metric spaces and give some corollaries of Theorem 3.1. Furthermore, we prove well-posedness of a coupled fixed point problem. We also provide some applications of our result to a mapping with a contraction of integral type. The results of findings in this paper extend and generalize several results from the existing literature.
Keywords: fixed point, metric space, contraction.

1. Introduction

In 2006, *Bhashkar and Lakshmikantham* [6] introduced the concept of coupled fixed point of a mapping $\mathcal{S}\colon \Xi\times\Xi \to \Xi$ (where $\Xi \neq \emptyset$ is a set) and investigated some coupled fixed point theorems in partially ordered complete metric spaces. They also proved mixed monotone property for the first time and gave their classical coupled fixed point theorem for mapping which satisfy the mixed monotone property. Later on, Ciric and Lakshmikantham [7], Sabetghadam *et al.* [29] and Olaleru *et al.* [25] proved some coupled fixed point theorems in metric spaces. Abbas *et al.* [1] proved common coupled fixed point results in the setting of cone metric spaces for weakly compatible mappings. Kim and Chandok [12] proved common coupled fixed

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point results for generalized nonlinear contraction mappings with mixed monotone property in partially ordered metric spaces (see, also [9, 16, 17, 21, 22, 24, 34, 35, 36]).

In 2011, *Aydi* [3] proved some coupled fixed point theorems on partial metric spaces. *Shatanawi et al.* [37] in 2012, proved coupled fixed point theorems for mixed monotone mappings satisfying nonlinear contraction involving two altering distance functions in ordered partial metric spaces. Recently, *Nashine* [23] proved coupled common fixed point results in ordered G-metric spaces and gave some examples to support the results. Very recently, *Kim et al.* [13] proved some common coupled fixed point theorems for weak compatible mappings in the setting of partial metric spaces (see, also [33]).

On the other hand, in [27, 30, 31, 32] the authors studied implicit relations in partial and weak partial metric spaces. Recently, *Kim* [14] introduced the coupled implicit relation and proved some fixed point theorems in the setting of Hilbert spaces.

The notion of partial metric space (PMS) was introduced by *Matthews* [19, 20] as a part of the study of denotational semantics of data flow networks. It is widely recognized that partial metric spaces play an important role in constructing models in the theory of computation (see, e.g., [10], [25] and some others). Introducing partial metric space, *Matthews* proved the partial metric version of Banach fixed point theorem ([5]). The PMS is a generalization of the usual metric spaces in which the distance of a point in the self may not be zero, that is, $d(u,u)$ may not be zero (for more details, see [2], [18], [26]).

Inspired by the idea of *Kim* [14] and some others, we study coupled and common coupled fixed point theorems in partial metric spaces by applying a new coupled implicit relation of three dimensions. The results obtained in this paper extend and generalize several previous works from the existing literature.

2. Preliminaries

We need the following definitions, lemmas and auxiliary results in partial metric spaces in the sequel.

Definition 2.1. ([20]) Let $\Xi \neq \emptyset$ and $\mathcal{P}\colon \Xi \times \Xi \to \mathbb{R}^+$ be a self mapping of Ξ such that for all $u, v, z \in \Xi$ the followings are satisfied:

($P1$) $u = v \Leftrightarrow \mathcal{P}(u,u) = \mathcal{P}(u,v) = \mathcal{P}(v,v)$,

($P2$) $\mathcal{P}(u,u) \leq \mathcal{P}(u,v)$,

($P3$) $\mathcal{P}(u,v) = \mathcal{P}(v,u)$,

($P4$) $\mathcal{P}(u,v) \leq \mathcal{P}(u,z) + \mathcal{P}(z,v) - \mathcal{P}(z,z)$.

Then $\mathcal{P}$ is called partial metric on Ξ and the pair $(\Xi, \mathcal{P})$ is called partial metric space (in short PMS).

Remark 2.1. It is clear that if $\mathcal{P}(u,v) = 0$, then from ($P1$), ($P2$), and ($P3$), $u = v$. But if $u = v$, $\mathcal{P}(u,v)$ may not be 0.

If $\mathcal{P}$ is a partial metric on Ξ, then the function $\mathcal{P}^s\colon \Xi \times \Xi \to \mathbb{R}^+$ given by

$$\mathcal{P}^s(u,v) = 2\mathcal{P}(u,v) - \mathcal{P}(u,u) - \mathcal{P}(v,v), \tag{2.1}$$

is a metric on Ξ.

Example 2.1. ([4]) Let $\Xi = \mathbb{R}^+$, where $\mathbb{R}^+ = [0,+\infty)$ and $\mathcal{P}\colon \Xi \times \Xi \to \mathbb{R}^+$ be given by $\mathcal{P}(u,v) = \max\{u,v\}$ for all $u,v \in \mathbb{R}^+$. Then $(\mathbb{R}^+,\mathcal{P})$ is a partial metric space.

Example 2.2. ([4]) Let I denote the set of all intervals $[u,v]$ for any real numbers $u \le v$. Let $\mathcal{P}\colon I \times I \to [0,\infty)$ be a function such that

$$\mathcal{P}\Big([u,v],[r,s]\Big) = \max\{v,s\} - \min\{u,r\}.$$

Then $(I,\mathcal{P})$ is a partial metric space.

Example 2.3. ([8]) Let $\Xi = \mathbb{R}$ and $\mathcal{P}\colon \Xi \times \Xi \to \mathbb{R}^+$ be given by $\mathcal{P}(u,v) = e^{\max\{u,v\}}$ for all $u,v \in \mathbb{R}$. Then $(\Xi,\mathcal{P})$ is a partial metric space.

Various applications of this space has been extensively investigated by many authors (see, [15], [38] for details).

Note also that each partial metric $\mathcal{P}$ on Ξ generates a T_0 topology $\tau_{\mathcal{P}}$ on Ξ, whose base is a family of open $\mathcal{P}$-balls $\{\mathcal{B}_{\mathcal{P}}(u,\varepsilon) : u \in \Xi, \varepsilon > 0\}$ where

$$\mathcal{B}_{\mathcal{P}}(u,\varepsilon) = \{v \in \Xi : \mathcal{P}(u,v) < \mathcal{P}(u,u) + \varepsilon\},$$

for all $u \in \Xi$ and $\varepsilon > 0$.

Similarly, closed $\mathcal{P}$-ball is defined as

$$\mathcal{B}_{\mathcal{P}}[u,\varepsilon] = \{v \in \Xi : \mathcal{P}(u,v) \le \mathcal{P}(u,u) + \varepsilon\},$$

for all $u \in \Xi$ and $\varepsilon > 0$.

Definition 2.2. ([19]) Let $(\Xi,\mathcal{P})$ be a partial metric space. Then

• a sequence $\{\alpha_n\}$ in $(\Xi,\mathcal{P})$ is said to be convergent to a point $\alpha \in \Xi$ if and only if $\mathcal{P}(\alpha,\alpha) = \lim_{n\to\infty} \mathcal{P}(\alpha_n,\alpha)$;

• a sequence $\{\alpha_n\}$ is called a Cauchy sequence if $\lim_{m,n\to\infty} \mathcal{P}(\alpha_m,\alpha_n)$ exists and is finite;

• $(\Xi,\mathcal{P})$ is said to be complete if every Cauchy sequence $\{\alpha_n\}$ in Ξ converges to a point $\alpha \in \Xi$ with respect to $\tau_{\mathcal{P}}$. Furthermore,

$$\lim_{m,n\to\infty} \mathcal{P}(\alpha_m,\alpha_n) = \lim_{n\to\infty} \mathcal{P}(\alpha_n,\alpha) = \mathcal{P}(\alpha,\alpha).$$

• A mapping $\mathcal{T}\colon \Xi \to \Xi$ is said to be continuous at $\alpha_0 \in \Xi$ if for every $\varepsilon > 0$, there exists $r > 0$ such that $\mathcal{T}\big(\mathcal{B}_{\mathcal{P}}(\alpha_0,r)\big) \subset \mathcal{B}_{\mathcal{P}}\big(\mathcal{T}(\alpha_0),\varepsilon\big)$.

Lemma 2.1. *([19, 20, 3]) Let $(\Xi, \mathcal{P})$ be a partial metric space. Then*

($\mathbf{\Delta_1}$) *a sequence $\{\alpha_n\}$ in $(\Xi, \mathcal{P})$ is a Cauchy sequence if and only if it is a Cauchy sequence in the metric space $(\Xi, \mathcal{P}^s)$,*

($\mathbf{\Delta_2}$) *a partial metric space $(\Xi, \mathcal{P})$ is complete if and only if the metric space $(\Xi, \mathcal{P}^s)$ is complete, furthermore, $\lim_{n\to\infty} \mathcal{P}^s(\alpha_n, \alpha) = 0$ if and only if*

$$\mathcal{P}(\alpha, \alpha) = \lim_{n\to\infty} \mathcal{P}(\alpha_n, \alpha) = \lim_{n,m\to\infty} \mathcal{P}(\alpha_n, \alpha_m). \tag{2.2}$$

Lemma 2.2. *(see [11]) Let $(\Xi, \mathcal{P})$ be a partial metric space. The following statements hold:*

(i) If $u, v \in \Xi$, $\mathcal{P}(u, v) = 0$, then $u = v$;

(ii) If $u \neq v$, then $\mathcal{P}(u, v) > 0$.

Definition 2.3. ([3]) An element $(u, v) \in \Xi \times \Xi$ is said to be a coupled fixed point of the mapping $\mathcal{S}\colon \Xi \times \Xi \to \Xi$ if $\mathcal{S}(u, v) = u$ and $\mathcal{S}(v, u) = v$.

Example 2.4. Let $\Xi = [0, +\infty)$ and $\mathcal{S}\colon \Xi \times \Xi \to \Xi$ defined by $\mathcal{S}(u, v) = \frac{u+v}{6}$ for all $u, v \in \Xi$. One can easily see that $\mathcal{S}$ has a unique coupled fixed point $(0, 0)$.

Example 2.5. Let $\Xi = [0, +\infty)$ and $\mathcal{S}\colon \Xi \times \Xi \to \Xi$ be defined by $\mathcal{S}(u, v) = \frac{u+v}{2}$ for all $u, v \in \Xi$. Then we see that $\mathcal{S}$ has two coupled fixed point $(0, 0)$ and $(1, 1)$, that is, the coupled fixed point is not unique.

