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ON CONFORMAL QUASI HEMI-SLANT SUBMERSIONS FROM LORENTZIAN PARA SASAKIAN MANIFOLDS ONTP RIEMANNIAN MANIFOLDS

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Abstract. In the present article, our purpose is to define and study conformal quasi hemi-slant submersions (cqhs submersions, in short) from Lorentzian para Sasakian manifolds onto Riemannian manifolds. Its geometric properties are also investigated. Lastly, we give a non-trivial example for this type of submersion.
Keywords: Lorentzian para Sasakian manifolds, Conformal submersion, Conformal quasi hemi-slant submersion.

1. Introductions

The concept of Riemannian submersions between Riemannian manifolds was initiated by O' Neill [25] in 1966 and it has been further defined by Gray [14] in 1967. The Riemannian submersions play an essential role not only in differential geometry but also in science and technology. Expanding the study of Riemannian submersions between almost complex manifolds, Watson [38] defined almost Hermitian submersions in 1976. Later, the concept of anti-invariant submersions and Lagrangian submersion were defined by Sahin [35] and explained by Tastan [37] and Gunduzalp [13]. In 2011, Sahin defined semi-invariant submersions in complex

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geometry [34]. The theory of submersions, immersions and Riemannian maps are a recent geometry that plays a very important character in several disciplines of mathematics. Various results have been derived by distinguished geometers in this area. There are copious applications of Riemannian submersions in modern era. These are capable to handle many issues of Yang-Mills theory [6], Kaluza-Klein theory [7], supergravity and superstring theories ([15], [16]), Robotics [4] and theory of relativity [26].

Let $\pi : (N_1, g_1) \to (N_2, g_2)$ be a Riemannian submersion between Riemannian or semi-Riemannian manifolds. Different new subclasses of Riemannian submersions were introduced and studied such as slant submersion [33], semi-slant submersion [27], hemi-slant submersion [36] and quasi bi-slant submersion ([29], [32]). In 1997, Gundmundsson and Wood [12] generalized Riemannian submersion and presented a new class of Riemannian submersion horizontally conformal submersion. Afterwards, several geometers studied conformal anti-invariant submersion ([20], [22], [31]), conformal semi-invariant submersion ([3], [28]), conformal slant submersion [1], conformal semi-slant submersion ([2], [18], [19], [23], [30]), conformal hemi-slant submersion [17] and conformal quasi bi-slant submersion [21].

Inspired by the affirmative works, we characterize cqhs submersions in Lorentzian almost para contact geometry. Therefore, we choose Lorentzian para Sasakian manifold. We give some basics about conformal submersions, then we introduce cqhs submersions and explore the geometry of discussed submersions with an example.

2. Preliminaries

We present the definitions of Lorentzian para Sasakian manifold.

Let N_1 be a differentiable manifold of dimension $(2n+1)$, ϕ be a $(1,1)$ tensor field, ξ be a contravariant vector field, η be a 1-form and g_1 be a Lorentzian metric, then $(N_1, \phi, \xi, \eta, g_1)$ is called a Lorentzian para Sasakian manifold with conditions:

$$\phi^2 Z_1 = Z_1 + \eta(Z_1)\xi, \eta(\xi) = -1, \eta(\phi Z_1) = 0, \phi\xi = 0, \tag{2.1}$$

$$g_1(Z_1, \xi) = \eta(Z_1), g_1(\phi Z_1, \phi V_2) = g_1(Z_1, V_2) + \eta(Z_1)\eta(V_2), \tag{2.2}$$

$\forall\ Z_1, V_2 \in \Gamma(TN_1)$, is known as a Lorentzian para-contact metric structure and the manifold N_1 associated with the metric g_1 is called the Lorentzian para-contact metric manifold ([8], [9], [10], [24]). If moreover,

$$\nabla_{Z_1}\xi = \phi Z_1 \Leftrightarrow (\nabla_{Z_1}\eta)V_2 = g_1(\phi Z_1, V_2), \tag{2.3}$$

$$(\nabla_{Z_1}\phi)V_2 = \eta(V_2)Z_1 + g_1(Z_1, V_2)\xi + 2\eta(Z_1)\eta(V_2)\xi \tag{2.4}$$

hold on N_1 for all $Z_1, V_2 \in \Gamma(TN_1)$, then the Lorentzian para-contact metric manifold N_1 is known as a Lorentzian para-Sasakian (briefly, LP-Sasakian) manifold.

3. Conformal Submersions

We present background concepts, various definitions and useful results for the study of the cqhs submersions.