Definition 2.4. ([1, 12]) An element $(u, v) \in \Xi \times \Xi$ is called

($\mathbf{\Lambda_1}$) a coupled coincidence point of mappings $\mathcal{S}\colon \Xi \times \Xi \to \Xi$ and $\mathcal{A}\colon \Xi \to \Xi$ if $\mathcal{A}(u) = \mathcal{S}(u, v)$ and $\mathcal{A}(v) = \mathcal{S}(v, u)$, and $(\mathcal{A}u, \mathcal{A}v)$ is called a coupled point of coincidence.

($\mathbf{\Lambda_2}$) a common coupled fixed point of mappings $\mathcal{S}\colon \Xi \times \Xi \to \Xi$ and $\mathcal{A}\colon \Xi \to \Xi$ if $u = \mathcal{A}(u) = \mathcal{S}(u, v)$ and $v = \mathcal{A}(v) = \mathcal{S}(v, u)$.

Definition 2.5. ([1]) The mappings $\mathcal{S}\colon \Xi \times \Xi \to \Xi$ and $\mathcal{A}\colon \Xi \to \Xi$ are called weakly compatible if $\mathcal{A}(\mathcal{S}(u, v)) = \mathcal{S}(\mathcal{A}u, \mathcal{A}v)$ and $\mathcal{A}(\mathcal{S}(v, u)) = \mathcal{S}(\mathcal{A}v, \mathcal{A}u)$ for all $u, v \in \Xi$, whenever $\mathcal{A}(u) = \mathcal{S}(u, v)$ and $\mathcal{A}(v) = \mathcal{S}(v, u)$.

Example 2.6. Let $\Xi = [0, 3]$ endowed with $\mathcal{P}(u, v) = \max\{u, v\}$ for all $u, v \in \Xi$. Define $\mathcal{S}\colon \Xi \times \Xi \to \Xi$ and $\mathcal{A}\colon \Xi \to \Xi$ by

$$\mathcal{S}(u, v) = \begin{cases} u + v, & \text{if } u, v \in [0, 1) \\ 3, & \text{otherwise,} \end{cases}$$

for all $u, v \in \Xi$ and

$$\mathcal{A}(u) = \begin{cases} u, & \text{if } u \in [0, 1) \\ 3, & \text{if } u \in [1, 3], \end{cases}$$

for all $u \in \Xi$. Then for any $u, v \in [1, 3]$,

$$\mathcal{S}(\mathcal{A}u, \mathcal{A}v) = \mathcal{S}(3, 3) = 3 = \mathcal{A}(\mathcal{S}(u, v)) = \mathcal{A}(3) = 3.$$

Similarly, we have

$$\mathcal{S}(\mathcal{A}v, \mathcal{A}u) = \mathcal{S}(3, 3) = 3 = \mathcal{A}(\mathcal{S}(v, u)) = \mathcal{A}(3) = 3.$$

Thus,

$$\mathcal{S}(\mathcal{A}u, \mathcal{A}v) = \mathcal{A}(\mathcal{S}(u, v)) \text{ and } \mathcal{S}(\mathcal{A}v, \mathcal{A}u) = \mathcal{A}(\mathcal{S}(v, u)).$$

This shows that the mappings $\mathcal{S}$ and $\mathcal{A}$ are weakly compatible on $[0, 3]$.

Example 2.7. Let $\Xi = \mathbb{R}$ endowed with the usual metric $\mathcal{P}(u, v) = \max\{u, v\}$ for all $u, v \in \Xi$. Define $\mathcal{S}: \Xi \times \Xi \to \Xi$ and $\mathcal{A}: \Xi \to \Xi$ by $\mathcal{S}(u, v) = u + v$ and $\mathcal{A}(u) = u^2$ for all $u, v \in \Xi$. Then $\mathcal{S}$ and $\mathcal{A}$ are not weakly compatible maps on $\mathbb{R}$, since

$$\mathcal{S}(\mathcal{A}u, \mathcal{A}v) = \mathcal{S}(u^2, v^2) = u^2 + v^2, \text{ but } \mathcal{A}(\mathcal{S}(u, v)) = \mathcal{A}(u + v) = (u + v)^2.$$

Therefore,

$$\mathcal{S}(\mathcal{A}u, \mathcal{A}v) \neq \mathcal{A}(\mathcal{S}(u, v)).$$

Hence the mappings $\mathcal{S}$ and $\mathcal{A}$ are not weakly compatible on $\mathbb{R}$.

Definition 2.6. (Coupled implicit relation) Let $\mathbb{R}_+$ (where $\mathbb{R}_+ = [0, \infty)$) be the set of all nonnegative real numbers, Φ be the class of all continuous real valued functions $\phi: \mathbb{R}_+^3 \to \mathbb{R}_+$ non-decreasing in the third argument satisfying the following conditions: for $\zeta, \eta, \theta, \vartheta > 0$,

(CIR_1) $\zeta \leq \phi\Big(\frac{\theta+\vartheta}{2}, \frac{\zeta+\theta}{2}, \frac{\eta+\vartheta}{2}\Big)$ and $\eta \leq \phi\Big(\frac{\theta+\vartheta}{2}, \frac{\eta+\vartheta}{2}, \frac{\zeta+\theta}{2}\Big)$,

or

(CIR_2) $\zeta \leq \phi\Big(\frac{\theta+\vartheta}{2}, 0, \vartheta\Big)$ and $\eta \leq \phi\Big(\frac{\theta+\vartheta}{2}, 0, \theta\Big)$,

there exists a real number $0 < k < 1$ such that $\zeta + \eta \leq k(\theta + \vartheta)$.

Sabetghadam et al. [29] obtained the following result in cone metric space.

Theorem 2.1. *Let (Ξ, d) be a complete cone metric space. Suppose that the mapping $\mathcal{S}: \Xi \times \Xi \to \Xi$ satisfies the following contractive condition for all $\zeta, \eta, \lambda, \mu \in \Xi$*

$$d(\mathcal{S}(\zeta, \eta), \mathcal{S}(\lambda, \mu)) \leq l_1 d(\zeta, \lambda) + l_2 d(\eta, \nu), \tag{2.3}$$

where l_1, l_2 are nonnegative constants with $l_1 + l_2 < 1$. Then $\mathcal{S}$ has a unique coupled fixed point.

Recently, *Aydi* [3] obtained the following results in partial metric space.

Theorem 2.2. *Let $(\Xi, \mathcal{P})$ be a complete partial metric space. Suppose that the mapping $\mathcal{S}: \Xi \times \Xi \to \Xi$ satisfies one of the following contractive conditions $(\mathbf{\Theta_1})$, $(\mathbf{\Theta_2})$, $(\mathbf{\Theta_3})$:*

($\mathbf{\Theta_1}$) *for all* $\zeta, \eta, \lambda, \mu \in \Xi$ *and nonnegative constants* l_1, l_2 *with* $l_1 + l_2 < 1$,

$$\mathcal{P}(\mathcal{S}(\zeta, \eta), \mathcal{S}(\lambda, \mu)) \leq l_1 \mathcal{P}(\zeta, \lambda) + l_2 \mathcal{P}(\eta, \mu), \tag{2.4}$$

($\mathbf{\Theta_2}$) *for all* $\zeta, \eta, \lambda, \mu \in \Xi$ *and nonnegative constants* l_1, l_2 *with* $l_1 + l_2 < 1$,

$$\mathcal{P}(\mathcal{S}(\zeta, \eta), \mathcal{S}(\lambda, \mu)) \leq l_1 \mathcal{P}(\mathcal{S}(\zeta, \eta), \zeta) + l_2 \mathcal{P}(\mathcal{S}(\lambda, \mu), \mu), \tag{2.5}$$

($\mathbf{\Theta_3}$) *for all* $\zeta, \eta, \lambda, \mu \in \Xi$ *and nonnegative constants* l_1, l_2 *with* $l_1 + 2l_2 < 1$,

$$\mathcal{P}(\mathcal{S}(\zeta, \eta), \mathcal{S}(\lambda, \mu)) \leq l_1 \mathcal{P}(\mathcal{S}(\zeta, \eta), \lambda) + l_2 \mathcal{P}(\mathcal{S}(\lambda, \mu), \zeta). \tag{2.6}$$

Then $\mathcal{S}$ *has a unique coupled fixed point.*

Quite recently, *Kim et al.* [13] obtained the following results in partial metric space.

Theorem 2.3. *Let* $(\Xi, \mathcal{P})$ *be a complete partial metric space. Suppose that the mappings* $\mathcal{S}\colon \Xi \times \Xi \to \Xi$ *and* $g\colon \Xi \to \Xi$ *satisfying one of the following contractive conditions: for all* $\zeta, \eta, \lambda, \mu \in \Xi$,

($\mathbf{\Gamma_1}$)

$$\begin{aligned} \mathcal{P}(\mathcal{S}(\zeta, \eta), \mathcal{S}(\lambda, \mu)) \leq\ & a_1 \mathcal{P}(g\zeta, g\lambda) + a_2 \mathcal{P}(g\eta, g\mu) + a_3 \mathcal{P}(\mathcal{S}(\zeta, \eta), g\zeta) \\ & + a_4 \mathcal{P}(\mathcal{S}(\lambda, \mu), g\lambda) + a_5 \mathcal{P}(\mathcal{S}(\zeta, \eta), g\lambda) \\ & + a_6 \mathcal{P}(\mathcal{S}(\lambda, \mu), g\zeta)], \end{aligned} \tag{2.7}$$

where $a_1, a_2, a_3, a_4, a_5, a_6$ *are nonnegative constants with* $a_1 + a_2 + a_3 + a_4 + a_5 + 2a_6 < 1$.

($\mathbf{\Gamma_2}$)

$$\mathcal{P}(\mathcal{S}(\zeta, \eta), \mathcal{S}(\lambda, \mu)) \leq k\, l(\zeta, \eta, \lambda, \mu), \tag{2.8}$$

where

$$\begin{aligned} l(\zeta, \eta, \lambda, \mu) = \max \Big\{ & \mathcal{P}(g\zeta, g\mu), \mathcal{P}(\mathcal{S}(\zeta, \eta), g\zeta), \mathcal{P}(\mathcal{S}(\lambda, \mu), g\lambda), \\ & \frac{\mathcal{P}(\mathcal{S}(\zeta, \eta), g\lambda) + \mathcal{P}(\mathcal{S}(\lambda, \mu), g\zeta)}{2} \Big\} \end{aligned}$$

and $k \in (0, 1)$ *is a constant. If* $\mathcal{S}(\Xi \times \Xi) \subseteq g(\Xi)$ *and* $g(\Xi)$ *is a complete subset of* Ξ*, then* $\mathcal{S}$ *and* g *have a coupled coincidence point in* Ξ*. Moreover, if* $\mathcal{S}$ *and* g *are weakly compatible, then* $\mathcal{S}$ *and* g *have a unique common coupled fixed point.*

3. Main Results

In this section, we shall prove a unique coupled fixed point and a unique common coupled fixed point theorems under a new coupled implicit relation in the setting of partial metric spaces.

Theorem 3.1. *Let $(\Xi, \mathcal{P})$ be a complete partial metric space. Suppose that the mapping $\mathcal{S}\colon \Xi\times\Xi \to \Xi$ satisfying the following contractive condition: for all $\zeta, \eta, \lambda, \mu \in \Xi$*

$$\begin{aligned}\mathcal{P}(\mathcal{S}(\zeta,\eta),\mathcal{S}(\lambda,\mu)) \quad &\leq \quad \phi\Big(\frac{\mathcal{P}(\zeta,\lambda)+\mathcal{P}(\eta,\mu)}{2}, \frac{\mathcal{P}(\mathcal{S}(\zeta,\eta),\zeta)+\mathcal{P}(\mathcal{S}(\lambda,\mu),\lambda)}{2}, \\ &\qquad \frac{\mathcal{P}(\mathcal{S}(\eta,\zeta),\mu)+\mathcal{P}(\mathcal{S}(\mu,\lambda),\eta)}{2}\Big), \qquad (3.1)\end{aligned}$$

where ϕ is as defined in Definition 2.6. Then $\mathcal{S}$ has a unique coupled fixed point.

Proof. Choose $\zeta_0, \eta_0 \in \Xi$. Set $\zeta_1 = \mathcal{S}(\zeta_0, \eta_0)$ and $\eta_1 = \mathcal{S}(\eta_0, \zeta_0)$. Repeating this process, we obtain two sequences $\{\zeta_n\}$ and $\{\eta_n\}$ in Ξ such that $\zeta_{n+1} = \mathcal{S}(\zeta_n, \eta_n)$ and $\eta_{n+1} = \mathcal{S}(\eta_n, \zeta_n)$. Then, from equations (3.1) and using $(P2)$, $(P4)$ and Definition 2.6, we have

$$\begin{aligned}\mathcal{P}(\zeta_n,\zeta_{n+1}) &= \mathcal{P}(\mathcal{S}(\zeta_{n-1},\eta_{n-1}),\mathcal{S}(\zeta_n,\eta_n)) \\ &\leq \phi\Big(\frac{\mathcal{P}(\zeta_{n-1},\zeta_n)+\mathcal{P}(\eta_{n-1},\eta_n)}{2}, \\ &\qquad \frac{\mathcal{P}(\mathcal{S}(\zeta_{n-1},\eta_{n-1}),\zeta_{n-1})+\mathcal{P}(\mathcal{S}(\zeta_n,\eta_n),\zeta_n)}{2}, \\ &\qquad \frac{\mathcal{P}(\mathcal{S}(\eta_{n-1},\zeta_{n-1}),\eta_n)+\mathcal{P}(\mathcal{S}(\eta_n,\zeta_n),\eta_{n-1})}{2}\Big) \\ &= \phi\Big(\frac{\mathcal{P}(\zeta_{n-1},\zeta_n)+\mathcal{P}(\eta_{n-1},\eta_n)}{2}, \frac{\mathcal{P}(\zeta_n,\zeta_{n-1})+\mathcal{P}(\zeta_{n+1},\zeta_n)}{2}, \\ &\qquad \frac{\mathcal{P}(\eta_n,\eta_n)+\mathcal{P}(\eta_{n+1},\eta_{n-1})}{2}\Big) \\ &\leq \phi\Big(\frac{\mathcal{P}(\zeta_{n-1},\zeta_n)+\mathcal{P}(\eta_{n-1},\eta_n)}{2}, \frac{\mathcal{P}(\zeta_{n-1},\zeta_n)+\mathcal{P}(\zeta_n,\zeta_{n+1})}{2}, \\ &\qquad \frac{\mathcal{P}(\eta_n,\eta_n)+\mathcal{P}(\eta_{n-1},\eta_n)+\mathcal{P}(\eta_n,\eta_{n+1})-\mathcal{P}(\eta_n,\eta_n)}{2}\Big) \\ &= \phi\Big(\frac{\mathcal{P}(\zeta_{n-1},\zeta_n)+\mathcal{P}(\eta_{n-1},\eta_n)}{2}, \frac{\mathcal{P}(\zeta_{n-1},\zeta_n)+\mathcal{P}(\zeta_n,\zeta_{n+1})}{2}, \\ &\qquad \frac{\mathcal{P}(\eta_{n-1},\eta_n)+\mathcal{P}(\eta_n,\eta_{n+1})}{2}\Big). \qquad (3.2)\end{aligned}$$

Likewise, we have

$$\begin{aligned}\mathcal{P}(\eta_n,\eta_{n+1}) &\leq \phi\Big(\frac{\mathcal{P}(\eta_{n-1},\eta_n)+\mathcal{P}(\zeta_{n-1},\zeta_n)}{2}, \frac{\mathcal{P}(\eta_{n-1},\eta_n)+\mathcal{P}(\eta_n,\eta_{n+1})}{2}, \\ &\qquad \frac{\mathcal{P}(\zeta_{n-1},\zeta_n)+\mathcal{P}(\zeta_n,\zeta_{n+1})}{2}\Big). \qquad (3.3)\end{aligned}$$

Hence from Definition 2.6 (CIR_1), there exists $0 < k < 1$ such that

$$\mathcal{P}(\zeta_n, \zeta_{n+1}) + \mathcal{P}(\eta_n, \eta_{n+1}) \le k[\mathcal{P}(\zeta_{n-1}, \zeta_n) + \mathcal{P}(\eta_{n-1}, \eta_n)]. \tag{3.4}$$

Set $\rho_n = \mathcal{P}(\zeta_n, \zeta_{n+1}) + \mathcal{P}(\eta_n, \eta_{n+1})$. Then equation (3.4) implies that

$$\rho_n \le k\, \rho_{n-1}. \tag{3.5}$$

Then for each $n \in \mathbb{N}$, we have

$$\rho_n \le k\rho_{n-1} \le k^2\rho_{n-2} \le \dots \le k^n \rho_0. \tag{3.6}$$

If $\rho_0 = 0$, then $\mathcal{P}(\zeta_0, \zeta_1) + \mathcal{P}(\eta_0, \eta_1) = 0$. Hence, from Remark 2.1, we get $\zeta_0 = \zeta_1 = \mathcal{S}(\zeta_0, \eta_0)$ and $\eta_0 = \eta_1 = \mathcal{S}(\eta_0, \zeta_0)$, means that (ζ_0, η_0) is a coupled fixed point of $\mathcal{S}$. Now, we assume that $\rho_0 > 0$. For each $n \ge m$, where $n, m \in \mathbb{N}$, we have, by using condition $(P4)$

$$\begin{aligned}\mathcal{P}(\zeta_n, \zeta_m) &\le \mathcal{P}(\zeta_n, \zeta_{n-1}) + \mathcal{P}(\zeta_{n-1}, \zeta_{n-2}) + \dots \\ &\quad + \mathcal{P}(\zeta_{m+1}, \zeta_m) - \mathcal{P}(\zeta_{n-1}, \zeta_{n-1}) - \mathcal{P}(\zeta_{n-2}, \zeta_{n-2}) \\ &\quad - \dots - \mathcal{P}(\zeta_{m+1}, \zeta_{m+1}) \\ &\le \mathcal{P}(\zeta_n, \zeta_{n-1}) + \mathcal{P}(\zeta_{n-1}, \zeta_{n-2}) + \dots + \mathcal{P}(\zeta_{m+1}, \zeta_m).\end{aligned} \tag{3.7}$$

Likewise, we have

$$\begin{aligned}\mathcal{P}(\eta_n, \eta_m) &\le \mathcal{P}(\eta_n, \eta_{n-1}) + \mathcal{P}(\eta_{n-1}, \eta_{n-2}) + \dots \\ &\quad + \mathcal{P}(\eta_{m+1}, \eta_m) - \mathcal{P}(\eta_{n-1}, \eta_{n-1}) - \mathcal{P}(\eta_{n-2}, \eta_{n-2}) \\ &\quad - \dots - \mathcal{P}(\eta_{m+1}, \eta_{m+1}) \\ &\le \mathcal{P}(\eta_n, \eta_{n-1}) + \mathcal{P}(\eta_{n-1}, \eta_{n-2}) + \dots + \mathcal{P}(\eta_{m+1}, \eta_m).\end{aligned} \tag{3.8}$$

Thus,

$$\begin{aligned}\mathcal{P}(\zeta_n, \zeta_m) + \mathcal{P}(\eta_n, \eta_m) &\le \rho_{n-1} + \rho_{n-2} + \dots + \rho_m \\ &\le (k^{n-1} + k^{n-2} + \dots + k^m)\rho_0 \\ &\le \Big(\frac{k^m}{1-k}\Big)\rho_0.\end{aligned} \tag{3.9}$$