Definition 3.1. [5] Let (N_1, g_1) and (N_2, g_2) are two Riemannian manifolds of dimensions m and n respectively. A smooth map $\pi : (N_1, g_1) \to (N_2, g_2)$ is called a horizontally conformal submersion, if there is a positive function λ such that

$$g_1(V_1, V_2) = \frac{1}{\lambda^2} g_2(\pi_* V_1, \pi_* V_2), \tag{3.1}$$

$\forall\ V_1, V_2 \in \Gamma(ker\pi_*)^{\perp}$.

Hence, Riemannian submersion is the specific horizontally conformal submersion with $\lambda = 1$. Since, $\chi(y)$ represents the square dilation of π at y, so the dilation of π at y is represented by $\lambda(y) = \sqrt{\chi(y)}$. If a smooth map π is horizontally weakly conformal at every point on N_1, then π is called horizontally weakly conformal or semi-conformal on N_1 and if π is free from critical points on N_1, then it must be a (horizontally) conformal submersion.

We notice that a horizontally conformal submersion $\pi : (N_1, g_1, J) \to (N_2, g_2)$ is said to be horizontally homothetic if in this submersion the gradient of its dilation λ is vertical, i.e.,

$$\mathcal{H}(grad\lambda) = 0, \tag{3.2}$$

at $y \in N_1$, where $\mathcal{H}$ is the complement orthogonal distribution to $\mathcal{V} = \ker \pi_*$ in $\Gamma(T_y N_1)$.

Let $\pi : (N_1, g_1, J) \to (N_2, g_2)$ be a conformal Riemannian submersion. E and $\widehat{E}$ are vector fields on N_1 and N_2, respectively. E is said to be projectiable if there exist $\widehat{E}$ in such a manner that $\pi_*(E_q) = \widehat{E}_{\pi(q)}$ for any $q \in N_1$. Here E and $\widehat{E}$ are said to be $\pi-$related and any projectiable horizontal vector field Y_1 on N_1 is said to be basic.

O'Nill [25] defined two fundamental tensors $\mathcal{T}$ and $\mathcal{A}$ for vector fields Z_1 and Z_2 on N_1 such that

$$\mathcal{A}_{Z_1} Z_2 = \mathcal{H}\nabla^{N_1}_{\mathcal{H}Z_1} \mathcal{V}Z_2 + \mathcal{V}\nabla^{N_1}_{\mathcal{H}Z_1} \mathcal{H}Z_2, \tag{3.3}$$

$$\mathcal{T}_{Z_1} Z_2 = \mathcal{H}\nabla^{N_1}_{\mathcal{V}Z_1} \mathcal{V}Z_2 + \mathcal{V}\nabla^{N_1}_{\mathcal{V}Z_1} \mathcal{H}Z_2. \tag{3.4}$$

Now, from (3.3) and (3.4), we get

$$\nabla_{V_1} W_2 = \mathcal{T}_{V_1} W_2 + \mathcal{V}\nabla_{V_1} W_2, \tag{3.5}$$

$$\nabla_{V_1} X_1 = \mathcal{H}\nabla_{V_1} X_1 + \mathcal{T}_{V_1} X_1, \tag{3.6}$$

$$\nabla_{X_1} V_1 = \mathcal{A}_{X_1} V_1 + \mathcal{V}\nabla_{X_1} V_1, \tag{3.7}$$

$$\nabla_{X_1} Z_2 = \mathcal{H}\nabla_{X_1} Z_2 + \mathcal{A}_{X_1} Z_2 \tag{3.8}$$

for all $V_1, W_2 \in \Gamma(\ker \pi_*)$ and $X_1, Z_2 \in \Gamma(\ker \pi_*)^\perp$, here $\mathcal{V}\nabla_{V_1} W_2 = \widehat{\nabla}_{V_1} W_2$. If X_1 is basic, then $\mathcal{A}_{X_1} V_1 = \mathcal{H}\nabla_{X_1} V_1$.

Let $(N_1, \phi, \xi, \eta, g_1)$ be an almost contact metric manifold and (N_2, g_2) be a Riemannian manifold. Let $\pi : (N_1, \phi, \xi, \eta, g_1) \to (N_2, g_2)$ be a smooth map. Then the second fundamental form of π is given by

$$(\nabla \pi_*)(V_1, Z_2) = \nabla^\pi_{V_1} \pi_* Z_2 - \pi_*(\nabla_{V_1} Z_2), \textit{for all } V_1, Z_2 \in \Gamma(TN_1), \tag{3.9}$$

where the Levi-Civita connection of the metric g_1 and the pullback connection of metric g_2 are given by ∇ and ∇^π, respectively ([5],[11]). Here π is called totally geodesic map if $(\nabla \pi_*)(V_1, Z_2) = 0$, for all $V_1, Z_2 \in \Gamma(TN_1)$.