By definition of metric $\mathcal{P}^s$, we have $\mathcal{P}^s(\zeta, \eta) \le 2\mathcal{P}(\zeta, \eta)$, therefore for any $n \ge m$

$$\begin{aligned}\mathcal{P}^s(\zeta_n, \zeta_m) + \mathcal{P}^s(\eta_n, \eta_m) &\le 2\mathcal{P}(\zeta_n, \zeta_m) + 2\mathcal{P}(\eta_n, \eta_m) \\ &\le \Big(\frac{2k^m}{1-k}\Big)\rho_0,\end{aligned} \tag{3.10}$$

which implies that $\{\zeta_n\}$ and $\{\eta_n\}$ are Cauchy sequences in $(\Xi, \mathcal{P}^s)$ because $0 \le k < 1$. Since the partial metric space $(\Xi, \mathcal{P})$ is complete, by Lemma 2.1, the metric space $(\Xi, \mathcal{P}^s)$ is complete, so there exist $d_1, d_2 \in \Xi$ such that

$$\lim_{n\to\infty} \mathcal{P}^s(\zeta_n, d_1) = \lim_{n\to\infty} \mathcal{P}^s(\eta_n, d_2) = 0. \tag{3.11}$$

From Lemma 2.1, we obtain

$$\mathcal{P}(d_1, d_1) = \lim_{n\to\infty} \mathcal{P}(\zeta_n, d_1) = \lim_{n\to\infty} \mathcal{P}(\zeta_n, \zeta_n), \tag{3.12}$$

and

$$\mathcal{P}(d_2, d_2) = \lim_{n\to\infty} \mathcal{P}(\eta_n, d_2) = \lim_{n\to\infty} \mathcal{P}(\eta_n, \eta_n). \tag{3.13}$$

But, from condition $(P2)$ and equation (3.6), we have

$$\mathcal{P}(\zeta_n, \zeta_n) \le \mathcal{P}(\zeta_n, \zeta_{n+1}) \le \rho_n \le k^n \rho_0, \tag{3.14}$$

and since $0 \le k < 1$, hence letting $n \to \infty$, we get $\lim_{n\to\infty} \mathcal{P}(\zeta_n, \zeta_n) = 0$. It follows that

$$\mathcal{P}(d_1, d_1) = \lim_{n\to\infty} \mathcal{P}(\zeta_n, d_1) = \lim_{n\to\infty} \mathcal{P}(\zeta_n, \zeta_n) = 0. \tag{3.15}$$

Likewise, we obtain

$$\mathcal{P}(d_2, d_2) = \lim_{n\to\infty} \mathcal{P}(\eta_n, d_2) = \lim_{n\to\infty} \mathcal{P}(\eta_n, \eta_n) = 0. \tag{3.16}$$

Now, using equations (3.1), the conditions $(P3)$ and $(P4)$, we have

$$\begin{aligned}
\mathcal{P}(\mathcal{S}(d_1, d_2), d_1) &\le \mathcal{P}(\mathcal{S}(d_1, d_2), \zeta_{n+1}) + \mathcal{P}(\zeta_{n+1}, d_1) - \mathcal{P}(\zeta_{n+1}, \zeta_{n+1}) \\
&\le \mathcal{P}(\mathcal{S}(d_1, d_2), \zeta_{n+1}) + \mathcal{P}(\zeta_{n+1}, d_1) \\
&= \mathcal{P}(\mathcal{S}(d_1, d_2), \mathcal{S}(\zeta_n, \eta_n)) + \mathcal{P}(\zeta_{n+1}, d_1) \\
&= \mathcal{P}(\mathcal{S}(\zeta_n, \eta_n), \mathcal{S}(d_1, d_2)) + \mathcal{P}(\zeta_{n+1}, d_1) \\
&\le \phi\Big(\frac{\mathcal{P}(\zeta_n, d_1) + \mathcal{P}(\eta_n, d_2)}{2}, \frac{\mathcal{P}(\mathcal{S}(\zeta_n, \eta_n), \zeta_n) + \mathcal{P}(\mathcal{S}(d_1, d_2), d_1)}{2}, \\
&\qquad \frac{\mathcal{P}(\mathcal{S}(\eta_n, \zeta_n), d_2) + \mathcal{P}(\mathcal{S}(d_2, d_1), \eta_n)}{2}\Big) + \mathcal{P}(\zeta_{n+1}, d_1) \\
&= \phi\Big(\frac{\mathcal{P}(\zeta_n, d_1) + \mathcal{P}(\eta_n, d_2)}{2}, \frac{\mathcal{P}(\zeta_{n+1}, \zeta_n) + \mathcal{P}(d_1, d_1)}{2}, \\
&\qquad \frac{\mathcal{P}(\eta_{n+1}, d_2) + \mathcal{P}(d_2, \eta_n)}{2}\Big) + \mathcal{P}(\zeta_{n+1}, d_1).
\end{aligned} \tag{3.17}$$

Passing to the limit as $n \to \infty$ in equation (3.17) and using equations (3.15), (3.16), we obtain

$$\mathcal{P}(\mathcal{S}(d_1, d_2), d_1) \le \phi(0, 0, 0). \tag{3.18}$$

Hence from Definition 2.6 (CIR_2), there exists $0 < k < 1$ such that

$$\mathcal{P}(\mathcal{S}(d_1, d_2), d_1) \le k.0 = 0. \tag{3.19}$$

Hence, we have $\mathcal{P}(\mathcal{S}(d_1, d_2), d_1) = 0$, that is, $\mathcal{S}(d_1, d_2) = d_1$. Likewise, we can prove that $\mathcal{S}(d_2, d_1) = d_2$. This shows that (d_1, d_2) is a coupled fixed point of $\mathcal{S}$.

For the uniqueness, let (e_1, e_2) be another coupled fixed point of $\mathcal{S}$ such that $(d_1, d_2) \neq (e_1, e_2)$, then from equation (3.1) and using equations (3.15), (3.16) and $(P3)$, we have

$$\begin{aligned}
\mathcal{P}(d_1, e_1) &= \mathcal{P}(\mathcal{S}(d_1, d_2), \mathcal{S}(e_1, e_2)) \\
&\leq \phi\Big(\frac{\mathcal{P}(d_1, e_1) + \mathcal{P}(d_2, e_2)}{2}, \frac{\mathcal{P}(\mathcal{S}(d_1, d_2), d_1) + \mathcal{P}(\mathcal{S}(e_1, e_2), e_1)}{2}, \\
&\qquad \frac{\mathcal{P}(\mathcal{S}(d_2, d_1), e_2) + \mathcal{P}(\mathcal{S}(e_2, e_1), d_2)}{2}\Big) \\
&= \phi\Big(\frac{\mathcal{P}(d_1, e_1) + \mathcal{P}(d_2, e_2)}{2}, \frac{\mathcal{P}(d_1, d_1) + \mathcal{P}(e_1, e_1)}{2}, \\
&\qquad \frac{\mathcal{P}(d_2, e_2) + \mathcal{P}(e_2, d_2)}{2}\Big) \\
&= \phi\Big(\frac{\mathcal{P}(d_1, e_1) + \mathcal{P}(d_2, e_2)}{2}, 0, \frac{\mathcal{P}(d_2, e_2) + \mathcal{P}(d_2, e_2)}{2}\Big) \\
&= \phi\Big(\frac{\mathcal{P}(d_1, e_1) + \mathcal{P}(d_2, e_2)}{2}, 0, \mathcal{P}(d_2, e_2)\Big). \qquad (3.20)
\end{aligned}$$

Likewise, we have

$$\begin{aligned}
\mathcal{P}(d_2, e_2) &= \mathcal{P}(\mathcal{S}(d_2, d_1), \mathcal{S}(e_2, e_1)) \\
&\leq \phi\Big(\frac{\mathcal{P}(d_2, e_2) + \mathcal{P}(d_1, e_1)}{2}, \frac{\mathcal{P}(\mathcal{S}(d_2, d_1), d_2) + \mathcal{P}(\mathcal{S}(e_2, e_1), e_2)}{2}, \\
&\qquad \frac{\mathcal{P}(\mathcal{S}(d_1, d_2), e_1) + \mathcal{P}(\mathcal{S}(e_1, e_2), d_1)}{2}\Big) \\
&= \phi\Big(\frac{\mathcal{P}(d_2, e_2) + \mathcal{P}(d_1, e_1)}{2}, \frac{\mathcal{P}(d_2, d_2) + \mathcal{P}(e_2, e_2)}{2}, \\
&\qquad \frac{\mathcal{P}(d_1, e_1) + \mathcal{P}(e_1, d_1)}{2}\Big) \\
&= \phi\Big(\frac{\mathcal{P}(d_1, e_1) + \mathcal{P}(d_2, e_2)}{2}, 0, \frac{\mathcal{P}(d_1, e_1) + \mathcal{P}(d_1, e_1)}{2}\Big) \\
&= \phi\Big(\frac{\mathcal{P}(d_1, e_1) + \mathcal{P}(d_2, e_2)}{2}, 0, \mathcal{P}(d_1, e_1)\Big). \qquad (3.21)
\end{aligned}$$

Hence from Definition 2.6 (CIR_2), there exists $0 < k < 1$ such that

$$\mathcal{P}(d_1, e_1) + \mathcal{P}(d_2, e_2) \leq k[\mathcal{P}(d_1, e_1) + \mathcal{P}(d_2, e_2)], \qquad (3.22)$$

which is a contradiction, since $0 < k < 1$. Hence, we get $\mathcal{P}(d_1, e_1) + \mathcal{P}(d_2, e_2) = 0$ and so $d_1 = e_1$ and $d_2 = e_2$. Therefore, $(d_1, d_2) = (e_1, e_2)$ which shows that (d_1, d_2) is a unique coupled fixed point of $\mathcal{S}$. This completes the proof. □

Remark 3.1. If $\zeta = \eta$ and $\theta = \vartheta$ in Definition 2.6, the coupled implicit relation conditions are restricted to the following:

Let $\mathbb{R}_+$ be the set of all nonnegative real numbers, Φ be the class of all continuous real valued functions $\phi\colon \mathbb{R}_+^3 \to \mathbb{R}_+$ non-decreasing in the third argument and satisfying the following conditions: for $\zeta, \theta > 0$,

(IR_1) $\zeta \leq \phi\Big(\theta, \frac{\zeta+\theta}{2}, \frac{\zeta+\theta}{2}\Big)$,

or

(IR_2) $\zeta \leq \phi\Big(\theta, 0, \theta\Big)$,

there exists a real number $0 < k < 1$ such that $\zeta \leq k\theta$.

Now, we prove a common coupled fixed point theorem in partial metric space.