Lemma 3.1. *[5] Let $\pi : (N_1, \phi, \xi, \eta, g_1) \to (N_2, g_2)$ is a horizontal conformal submersion. Then, for any horizontal V_1, Y_2 and vertical vector fields X_1, Z_2, we get*

(a) $(\nabla \pi_*)(V_1, Y_2) = V_1(\ln \lambda)\pi_*(Y_2) + Y_2(\ln \lambda)\pi_*(V_1) - g_1(V_1, Y_2)\pi_*(grad \ln \lambda)$,

(b) $(\nabla \pi_*)(X_1, Z_2) = -\pi_*(\mathcal{T}_{X_1} Z_2)$,

(c) $(\nabla \pi_*)(V_1, X_1) = -\pi_*(\nabla^{N_1}_{V_1} X_1) = -\pi_*(\mathcal{A}_{V_1} X_1)$.

4. cqhs Submersions

We define cqhs submersions from LP-Sasakian manifolds and examine integrability conditions for horizontal and vertical distributions.

Definition 4.1. Let $(N_1, \phi, \xi, \eta, g_1)$ be a LP-Sasakian manifold and (N_2, g_2) be a Riemannian manifold. A horizontal conformal submersion $\pi : (N_1, \phi, \xi, \eta, g_1) \to (N_2, g_2)$ is called cqhs submersion if $\ker \pi_*$ has four orthogonal complementary distributions D, $D^\theta, D^\perp$ and $< \xi >$ such that D is invariant, D^θ is slant with angle θ and $D^\perp$ is anti-invariant, i.e.,

$$\ker \pi_* = D \oplus D^\theta \oplus D^\perp \oplus < \xi > . \tag{4.1}$$

In the above definition, the angle θ is known as the quasi hemi-slant angle of π.

We say that the cqhs $\pi : (N_1, \phi, \xi, \eta, g_1) \to (N_2, g_2)$ is proper if $D \neq 0, D^\theta \neq 0$, $D^\perp \neq \{0\}$ and $\theta \neq 0, \frac{\pi}{2}$.

Let π be a cqhs submersion from a LP-Sasakian manifold $(N_1, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold (N_2, g_2). Then,

$$TN_1 = (\ker \pi_*) \oplus (\ker \pi_*)^\perp.$$

For $W_1 \in \Gamma(\ker \pi_*)$, we get

$$W_1 = PW_1 + QW_1 + RW_1 - \eta(W_1)\xi, \tag{4.2}$$

where $PW_1 \in \Gamma(D), QW_1 \in \Gamma(D^\theta)$ and $RW_1 \in \Gamma(D^\perp)$. For all $U_1 \in \Gamma(\ker \pi_*)$, we get

$$\phi U_1 = \psi U_1 + \omega U_1, \tag{4.3}$$

where ωU_1 and ψU_1 are respectively horizontal and vertical components of ϕU_1.

Also for $Y_1 \in \Gamma(\ker \pi_*)^\perp$, we have

$$\phi Y_1 = BY_1 + CY_1, \tag{4.4}$$

where $BY_1 \in \Gamma(\ker \pi_*)$ and $CY_1 \in \Gamma(\mu)$. From (4.2) and (4.3), we have

$$\phi W_1 = \psi PW_1 + \omega PW_1 + \psi QW_1 + \omega QW_1 + \psi RW_1 + \omega QW_1. \tag{4.5}$$

Since $\phi D = D$ and $\phi D^\perp \subset \Gamma(\ker \pi_*)^\perp$, we get $\omega PW_1 = 0$ and $\psi RW_1 = 0$, and so

$$\phi W_1 = \psi PW_1 + \psi QW_1 + \omega QW_1 + \omega RW_1. \tag{4.6}$$

This means that, $\phi(\ker \pi_*) = \psi D \oplus \psi D^\theta \oplus \omega D^\theta \oplus \phi D^\perp$.

Here,

$$\Gamma(\ker \pi_*)^\perp = \omega \mathcal{D}^\theta \oplus \phi(\mathcal{D}^\perp) \oplus \mu. \tag{4.7}$$

Now, we will denote a cqhs submersion from a LP-Sasakian manifold $(N_1, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold (N_2, g_2) by π.

Lemma 4.1. *If π is a cqhs submersion, then we get*

$$\psi^2 Y_1 + B\omega Y_1 = Y_1 + \eta(Y_1)\xi, \qquad \omega\psi Y_1 + C\omega Y_1 = 0, \tag{4.8}$$

$$\psi BY_2 + BCY_2 = 0, \qquad \omega BY_2 + C^2 Y_2 = Y_2, \tag{4.9}$$

$\forall$ $Y_1 \in \Gamma(\ker \pi_*)$ and $Y_2 \in \Gamma(\ker \pi_*)^\perp$.