Theorem 3.2. *Let $(\Xi, \mathcal{P})$ be a complete partial metric space. Suppose that the mappings $\mathcal{S}: \Xi \times \Xi \to \Xi$ and $g: \Xi \to \Xi$ satisfy the following contractive condition: for all $\zeta, \eta, \lambda, \mu \in \Xi$*

$$\begin{aligned}\mathcal{P}(\mathcal{S}(\zeta,\eta),\mathcal{S}(\lambda,\mu)) \quad \leq \quad & \phi\Big(\frac{\mathcal{P}(g\zeta,g\lambda)+\mathcal{P}(g\eta,g\mu)}{2}, \frac{\mathcal{P}(\mathcal{S}(\zeta,\eta),g\zeta)+\mathcal{P}(\mathcal{S}(\lambda,\mu),g\lambda)}{2}, \\ & \frac{\mathcal{P}(\mathcal{S}(\eta,\zeta),g\mu)+\mathcal{P}(\mathcal{S}(\mu,\lambda),g\eta)}{2}\Big), \qquad (3.23)\end{aligned}$$

where ϕ is as defined in Definition 2.6. If $\mathcal{S}(\Xi \times \Xi) \subseteq g(\Xi)$ and $g(\Xi)$ is a complete subset of Ξ, then $\mathcal{S}$ and g have a coupled coincidence point in Ξ. Moreover, if $\mathcal{S}$ and g are weakly compatible, then $\mathcal{S}$ and g have a unique common coupled fixed point in Ξ.

Proof. Since $\mathcal{S}(\Xi \times \Xi) \subseteq g(\Xi)$, for $\zeta_0, \eta_0 \in \Xi$, we can define $g\zeta_1 = \mathcal{S}(\zeta_0, \eta_0)$ and $g\eta_1 = \mathcal{S}(\eta_0, \zeta_0)$. Repeating this process, we obtain two sequences $\{\zeta_n\}$ and $\{\eta_n\}$ in Ξ such that $g\zeta_{n+1} = \mathcal{S}(\zeta_n, \eta_n)$ and $g\eta_{n+1} = \mathcal{S}(\eta_n, \zeta_n)$. Then, from equations (3.23) and using $(P3)$, $(P4)$ and Definition 2.6, we have

$$\begin{aligned}\mathcal{P}(g\zeta_n, g\zeta_{n+1}) \quad &= \quad \mathcal{P}(\mathcal{S}(\zeta_{n-1},\eta_{n-1}),\mathcal{S}(\zeta_n,\eta_n)) \\ &\leq \quad \phi\Big(\frac{\mathcal{P}(g\zeta_{n-1},g\zeta_n)+\mathcal{P}(g\eta_{n-1},g\eta_n)}{2}, \\ &\qquad \frac{\mathcal{P}(\mathcal{S}(\zeta_{n-1},\eta_{n-1}),g\zeta_{n-1})+\mathcal{P}(\mathcal{S}(\zeta_n,\eta_n),g\zeta_n)}{2}, \\ &\qquad \frac{\mathcal{P}(\mathcal{S}(\eta_{n-1},\zeta_{n-1}),g\eta_n)+\mathcal{P}(\mathcal{S}(\eta_n,\zeta_n),g\eta_{n-1})}{2}\Big) \\ &= \quad \phi\Big(\frac{\mathcal{P}(g\zeta_{n-1},g\zeta_n)+\mathcal{P}(g\eta_{n-1},g\eta_n)}{2}, \\ &\qquad \frac{\mathcal{P}(g\zeta_n,g\zeta_{n-1})+\mathcal{P}(g\zeta_{n+1},g\zeta_n)}{2}, \\ &\qquad \frac{\mathcal{P}(g\eta_n,g\eta_n)+\mathcal{P}(g\eta_{n+1},g\eta_{n-1})}{2}\Big) \\ &\leq \quad \phi\Big(\frac{\mathcal{P}(g\zeta_{n-1},g\zeta_n)+\mathcal{P}(g\eta_{n-1},g\eta_n)}{2}, \\ &\qquad \frac{\mathcal{P}(g\zeta_{n-1},g\zeta_n)+\mathcal{P}(g\zeta_n,g\zeta_{n+1})}{2},\end{aligned}$$

$$\frac{\mathcal{P}(g\eta_n, g\eta_n) + \mathcal{P}(g\eta_{n-1}, g\eta_n) + \mathcal{P}(g\eta_n, g\eta_{n+1}) - \mathcal{P}(g\eta_n, g\eta_n)}{2}\Big)$$

$$\begin{aligned} &= \phi\Big(\frac{\mathcal{P}(g\zeta_{n-1}, g\zeta_n) + \mathcal{P}(g\eta_{n-1}, g\eta_n)}{2}, \\ &\quad \frac{\mathcal{P}(g\zeta_{n-1}, g\zeta_n) + \mathcal{P}(g\zeta_n, g\zeta_{n+1})}{2}, \\ &\quad \frac{\mathcal{P}(g\eta_{n-1}, g\eta_n) + \mathcal{P}(g\eta_n, g\eta_{n+1})}{2}\Big). \end{aligned} \tag{3.24}$$

Likewise, we have

$$\begin{aligned} \mathcal{P}(g\eta_n, g\eta_{n+1}) &= \mathcal{P}(\mathcal{S}(\eta_{n-1}, \zeta_{n-1}), \mathcal{S}(\eta_n, \zeta_n)) \\ &\leq \phi\Big(\frac{\mathcal{P}(g\eta_{n-1}, g\eta_n) + \mathcal{P}(g\zeta_{n-1}, g\zeta_n)}{2}, \\ &\quad \frac{\mathcal{P}(g\eta_{n-1}, g\eta_n) + \mathcal{P}(g\eta_n, g\eta_{n+1})}{2}, \\ &\quad \frac{\mathcal{P}(g\zeta_{n-1}, g\zeta_n) + \mathcal{P}(g\zeta_n, g\zeta_{n+1})}{2}\Big). \end{aligned} \tag{3.25}$$

Hence from Definition 2.6 (CIR_1), there exists $0 < k < 1$ such that

$$\mathcal{P}(g\zeta_n, g\zeta_{n+1}) + \mathcal{P}(g\eta_n, g\eta_{n+1}) \leq k[\mathcal{P}(g\zeta_{n-1}, g\zeta_n) + \mathcal{P}(g\eta_{n-1}, g\eta_n)]. \tag{3.26}$$

Set $A_n = \mathcal{P}(g\zeta_n, g\zeta_{n+1}) + \mathcal{P}(g\eta_n, g\eta_{n+1})$. Then equation (3.26) implies that

$$A_n \leq k\, A_{n-1}. \tag{3.27}$$

Then for each $n \in \mathbb{N}$, we have

$$A_n \leq kA_{n-1} \leq k^2 A_{n-2} \leq \ldots \leq k^n A_0. \tag{3.28}$$

If $A_0 = 0$, then $\mathcal{P}(g\zeta_0, g\zeta_1) + \mathcal{P}(g\eta_0, g\eta_1) = 0$. Hence, from Remark 2.1, we get $g\zeta_0 = g\zeta_1 = \mathcal{S}(\zeta_0, \eta_0)$ and $g\eta_0 = g\eta_1 = \mathcal{S}(\eta_0, \zeta_0)$, meaning that $(g\zeta_0, g\eta_0)$ is a coupled fixed point of $\mathcal{S}$ and g. Now, we assume that $A_0 > 0$. For each $n \geq m$, where $n, m \in \mathbb{N}$, we have, by using condition $(P4)$

$$\begin{aligned} \mathcal{P}(g\zeta_n, g\zeta_m) &\leq \mathcal{P}(g\zeta_n, g\zeta_{n-1}) + \mathcal{P}(g\zeta_{n-1}, g\zeta_{n-2}) + \ldots \\ &\quad + \mathcal{P}(g\zeta_{m+1}, g\zeta_m) - \mathcal{P}(g\zeta_{n-1}, g\zeta_{n-1}) - \mathcal{P}(g\zeta_{n-2}, g\zeta_{n-2}) \\ &\quad - \ldots - \mathcal{P}(g\zeta_{m+1}, g\zeta_{m+1}) \\ &\leq \mathcal{P}(g\zeta_n, g\zeta_{n-1}) + \mathcal{P}(g\zeta_{n-1}, g\zeta_{n-2}) + \ldots \\ &\quad + \mathcal{P}(g\zeta_{m+1}, g\zeta_m). \end{aligned} \tag{3.29}$$

Likewise, we have

$$\begin{aligned} \mathcal{P}(g\eta_n, g\eta_m) &\leq \mathcal{P}(g\eta_n, g\eta_{n-1}) + \mathcal{P}(g\eta_{n-1}, g\eta_{n-2}) + \ldots \\ &\quad + \mathcal{P}(g\eta_{m+1}, g\eta_m) - \mathcal{P}(g\eta_{n-1}, g\eta_{n-1}) - \mathcal{P}(g\eta_{n-2}, g\eta_{n-2}) \\ &\quad - \ldots - \mathcal{P}(g\eta_{m+1}, g\eta_{m+1}) \\ &\leq \mathcal{P}(g\eta_n, g\eta_{n-1}) + \mathcal{P}(g\eta_{n-1}, g\eta_{n-2}) + \ldots \\ &\quad + \mathcal{P}(g\eta_{m+1}, g\eta_m). \end{aligned} \tag{3.30}$$

Thus,

$$\begin{aligned}\mathcal{P}(g\zeta_n, g\zeta_m)+\mathcal{P}(g\eta_n, g\eta_m) &\leq A_{n-1}+A_{n-2}+\ldots+A_m\\ &\leq (k^{n-1}+k^{n-2}+\ldots+k^m)A_0\\ &\leq \Big(\frac{k^m}{1-k}\Big)A_0. \end{aligned} \tag{3.31}$$

By definition of metric $\mathcal{P}^s$, we have $\mathcal{P}^s(g\zeta, g\eta) \leq 2\mathcal{P}(g\zeta, g\eta)$, therefore for any $n \geq m$

$$\begin{aligned}\mathcal{P}^s(g\zeta_n, g\zeta_m)+\mathcal{P}^s(g\eta_n, g\eta_m) &\leq 2\mathcal{P}(g\zeta_n, g\zeta_m)+2\mathcal{P}(g\eta_n, g\eta_m)\\ &\leq \left(\frac{2k^m}{1-k}\right)A_0, \end{aligned} \tag{3.32}$$

which implies that $\{g\zeta_n\}$ and $\{g\eta_n\}$ are Cauchy sequences in $(\Xi, \mathcal{P}^s)$ because $0 \leq k < 1$. Since the partial metric space $(\Xi, \mathcal{P})$ is complete, by Lemma 2.1, the metric space $(\Xi, \mathcal{P}^s)$ is complete, so there exist $z, r \in \Xi$ such that

$$\lim_{n\to\infty}\mathcal{P}^s(g\zeta_n, gz) = \lim_{n\to\infty}\mathcal{P}^s(g\eta_n, gr) = 0. \tag{3.33}$$

From Lemma 2.1, we obtain

$$\mathcal{P}(gz, gz) = \lim_{n\to\infty}\mathcal{P}(g\zeta_n, gz) = \lim_{n\to\infty}\mathcal{P}(g\zeta_n, g\zeta_n), \tag{3.34}$$

and

$$\mathcal{P}(gr, gr) = \lim_{n\to\infty}\mathcal{P}(g\eta_n, gr) = \lim_{n\to\infty}\mathcal{P}(g\eta_n, g\eta_n). \tag{3.35}$$

But, from condition $(P2)$ and equation (3.28), we have

$$\mathcal{P}(g\zeta_n, g\zeta_n) \leq \mathcal{P}(g\zeta_n, g\zeta_{n+1}) \leq A_n \leq k^n A_0, \tag{3.36}$$

and since $0 \leq k < 1$, hence letting $n \to \infty$, we get $\lim_{n\to\infty}\mathcal{P}(g\zeta_n, g\zeta_n) = 0$. It follows that

$$\mathcal{P}(gz, gz) = \lim_{n\to\infty}\mathcal{P}(g\zeta_n, gz) = \lim_{n\to\infty}\mathcal{P}(g\zeta_n, g\zeta_n) = 0. \tag{3.37}$$

Likewise, we obtain

$$\mathcal{P}(gr, gr) = \lim_{n\to\infty}\mathcal{P}(g\eta_n, gr) = \lim_{n\to\infty}\mathcal{P}(g\eta_n, g\eta_n) = 0. \tag{3.38}$$

Now, using equation (3.23), we have

$$\begin{aligned}\mathcal{P}(\mathcal{S}(z,r), gz) &\leq \mathcal{P}(\mathcal{S}(z,r), g\zeta_{n+1})+\mathcal{P}(g\zeta_{n+1}, gz)-\mathcal{P}(g\zeta_{n+1}, g\zeta_{n+1})\\ &\leq \mathcal{P}(\mathcal{S}(z,r), g\zeta_{n+1})+\mathcal{P}(g\zeta_{n+1}, gz)\\ &= \mathcal{P}(\mathcal{S}(z,r), \mathcal{S}(\zeta_n, \eta_n))+\mathcal{P}(g\zeta_{n+1}, gz)\end{aligned}$$

$$
\begin{aligned}
&= \mathcal{P}(\mathcal{S}(\zeta_n,\eta_n),\mathcal{S}(z,r)) + \mathcal{P}(g\zeta_{n+1},gz) \\
&\leq \phi\Big(\frac{\mathcal{P}(g\zeta_n,gz)+\mathcal{P}(g\eta_n,gr)}{2}, \frac{\mathcal{P}(\mathcal{S}(\zeta_n,\eta_n),g\zeta_n)+\mathcal{P}(\mathcal{S}(z,r),gz)}{2}, \\
&\qquad \frac{\mathcal{P}(\mathcal{S}(\eta_n,\zeta_n),gr)+\mathcal{P}(\mathcal{S}(r,z),g\eta_n)}{2}\Big) + \mathcal{P}(g\zeta_{n+1},gz) \\
&= \phi\Big(\frac{\mathcal{P}(g\zeta_n,gz)+\mathcal{P}(g\eta_n,gr)}{2}, \frac{\mathcal{P}(g\zeta_{n+1},g\zeta_n)+p(gz,gz)}{2}, \\
&\qquad \frac{\mathcal{P}(g\eta_{n+1},gr)+\mathcal{P}(gr,g\eta_n)}{2}\Big) + \mathcal{P}(g\zeta_{n+1},gz). \qquad (3.39)
\end{aligned}
$$

Letting $n \to \infty$ in equation (3.39) and using equations (3.37), (3.38), we obtain

$$\mathcal{P}(\mathcal{S}(z,r),gz) \quad \leq \quad \phi(0,0,0). \qquad (3.40)$$

Hence from Definition 2.6 (CIR_2), there exists $0 < k < 1$ such that

$$\mathcal{P}(\mathcal{S}(z,r),gz) \leq k.0 = 0. \qquad (3.41)$$

Hence, we have $\mathcal{P}(\mathcal{S}(z,r),gz) = 0$, that is, $\mathcal{S}(z,r) = gz$. Since the pair $(\mathcal{S},g)$ is weakly compatible, so by weak compatibility of $\mathcal{S}$ and g, we have

$$g(\mathcal{S}(z,r)) = \mathcal{S}(gz,gr) \text{ and } g(\mathcal{S}(r,z)) = \mathcal{S}(gr,gz).$$

Hence (gz,gr) is a common coupled fixed point of $\mathcal{S}$ and g.

Now, we show the uniqueness of the common coupled fixed point of $\mathcal{S}$ and g. Assume that (gz_1,gr_1) is another common coupled fixed point of $\mathcal{S}$ and g with $gz \neq gz_1$ and $gr \neq gr_1$, that is, $(gz,gr) \neq (gz_1,gr_1)$. Then from equation (3.23) and using equations (3.37), (3.38), $(P3)$, we have

$$
\begin{aligned}
\mathcal{P}(gz,gz_1) &= \mathcal{P}(\mathcal{S}(z,r),\mathcal{S}(z_1,r_1)) \\
&\leq \phi\Big(\frac{\mathcal{P}(gz,gz_1)+\mathcal{P}(gr,gr_1)}{2}, \frac{\mathcal{P}(\mathcal{S}(z,r),gz)+\mathcal{P}(\mathcal{S}(z_1,r_1),gz_1)}{2}, \\
&\qquad \frac{\mathcal{P}(\mathcal{S}(r,z),gr_1)+\mathcal{P}(\mathcal{S}(r_1,z_1),gr)}{2}\Big) \\
&= \phi\Big(\frac{\mathcal{P}(gz,gz_1)+\mathcal{P}(gr,gr_1)}{2}, \frac{\mathcal{P}(gz,gz)+\mathcal{P}(gz_1,gz_1)}{2}, \\
&\qquad \frac{\mathcal{P}(gr,gr_1)+\mathcal{P}(gr_1,gr)}{2}\Big) \\
&= \phi\Big(\frac{\mathcal{P}(gz,gz_1)+\mathcal{P}(gr,gr_1)}{2}, 0, \frac{\mathcal{P}(gr,gr_1)+\mathcal{P}(gr,gr_1)}{2}\Big) \\
&= \phi\Big(\frac{\mathcal{P}(gz,gz_1)+\mathcal{P}(gr,gr_1)}{2}, 0, \mathcal{P}(gr,gr_1)\Big). \qquad (3.42)
\end{aligned}
$$

Likewise, we obtain

$$
\begin{aligned}
\mathcal{P}(gr,gr_1) &= \mathcal{P}(\mathcal{S}(r,z),\mathcal{S}(r_1,z_1)) \\
&\leq \phi\Big(\frac{\mathcal{P}(gr,gr_1)+\mathcal{P}(gz,gz_1)}{2}, 0, \mathcal{P}(gz,gz_1)\Big). \qquad (3.43)
\end{aligned}
$$

Hence from Definition 2.6 (CIR_2), there exists $0 < k < 1$ such that

$$\mathcal{P}(gz, gz_1) + \mathcal{P}(gr, gr_1) \leq k[\mathcal{P}(gz, gz_1) + \mathcal{P}(gr, gr_1)], \tag{3.44}$$

which is a contradiction, since $0 < k < 1$. Hence, we get $\mathcal{P}(gz, gz_1) + \mathcal{P}(gr, gr_1) = 0$ and so $gz = gz_1$ and $gr = gr_1$. Therefore, $(gz, gr) = (gz_1, gr_1)$ which shows that (gz, gr) is a unique common coupled fixed point of $\mathcal{S}$ and g. This completes the proof.

□

Next, we give some analogues of coupled fixed point theorems in metric spaces for partial metric spaces by combining Theorem 3.1 with $\phi \in \Phi$ and ϕ satisfies the conditions (CIR_1) and (CIR_2). The following corollary is a Corollary 2.2 of Aydi [3].

Corollary 3.1. *Let $(\Xi, \mathcal{P})$ be a complete partial metric space. Suppose that the mapping $\mathcal{S}: \Xi \times \Xi \to \Xi$ satisfying the following contractive condition: for all $\zeta, \eta, \lambda, \mu \in \Xi$:*

$$\mathcal{P}(\mathcal{S}(\zeta, \eta), \mathcal{S}(\lambda, \mu)) \leq \frac{k}{2}[\mathcal{P}(\zeta, \lambda) + \mathcal{P}(\eta, \mu)],$$

where $k \in [0, 1)$ is a constant. Then $\mathcal{S}$ has a unique coupled fixed point.

Proof. The assertion follows using Theorem 3.1 with $\phi(\alpha, \beta, \gamma) = k\alpha$ for some $k \in [0, 1)$ and all $\alpha, \beta, \gamma \in \mathbb{R}_+$. □

The following corollary is an analogue of Corollary 2.6 of Aydi [3].

Corollary 3.2. *Let $(\Xi, \mathcal{P})$ be a complete partial metric space. Suppose that the mapping $\mathcal{S}: \Xi \times \Xi \to \Xi$ satisfying the following contractive condition: for all $\zeta, \eta, \lambda, \mu \in \Xi$:*

$$\mathcal{P}(\mathcal{S}(\zeta, \eta), \mathcal{S}(\lambda, \mu)) \leq \frac{k}{2}[\mathcal{P}(\mathcal{S}(\zeta, \eta), \zeta) + \mathcal{P}(\mathcal{S}(\lambda, \mu), \lambda)],$$

where $k \in [0, 1)$ is a constant. Then $\mathcal{S}$ has a unique coupled fixed point.