Proof. The proof follows by using (2.1), (4.3), and (4.4). □

Lemma 4.2. *If π is a cqhs submersion, then we get*

$$\mathcal{V}\nabla_{U_1}\phi U_2 + \mathcal{T}_{U_1}\omega U_2 = B\mathcal{T}_{U_1}U_2 + \phi\mathcal{V}\nabla_{U_1}U_2 + \eta(U_2)U_1 + g_1(U_1, U_2)\xi + 2\eta(U_1)\eta(U_2)\xi, \tag{4.10}$$

$$\mathcal{T}_{U_1}\phi U_2 + \mathcal{H}\nabla_{U_1}\omega U_2 = C\mathcal{T}_{U_1}U_2 + \omega\mathcal{V}\nabla_{U_1}U_2, \tag{4.11}$$

$$\mathcal{T}_{U_1}BY_1 + \mathcal{H}\nabla_{U_1}CY_1 = C\mathcal{H}\nabla_{U_1}Y_1 + \omega\mathcal{T}_{U_1}Y_1, \tag{4.12}$$

$$\mathcal{V}\nabla_{U_1}BY_1 + \mathcal{T}_{U_1}CY_1 = B\mathcal{H}\nabla_{U_1}Y_1 + \phi\mathcal{T}_{U_1}Y_1, \tag{4.13}$$

$$\mathcal{V}\nabla_{Y_1}\phi U_1 + \mathcal{A}_{Y_1}\omega U_1 = B\mathcal{A}_{Y_1}U_1 + \phi\mathcal{V}\nabla_{Y_1}U_1, \tag{4.14}$$

$$\mathcal{A}_{Y_1}\phi U_1 + \mathcal{H}\nabla_{Y_1}\omega U_1 = C\mathcal{A}_{Y_1}U_1 + \omega\mathcal{V}\nabla_{Y_1}U_1 + \eta(U_1)Y_1, \tag{4.15}$$

$$\mathcal{A}_{Y_1}BY_2 + \mathcal{H}\nabla_{Y_1}CY_2 = C\mathcal{H}\nabla_{Y_1}Y_2 + \omega\mathcal{A}_{Y_1}Y_2, \tag{4.16}$$

$$\mathcal{V}\nabla_{Y_1}BY_2 + \mathcal{A}_{Y_1}CY_2 = B\mathcal{H}\nabla_{Y_1}Y_2 + \phi\mathcal{A}_{Y_1}Y_2 + g_1(Y_1, Y_2)\xi \tag{4.17}$$

$\forall$ $U_1, U_2 \in \Gamma(\ker \pi_*)$ and $Y_1, Y_2 \in \Gamma(\ker \pi_*)^\perp$.

Proof. Using (2.4),(3.5)-(3.8), (4.3) and (4.4), we can easily get (4.10)-(4.17). □

Now, we define

$$(\nabla_{Z_1}\phi)Z_2 = \mathcal{V}\nabla_{Z_1}\phi Z_2 - \phi\mathcal{V}\nabla_{Z_1}Z_2, \tag{4.18}$$

$$(\nabla_{Z_1}\omega)Z_2 = \mathcal{H}\nabla_{Z_1}\omega Z_2 - \omega\mathcal{V}\nabla_{Z_1}Z_2, \tag{4.19}$$

$$(\nabla_{X_1}B)X_2 = \mathcal{V}\nabla_{X_1}BX_2 - B\mathcal{H}\nabla_{X_1}X_2, \tag{4.20}$$

$$(\nabla_{X_1}C)X_2 = \mathcal{H}\nabla_{X_1}BX_2 - C\mathcal{H}\nabla_{X_1}X_2 \tag{4.21}$$

$\forall\ Z_1, Z_2 \in \Gamma(\ker\pi_*)$ and $X_1, X_2 \in \Gamma(\ker\pi_*)^\perp$.

Lemma 4.3. *If π is a cqhs submersion, then we have*

$$(\nabla_{Z_1}\phi)Z_2 = B\mathcal{T}_{Z_1}Z_2 - \mathcal{T}_{Z_1}\omega Z_2 + \eta(Z_2)Z_1 + g_1(Z_1, Z_2)\xi + 2\eta(Z_1)\eta(Z_2)\xi, \tag{4.22}$$

$$(\nabla_{Z_1}\omega)Z_2 = C\mathcal{T}_{Z_1}Z_2 - \mathcal{T}_{Z_1}\phi Z_2, \tag{4.23}$$

$$(\nabla_{W_1}B)W_2 = \omega\mathcal{A}_{W_1}W_2 - \mathcal{A}_{W_1}BW_2, \tag{4.24}$$

$$(\nabla_{W_1}C)W_2 = \phi\mathcal{A}_{W_1}W_2 - \mathcal{A}_{W_1}CW_2 + g_1(W_1, W_2)\xi, \tag{4.25}$$

$\forall\ Z_1, Z_2 \in \Gamma(\ker\pi_*)$ *and* $W_1, W_2 \in \Gamma(\ker\pi_*)^\perp$.

Proof. Using (4.10),(4.11) and (4.16)-(4.21), we can easily get (4.22)-(4.25). □

Lemma 4.4. *Let π is cqhs submersion. Then, we have*

(a) $\phi^2 W_1 = (\cos^2\theta)W_1$,
(b) $g_1(\phi W_1, \phi W_2) = \cos^2\theta g_1(W_1, W_2)$,
(c) $g_1(\omega W_1, \omega W_2) = \sin^2\theta g_1(W_1, W_2)$, $\forall\ W_1, W_2 \in \Gamma(D^\theta)$.

Proof. The proof of the this Lemma is similar to that of Lemma 5 of [18]. □

Theorem 4.1. *Let π be cqhs submersion. Then D is integrable if and only if*

$$\frac{1}{\lambda^2}g_2((\nabla\pi_*)(Z_1, \phi Z_2) - (\nabla\pi_*)(Z_2, \phi Z_1), \pi_*(\omega V_1)) = g_1(\mathcal{V}\nabla_{Z_1}\phi Z_2 - \mathcal{V}\nabla_{Z_2}\phi Z_1, \psi QV_1),$$

$\forall\ Z_1, Z_2 \in \Gamma(D)$ *and* $V_1 \in \Gamma(D^\theta \oplus D^\perp)$.

Proof. It is know that D is integrable if and only if $g_1([Z_1, Z_2], V_1) = 0, g_1([Z_1, Z_2], V_2) = 0$ and $g_1([Z_1, Z_2], \xi) = 0\ \forall\ Z_1, Z_2 \in \Gamma(D)$, $V_1 \in \Gamma(D^\theta \oplus D^\perp)$ and $V_2 \in \Gamma(\ker\pi_*)^\perp$. It is clear that $\ker\pi_*$ is integrable so $g_1([Z_1, Z_2], V_2) = 0$. Thus D is integrable if and only if $g_1([Z_1, Z_2], \xi) = 0$ and $g_1([Z_1, Z_2], V_1) = 0$. From (2.3), we get $g_1([Z_1, Z_2], \xi) = 0$.

Again using $(2.2), (2.4), (3.5), (4.2)$ and (4.3), we get

$$\begin{aligned} g_1([Z_1, Z_2], V_1) &= g_1(\nabla_{Z_1}\phi Z_2, \phi V_1) - g_1(\nabla_{Z_2}\phi Z_1, \phi V_1), \\ &= g_1(\mathcal{V}\nabla_{Z_1}\phi Z_2 - \mathcal{V}\nabla_{Z_2}\phi Z_1, \psi Q V_1) + \\ &\quad g_1(\mathcal{T}_{Z_1}\phi Z_2 - \mathcal{T}_{Z_2}\phi Z_1, \omega Q V_1 + \omega R V_1). \end{aligned}$$

Using $(3.1), (3.9)$ and Lemma (3.1), we have

$$\begin{aligned} g_1([Z_1, Z_2], V_1) &= g_1(\mathcal{V}\nabla_{Z_1}\phi Z_2 - \mathcal{V}\nabla_{Z_2}\phi Z_1, \psi Q V_1) - \\ &\quad \frac{1}{\lambda^2} g_2((\nabla \pi_*)(Z_1, \phi Z_2) - (\nabla \pi_*)(Z_2, \phi Z_1), \pi_*(\omega Q V_1 + \omega R V_1)). \end{aligned}$$

Now, since $\omega V_1 = \omega Q V_1 + \omega R V_1$, we get

$$\begin{aligned} g_1([Z_1, Z_2], V_1) &= g_1(\mathcal{V}\nabla_{Z_1}\phi Z_2 - \mathcal{V}\nabla_{Z_2}\phi Z_1, \psi Q V_1) - \\ &\quad \frac{1}{\lambda^2} g_2((\nabla \pi_*)(Z_1, \phi Z_2) - (\nabla \pi_*)(Z_2, \phi Z_1), \pi_*(\omega V_1)). \end{aligned}$$

□

Theorem 4.2. *Let π be cqhs submersion. Then D^θ is integrable if and only if*

$$\begin{aligned} & g_1(\mathcal{T}_{Z_1}\omega\psi Z_2 - \mathcal{T}_{Z_2}\omega\psi Z_1, W_1) + g_1(\mathcal{T}_{Z_1}\omega Z_2 - \mathcal{T}_{Z_2}\omega Z_1, \phi P W_1) \\ &= \frac{1}{\lambda^2} g_2((\nabla \pi_*)(Z_1, \omega Z_2) - (\nabla \pi_*)(Z_2, \omega Z_1), \pi_*(\phi R W_1)), \end{aligned}$$

$\forall$ $Z_1, Z_2 \in \Gamma(D^\theta)$ and $W_1 \in \Gamma(D \oplus D^\perp)$.