Proof. The assertion follows using Theorem 3.1 with $\phi(\alpha, \beta, \gamma) = k\beta$ for some $k \in [0, 1)$ and all $\alpha, \beta, \gamma \in \mathbb{R}_+$. □

The following corollary is an analogue of Corollary 2.7 of Aydi [3].

Corollary 3.3. *Let $(\Xi, \mathcal{P})$ be a complete partial metric space. Suppose that the mapping $\mathcal{S}: \Xi \times \Xi \to \Xi$ satisfying the following contractive condition: for all $\zeta, \eta, \lambda, \mu \in \Xi$:*

$$\mathcal{P}(\mathcal{S}(\zeta, \eta), \mathcal{S}(\lambda, \mu)) \leq \frac{k}{2}[\mathcal{P}(\mathcal{S}(\zeta, \eta), \lambda) + \mathcal{P}(\mathcal{S}(\lambda, \mu), \zeta)],$$

where $k \in [0, \frac{2}{3})$ is a constant. Then $\mathcal{S}$ has a unique coupled fixed point.

Proof. The assertion follows using Theorem 3.1 with $\phi(\alpha,\beta,\gamma) = k\gamma$ for some $k \in [0, \frac{2}{3})$ and all $\alpha, \beta, \gamma \in \mathbb{R}_+$. □

If, we define $\mathcal{T}\zeta = \mathcal{S}(\zeta,\zeta)$. Then, we have the following corollary.

Corollary 3.4. *Let $(\Xi, \mathcal{P})$ be a complete partial metric space. Suppose that the mapping $\mathcal{T}\colon \Xi \to \Xi$ satisfying the following contractive condition:*

$$\mathcal{P}(\mathcal{T}\zeta, \mathcal{T}\lambda) \leq \phi\Big(\mathcal{P}(\zeta,\lambda), \frac{\mathcal{P}(\mathcal{T}\zeta,\zeta) + \mathcal{P}(\mathcal{T}\lambda,\lambda)}{2}, \frac{\mathcal{P}(\mathcal{T}\zeta,\lambda) + \mathcal{P}(\mathcal{T}\lambda,\zeta)}{2}\Big), \tag{3.45}$$

for all $\zeta, \lambda \in \Xi$. Then $\mathcal{T}$ has a unique fixed point.

Proof. Taking $\zeta = \eta$ and $\lambda = \mu$ in Theorem 3.1, then (3.1) coincides with (3.45). Thus, we have the conclusion of the Corollary from Theorem 3.1. □

Remark 3.2. Our results extend and generalize the results of *Aydi* [3], *Kim et al.* [13] and many others from the existing literature.

Example 3.1. Let $\Xi = [0, +\infty)$ endowed with the usual partial metric $\mathcal{P}$ defined by $\mathcal{P}\colon \Xi \times \Xi \to [0, +\infty)$ with $\mathcal{P}(\zeta, \eta) = \max\{\zeta, \eta\}$. The partial metric space $(\Xi, \mathcal{P})$ is complete because $(\Xi, \mathcal{P}^s)$ is complete. Indeed, for any $\zeta, \eta \in \Xi$,

$$\begin{aligned} \mathcal{P}^s(\zeta,\eta) &= 2\mathcal{P}(\zeta,\eta) - \mathcal{P}(\zeta,\zeta) - \mathcal{P}(\eta,\eta) \\ &= 2\max\{\zeta,\eta\} - (\zeta+\eta) = |\zeta - \eta|. \end{aligned}$$

Thus, $(\Xi, \mathcal{P}^s)$ is the Euclidean metric space which is complete. Consider the mapping $\mathcal{S}\colon \Xi \times \Xi \to \Xi$ defined by $\mathcal{S}(\zeta,\eta) = \frac{\zeta+\eta}{6}$. Now, for any $\zeta, \eta, \lambda, \mu \in \Xi$, we have

$$\begin{aligned} \mathcal{P}(\mathcal{S}(\zeta,\eta), \mathcal{S}(\lambda,\mu)) &= \frac{1}{6}\max\{\zeta+\eta, \lambda+\mu\} \\ &\leq \frac{1}{6}[\max\{\zeta,\lambda\} + \max\{\eta,\mu\}] \\ &= \frac{1}{6}[\mathcal{P}(\zeta,\lambda) + \mathcal{P}(\eta,\mu)], \end{aligned}$$

which is the contractive condition of Corollary 3.1 for $k = 1/3 < 1$. Therefore, by Corollary 3.1, F has a unique coupled fixed point, which is $(0, 0)$. Note that if the mapping $\mathcal{S}\colon \Xi \times \Xi \to \Xi$ is given by $\mathcal{S}(\zeta,\eta) = \frac{\zeta+\eta}{2}$, then $\mathcal{S}$ satisfies contractive condition of Corollary 3.1 for $k = 1$, that is,

$$\begin{aligned} \mathcal{P}(\mathcal{S}(\zeta,\eta), \mathcal{S}(\lambda,\mu)) &= \frac{1}{2}\max\{\zeta+\eta, \lambda+\mu\} \\ &\leq \frac{1}{2}[\max\{\zeta,\lambda\} + \max\{\eta,\mu\}] \\ &= \frac{1}{2}[\mathcal{P}(\zeta,\lambda) + \mathcal{P}(\eta,\mu)]. \end{aligned}$$

In this case $(0, 0)$ and $(1, 1)$ are both coupled fixed points of $\mathcal{S}$, and hence, the coupled fixed point of $\mathcal{S}$ is not unique. This shows that the condition $k < 1$ in Corollary 3.1, and hence $l_1 + l_2 < 1$ in Theorem 2.2 (Θ_1) cannot be omitted in the statement of the aforesaid results. Likewise, we can verify the results of Corollary 3.2 and Corollary 3.3.

Example 3.2. Let $\Xi = \mathbb{R}$. Let $\mathcal{P}: \Xi \times \Xi \to \mathbb{R}$ be defined by $\mathcal{P}(\zeta, \eta) = \max\{\zeta, \eta\}$ for all $\zeta, \eta \in \Xi$. Then the partial metric space $(\Xi, \mathcal{P})$ is complete because $(\Xi, \mathcal{P}^s)$ is complete. Indeed, for any $\zeta, \eta \in \Xi$,

$$\begin{aligned}\mathcal{P}^s(\zeta, \eta) &= 2\mathcal{P}(\zeta, \eta) - \mathcal{P}(\zeta, \zeta) - \mathcal{P}(\eta, \eta)\\ &= 2\max\{\zeta, \eta\} - (\zeta + \eta) = |\zeta - \eta|.\end{aligned}$$

Thus, $(\Xi, \mathcal{P}^s)$ is the Euclidean metric space which is complete. Consider the mapping $\mathcal{S}: \Xi \times \Xi \to \Xi$ defined by $\mathcal{S}(\zeta, \eta) = \frac{3\zeta - \eta}{5}$ and let $\mathcal{P}(\zeta, \eta) = |\zeta - \eta|$ for all $\zeta, \eta \in \Xi$. Let us take $\zeta \neq \lambda$ and $\eta = \mu$ in the inequality of Corollary 3.1. Hence $t = |\zeta - \lambda| > 0$. Now, using inequality of Corollary 3.1, we have

$$\begin{aligned}\frac{3}{5}t &= \frac{3|\zeta - \lambda|}{5} = \mathcal{P}(\mathcal{S}(\zeta, \eta), \mathcal{S}(\lambda, \mu))\\ &\leq \frac{k}{2}\Big[|\zeta - \lambda| + |\eta - \mu|\Big] = \frac{k}{2}(|\zeta - \lambda|)\\ &= \frac{k}{2}t,\end{aligned}$$

or,

$$\frac{3}{5}t \leq \frac{k}{2}t,$$

that is,

$$k \geq \frac{6}{5},$$

which is a contradiction. Hence the Corollary 3.1 is not applicable to the operator $\mathcal{S}$ in order to prove that $(0, 0)$ is a unique coupled fixed point of $\mathcal{S}$.

4. Well-Posedness Theorem

In this section, we deal with the well-posedness of coupled fixed point problem studied in the previous section.

Definition 4.1. ([28]) Let (Ξ, d) be a metric space and let $f: \Xi \to \Xi$ be a mapping. The problem of finding a fixed point of f is said to be well posed if:

(1) f has a unique fixed point ζ_0,

(2) for any sequence $\{\zeta_n\} \in \Xi$ with $\lim_{n\to\infty} d(f\zeta_n, \zeta_n) = 0$, we have $\lim_{n\to\infty} d(\zeta_n, \zeta_0) = 0$.

Now, we generalize this definition for coupled fixed point in partial metric spaces.

Let $\mathcal{C}_0\mathcal{FP}(\mathcal{S}, \Xi \times \Xi)$ denote a coupled fixed point problem of mapping $\mathcal{S}$ and $\mathcal{C}_0\mathcal{F}(\mathcal{S})$ denote the set of all coupled fixed points of $\mathcal{S}$.

Definition 4.2. Let $(\Xi, \mathcal{P})$ be a partial metric space and let $\mathcal{S}: \Xi \times \Xi \to \Xi$ be a mapping. $\mathcal{C}_0\mathcal{FP}(\mathcal{S}, \Xi \times \Xi)$ is called well posed if:

(1) $\mathcal{C}_0\mathcal{F}(\mathcal{S})$ is unique,

(2) for any sequences $\{\zeta_n\}$, $\{\eta_n\}$ in Ξ with $(\bar{\zeta}, \bar{\eta}) \in \mathcal{C}_0\mathcal{F}(\mathcal{S})$ and

$$\lim_{n\to\infty} \mathcal{P}(\mathcal{S}(\zeta_n, \eta_n), \zeta_n) = 0 = \lim_{n\to\infty} \mathcal{P}(\mathcal{S}(\eta_n, \zeta_n), \eta_n)$$

implies

$$\bar{\zeta} = \lim_{n\to\infty} \zeta_n, \quad \bar{\eta} = \lim_{n\to\infty} \eta_n.$$

Theorem 4.1. *Let $(\Xi, \mathcal{P})$ be a complete partial metric space and $\mathcal{S}\colon \Xi\times\Xi \to \Xi$ be a mapping as in Theorem 3.1. For any sequences $\{\zeta_n\}$, $\{\eta_n\}$ in Ξ and $(\zeta, \eta) \in \mathcal{C}_0\mathcal{F}(\mathcal{S})$, if*

$$\lim_{n\to\infty} \mathcal{P}(\zeta, \mathcal{S}(\zeta_n, \eta_n)) = 0 = \lim_{n\to\infty} \mathcal{P}(\eta, \mathcal{S}(\eta_n, \zeta_n)),$$

then the coupled fixed point problem of $\mathcal{S}$ is well-posed with $\mathcal{P}(a, a) = 0$ for some $a \in \Xi$.