Proof. We observe here that D^θ is integrable if and only if $g_1([Z_1, Z_2], W_1) = 0, g_1([Z_1, Z_2], \xi) = 0$ and $g_1([Z_1, Z_2], W_2) = 0$, $\forall$ $Z_1, Z_2 \in \Gamma(D^\theta), W_1 \in \Gamma(D \oplus D^\perp)$ and $W_2 \in (\ker \pi_*)^\perp$. Since $\ker \pi_*$ is integrable then $g_1([Z_1, Z_2], W_2) = 0$. Thus, D^θ is integrable if and only if $g_1([Z_1, Z_2], \xi) = 0$ and $g_1([Z_1, Z_2], W_1) = 0$. From (2.3), we get $g_1([Z_1, Z_2], \xi) = 0$.

Next, using $(2.2), (2.4), (3.5), (3.6), (4.2), (4.3)$ and Lemma (4.4), we get

$$\begin{aligned} & g_1([Z_1, Z_2], W_1) \\ =\ & g_1(\nabla_{Z_1}\phi Z_2, \phi W_1) - g_1(\nabla_{Z_2}\phi Z_1, \phi W_1), \\ =\ & g_1(\nabla_{Z_1}\psi^2 Z_2, W_1) - g_1(\nabla_{Z_2}\psi^2 Z_1, W_1) + g_1(\nabla_{Z_1}\omega\psi Z_2, W_1) - \\ & g_1(\nabla_{Z_2}\omega\psi Z_1, W_1) + g_1(\nabla_{Z_1}\omega Z_2, \phi W_1) - g_1(\nabla_{Z_2}\omega Z_1, \phi W_1) \\ =\ & \cos^2\theta g_1([Z_1, Z_2], W_1) + g_1(\mathcal{T}_{Z_1}\omega\psi Z_2, W_1) - g_1(\mathcal{T}_{Z_2}\omega\psi Z_1, W_1) + \\ & g_1(\mathcal{T}_{Z_1}\omega Z_2 - \mathcal{T}_{Z_2}\omega Z_1, \phi P W_1) + g_1(\mathcal{H}\nabla_{Z_1}\omega Z_2 - \mathcal{H}\nabla_{Z_2}\omega Z_1, \phi R W_1). \end{aligned}$$

Now, we get

$$\begin{aligned} & \sin^2\theta g_1([Z_1, Z_2], W_1) \\ =\ & g_1(\mathcal{T}_{Z_1}\omega\psi Z_2 - \mathcal{T}_{Z_2}\omega\psi Z_1, W_1) + g_1(\mathcal{T}_{Z_1}\omega Z_2 - \mathcal{T}_{Z_2}\omega Z_1, \phi P W_1) + \\ & g_1(\mathcal{H}\nabla_{Z_1}\omega Z_2 - \mathcal{H}\nabla_{Z_2}\omega Z_1, \phi R W_1). \end{aligned}$$

Using $(3.1),(3.9)$ and Lemma (3.1), we have the result

$$\begin{aligned}&\sin^2\theta g_1([Z_1,Z_2],W_1)\\ =\ & g_1(\mathcal{T}_{Z_1}\omega\psi Z_2-\mathcal{T}_{Z_2}\omega\psi Z_1,W_1)+g_1(\mathcal{T}_{Z_1}\omega Z_2-\mathcal{T}_{Z_2}\omega Z_1,\phi PW_1)-\\ &\frac{1}{\lambda^2}g_2((\nabla\pi_*)(Z_1,\omega Z_2)-(\nabla\pi_*)(Z_2,\omega Z_1),\pi_*(\phi RW_1)).\end{aligned}$$

□

Theorem 4.3. *Let π be cqhs submersion. Then $D^{\perp}$ is always integrable.*

Proof. The proof of this Theorem is similar as Theorem 3.13 of ([36]). □

Theorem 4.4. *Let π be a cqhs submersion. Then $(\ker\pi_*)^{\perp}$ is integrable if and only if*

$$\begin{aligned}&\frac{1}{\lambda^2}g_2(\nabla_{Z_2}\pi_*(CZ_1)-\nabla_{Z_1}\pi_*(CZ_2),\pi_*(\omega W_1))\\ =\ & g_1(\mathcal{V}\nabla_{Z_1}BZ_2-\mathcal{V}\nabla_{Z_2}BZ_1+\mathcal{A}_{Z_1}CZ_2-\mathcal{A}_{Z_2}CZ_1,\phi PW_1+\psi QW_1)+\\ &g_1(\mathcal{A}_{Z_1}BZ_2-\mathcal{A}_{Z_2}BZ_1,\omega W_1)-g_1(Z_1,\omega W_1)g_1(grad\ln\lambda,CZ_2)+\\ &g_1(Z_2,\omega W_1)g_1(grad\ln\lambda,CZ_1)+2g_1(Z_1,CZ_2)g_1(grad\ln\lambda,\omega W_1),\end{aligned}$$

$\forall\ W_1\in\Gamma(\ker\pi_*)$ *and* $Z_1,Z_2\in\Gamma(\ker\pi_*)^{\perp}$.

Proof. For $Z_1,Z_2\in\Gamma(\ker\pi_*)^{\perp}$ and $W_1\neq\xi\in\Gamma(\ker\pi_*)$. Using (2.3), we have $g_1([Z_1,Z_2],\xi)=0$.

From $(2.2),(2.4),(3.7),(3.8),(4.2),(4.3)$ and (4.4), we have

$$\begin{aligned}&g_1([Z_1,Z_2],W_1)\\ =\ & g_1(\nabla_{Z_1}\phi Z_2,\phi W_1)-g_1(\nabla_{Z_2}\phi Z_1,\phi W_1),\\ =\ & g_1(\mathcal{V}\nabla_{Z_1}BZ_2-\mathcal{V}\nabla_{Z_2}BZ_1+\mathcal{A}_{Z_1}CZ_2-\mathcal{A}_{Z_2}CZ_1,\phi PW_1+\psi QW_1)+\\ &g_1(\mathcal{A}_{Z_1}BZ_2-\mathcal{A}_{Z_2}BZ_1,\omega QW_1+\phi RW_1)+g_1(\mathcal{H}\nabla_{Z_1}CZ_2,\omega QW_1+\phi RW_1)-\\ &g_1(\mathcal{H}\nabla_{Z_2}CZ_1,\omega QW_1+JRW_1).\end{aligned}$$

Since $\omega QW_1+\phi RW_1=\omega QW_1+\omega RW_1=\omega W_1$, we get

$$\begin{aligned}&g_1([Z_1,Z_2],W_1)\\ =\ & g_1(\mathcal{V}\nabla_{Z_1}BZ_2-\mathcal{V}\nabla_{Z_2}BZ_1+\mathcal{A}_{Z_1}CZ_2-\mathcal{A}_{Z_2}CZ_1,\phi PW_1+\psi QW_1)+\\ &g_1(\mathcal{A}_{Z_1}BZ_2-\mathcal{A}_{Z_2}BZ_1,\omega W_1)+g_1(\mathcal{H}\nabla_{Z_1}CZ_2,\omega W_1)-g_1(\mathcal{H}\nabla_{Z_2}CZ_1,\omega W_1).\end{aligned}$$

Using $(3.1),(3.9)$ and Lemma (4.4), we have

$$\begin{aligned}&g_1([Z_1,Z_2],W_1)\\ =\ & g_1(\mathcal{V}\nabla_{Z_1}BZ_2-\mathcal{V}\nabla_{Z_2}BZ_1+\mathcal{A}_{Z_1}CZ_2-\mathcal{A}_{Z_2}CZ_1,\phi PW_1+\psi QW_1)+\\ &g_1(\mathcal{A}_{Z_1}BZ_2-\mathcal{A}_{Z_2}BZ_1,\omega W_1)+\frac{1}{\lambda^2}g_2(\nabla_{Z_1}\pi_*(CZ_2)-\nabla_{Z_2}\pi_*(CZ_1),\pi_*(\omega W_1))-\\ &g_1(Z_1,\omega W_1)g_1(grad\ln\lambda,CZ_2)+g_1(Z_2,\omega W_1)g_1(grad\ln\lambda,CZ_1)+\\ &2g_1(Z_1,CZ_2)g_1(grad\ln\lambda,\omega W_1).\end{aligned}$$

□

Theorem 4.5. *Let π is a cqhs submersion. Then any of the first two statements below imply the third:*