Proof. From Theorem 3.1, the mapping $\mathcal{S}$ has a unique coupled fixed point $(\zeta_0, \eta_0) \in \Xi \times \Xi$. Let $\{\zeta_n\}$, $\{\eta_n\}$ in Ξ and

$$\lim_{n\to\infty} \mathcal{P}(\mathcal{S}(\zeta_n, \eta_n), \zeta_n) = 0 = \lim_{n\to\infty} \mathcal{P}(\mathcal{S}(\eta_n, \zeta_n), \eta_n).$$

Without loss of generality, we assume that $(\zeta_0, \eta_0) \neq (\zeta_n, \eta_n)$ for any non-negative integer n. Using $\mathcal{S}(\zeta_0, \eta_0) = \zeta_0$ and $\mathcal{S}(\eta_0, \zeta_0) = \eta_0$, we obtain

$$\begin{aligned}
\mathcal{P}(\zeta_0, \zeta_n) &= \mathcal{P}(\mathcal{S}(\zeta_0, \eta_0), \mathcal{S}(\zeta_n, \eta_n)) + \mathcal{P}(\mathcal{S}(\zeta_n, \eta_n), \zeta_n) \\
&\quad -\mathcal{P}(\mathcal{S}(\zeta_n, \eta_n), \mathcal{S}(\zeta_n, \eta_n)) \\
&\leq \mathcal{P}(\mathcal{S}(\zeta_0, \eta_0), \mathcal{S}(\zeta_n, \eta_n)) + \mathcal{P}(\mathcal{S}(\zeta_n, \eta_n), \zeta_n) \\
&\leq \phi\Big(\frac{\mathcal{P}(\zeta_0, \zeta_n) + \mathcal{P}(\eta_0, \eta_n)}{2}, \frac{\mathcal{P}(\mathcal{S}(\zeta_0, \eta_0), \zeta_0) + \mathcal{P}(\mathcal{S}(\zeta_n, \eta_n), \zeta_n)}{2}, \\
&\qquad \frac{\mathcal{P}(\mathcal{S}(\eta_0, \zeta_0), \eta_n) + \mathcal{P}(\mathcal{S}(\eta_n, \zeta_n), \eta_0)}{2}\Big) + \mathcal{P}(\mathcal{S}(\zeta_n, \eta_n), \zeta_n), \qquad (4.1)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{P}(\eta_0, \eta_n) &\leq \mathcal{P}(\mathcal{S}(\eta_0, \zeta_0), \mathcal{S}(\eta_n, \zeta_n)) + \mathcal{P}(\mathcal{S}(\eta_n, \zeta_n), \eta_n) \\
&\quad -\mathcal{P}(\mathcal{S}(\eta_n, \zeta_n), \mathcal{S}(\eta_n, \zeta_n)) \\
&\leq \mathcal{P}(\mathcal{S}(\eta_0, \zeta_0), \mathcal{S}(\eta_n, \zeta_n)) + \mathcal{P}(\mathcal{S}(\eta_n, \zeta_n), \eta_n) \\
&\leq \phi\Big(\frac{\mathcal{P}(\eta_0, \eta_n) + \mathcal{P}(\zeta_0, \zeta_n)}{2}, \frac{\mathcal{P}(\mathcal{S}(\eta_0, \zeta_0), \eta_0) + \mathcal{P}(\mathcal{S}(\eta_n, \zeta_n), \eta_n)}{2}, \\
&\qquad \frac{\mathcal{P}(\mathcal{S}(\zeta_0, \eta_0), \zeta_n) + \mathcal{P}(\mathcal{S}(\zeta_n, \eta_n), \zeta_0)}{2}\Big) + \mathcal{P}(\mathcal{S}(\eta_n, \zeta_n), \eta_n). \qquad (4.2)
\end{aligned}$$

Since

$$\lim_{n\to\infty} \mathcal{P}(\zeta_0, \mathcal{S}(\zeta_n, \eta_n)) = 0 = \lim_{n\to\infty} \mathcal{P}(\eta_0, \mathcal{S}(\eta_n, \zeta_n)),$$

for $(\zeta_0, \eta_0) \in \mathcal{C}_0\mathcal{F}(\mathcal{S})$, using $(P3)$, we obtain

$$\lim_{n\to\infty} \mathcal{P}(\zeta_0, \zeta_n) \leq \lim_{n\to\infty} \phi\Big(\frac{\mathcal{P}(\zeta_0, \zeta_n) + \mathcal{P}(\eta_0, \eta_n)}{2}, 0, \mathcal{P}(\eta_0, \eta_n)\Big), \tag{4.3}$$

and

$$\lim_{n\to\infty} \mathcal{P}(\eta_0, \eta_n) \leq \lim_{n\to\infty} \phi\Big(\frac{\mathcal{P}(\eta_0, \eta_n) + \mathcal{P}(\zeta_0, \zeta_n)}{2}, 0, \mathcal{P}(\zeta_0, \zeta_n)\Big). \tag{4.4}$$

By the definition of implicit relation (CIR_2), there exists $0 < k < 1$ such that

$$\lim_{n\to\infty} \mathcal{P}(\zeta_0, \zeta_n) + \lim_{n\to\infty} \mathcal{P}(\eta_0, \eta_n) \leq k\Big(\lim_{n\to\infty} \mathcal{P}(\zeta_0, \zeta_n) + \lim_{n\to\infty} \mathcal{P}(\eta_0, \eta_n)\Big), \tag{4.5}$$

which is a contradiction. Thus, $\lim_{n\to\infty} \mathcal{P}(\zeta_0, \zeta_n) + \lim_{n\to\infty} \mathcal{P}(\eta_0, \eta_n) = 0$ and hence

$$\lim_{n\to\infty} \zeta_n = \zeta_0 \quad \text{and} \quad \lim_{n\to\infty} \eta_n = \eta_0.$$

This completes the proof. □

5. Applications

Here, we apply our result to a mapping with a contraction of integral type.

Let us assume that Γ denote the collection of all functions ϑ defined on $[0, +\infty)$ satisfy the following conditions:

(ϑ1) Each ϑ is Lebesgue integrable mapping on every compact subset of $[0, +\infty)$.

(ϑ2) For any $\varepsilon > 0$, we have $\int_0^\varepsilon \vartheta(t) > 0$.

Theorem 5.1. *Let $(\Xi, \mathcal{P})$ be a complete partial metric space. Suppose that the mapping $\mathcal{S}\colon \Xi\times\Xi \to \Xi$ satisfying the following contractive condition: for all $\zeta, \eta, \lambda, \mu \in \Xi$:*

$$\int_0^{\mathcal{P}(\mathcal{S}(\zeta,\eta),\mathcal{S}(\lambda,\mu))} \vartheta(t)dt \leq \kappa \int_0^{[\mathcal{P}(\zeta,\lambda)+\mathcal{P}(\eta,\mu)]} \vartheta(t)dt,$$

where $\kappa \in [0, 1/2)$ is a constant and $\vartheta \in \Gamma$. Then $\mathcal{S}$ possesses a unique coupled fixed point.

Theorem 5.2. *Let $(\Xi, \mathcal{P})$ be a complete partial metric space. Suppose that the mapping $\mathcal{S}\colon \Xi\times\Xi \to \Xi$ satisfying the following contractive condition: for all $\zeta, \eta, \lambda, \mu \in \Xi$:*

$$\int_0^{\mathcal{P}(\mathcal{S}(\zeta,\eta),\mathcal{S}(\lambda,\mu))} \vartheta(t)dt \leq \kappa \int_0^{[\mathcal{P}(\mathcal{S}(\zeta,\eta),\zeta)+\mathcal{P}(\mathcal{S}(\lambda,\mu),\lambda)]} \vartheta(t)dt,$$

where $\kappa \in [0, 1/2)$ is a constant and $\vartheta \in \Gamma$. Then $\mathcal{S}$ possesses a unique coupled fixed point.

Theorem 5.3. *Let* $(\Xi, \mathcal{P})$ *be a complete partial metric space. Suppose that the mapping* $\mathcal{S}\colon \Xi\times\Xi \to \Xi$ *satisfying the following contractive condition: for all* $\zeta, \eta, \lambda, \mu \in \Xi$*:*

$$\int_0^{\mathcal{P}(\mathcal{S}(\zeta,\eta),\mathcal{S}(\lambda,\mu))} \vartheta(t)dt \leq \kappa \int_0^{[\mathcal{P}(\mathcal{S}(\zeta,\eta),\lambda)+\mathcal{P}(\mathcal{S}(\lambda,\mu),\zeta)]} \vartheta(t)dt,$$

where $\kappa \in [0, 1/3)$ *is a constant and* $\vartheta \in \Gamma$*. Then* $\mathcal{S}$ *possesses a unique coupled fixed point.*

Remark 5.1. Theorem 5.1, Theorem 5.2 and Theorem 5.3 extend and generalize Corollary 2.2, Corollary 2.6 and Corollary 2.7 respectively of *Aydi* [3] to the case of integral type contraction.

6. Conclusion

In this paper, we study coupled fixed point and common coupled fixed point for a newly proposed coupled implicit relation in the setting of partial metric spaces and give some corollaries of Theorem 3.1. Furthermore, we prove well-posedness of a coupled fixed point problem. We also provide some applications of our result to a mapping with a contraction of integral type. Our results extend and generalize several results from the existing literature (see, for example, [3, 13] and many others).

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