(a) $(\ker \pi_*)^{\perp}$ is integrable,

(b) π is a horizontally homothetic map,

(c)

$$\begin{aligned} & \frac{1}{\lambda^2} g_2(\nabla_{Z_2}\pi_*(CZ_1) - \nabla_{Z_1}\pi_*(CZ_2), \pi_*(\omega W_1)) \\ = \; & g_1(\mathcal{V}\nabla_{Z_1}BZ_2 - \mathcal{V}\nabla_{Z_2}BZ_1 + \mathcal{A}_{Z_1}CZ_2 - \mathcal{A}_{Z_2}CZ_1, \phi PW_1 + \psi QW_1) + \\ & g_1(\mathcal{A}_{Z_1}BZ_2 - \mathcal{A}_{Z_2}BZ_1, \omega W_1), \end{aligned}$$

$\forall\ W_1 \in \Gamma(\ker \pi_*)$ and $Z_1, Z_2 \in \Gamma(\ker \pi_*)^{\perp}$.

Proof. For $W_1 \neq \xi \in \Gamma(\ker \pi_*)$ and $Z_1, Z_2 \in \Gamma(\ker \pi_*)^{\perp}$, from Theorem 4.4, we have

$$\begin{aligned} & g_1([Z_1, Z_2], W_1) \\ = \; & g_1(\mathcal{V}\nabla_{Z_1}BZ_2 - \mathcal{V}\nabla_{Z_2}BZ_1 + \mathcal{A}_{Z_1}CZ_2 - \mathcal{A}_{Z_2}CZ_1, \phi PW_1 + \psi QW_1) + \\ & g_1(\mathcal{A}_{Z_1}BZ_2 - \mathcal{A}_{Z_2}BZ_1, \omega W_1) + \frac{1}{\lambda^2} g_2(\nabla_{Z_1}\pi_*(CZ_2) - \nabla_{Z_2}\pi_*(CZ_1), \pi_*(\omega W_1)) - \\ & g_1(Z_1, \omega W_1) g_1(grad \ln \lambda, CZ_2) + g_1(Z_2, \omega W_1) g_1(grad \ln \lambda, CZ_1) + \\ & 2 g_1(Z_1, CZ_2) g_1(grad \ln \lambda, \omega W_1), \end{aligned}$$

Now, using (3.2), we get Theorem 4.5(c)

$$\begin{aligned} & g_1([Z_1, Z_2], W_1) \\ = \; & g_1(\mathcal{V}\nabla_{Z_1}BZ_2 - \mathcal{V}\nabla_{Z_2}BZ_1 + \mathcal{A}_{Z_1}CZ_2 - \mathcal{A}_{Z_2}CZ_1, \phi PW_1 + \psi QW_1) + \\ & g_1(\mathcal{A}_{Z_1}BZ_2 - \mathcal{A}_{Z_2}BZ_1, \omega W_1) + \frac{1}{\lambda^2} g_2(\nabla_{Z_1}\pi_*(CZ_2) - \nabla_{Z_2}\pi_*(CZ_1), \pi_*(\omega W_1)). \end{aligned}$$

□

5. Totally Geodesic Conditions

We discuss totally geodesic conditions for cqhs submersions.

Proposition 5.1. *Let π is a cqhs submersion. Then D is not totally geodesic.*

Proof. For all $Z_1, Z_2 \in \Gamma(D)$, using (2.3), we have $g_1(\nabla_{Z_1}Z_2, \xi) = -g_1(Z_2, \phi Z_1)$, since $g_1(Z_2, \phi Z_1) \neq 0$, so D is not totally geodesic foliation on N_1. □

Theorem 5.1. *Let π is a cqhs submersion. Then $D\oplus < \xi >$ is totally geodesic if and only if*

$$g_1(\mathcal{V}\nabla_{Z_1}\phi Z_2, \psi Q V_1) = \frac{1}{\lambda^2} g_2((\nabla\pi_*)(Z_1, \phi Z_2), \pi_*(\omega V_1)),$$

$$g_1(\mathcal{V}\nabla_{Z_1}\phi Z_2, B V_2) = \frac{1}{\lambda^2} g_2((\nabla\pi_*)(Z_1, \phi Z_2), \pi_*(C V_2)),$$

$\forall\ Z_1, Z_2 \in \Gamma(D\oplus < \xi >)$, $V_1 \in \Gamma(D^\theta \oplus D^\perp)$ *and* $V_2 \in \Gamma(\ker\pi_*)^\perp$.

Proof. $D\oplus < \xi >$ is totally geodesic foliation on N_1 if and only if $g_1(\nabla_{Z_1} Z_2, V_1) = 0$ and $g_1(\nabla_{Z_1} Z_2, V_2) = 0$, $\forall\ Z_1, Z_2 \in \Gamma(D\oplus < \xi >)$, $V_1 \in \Gamma(D^\theta \oplus D^\perp)$ and $V_2 \in \Gamma(\ker\pi_*)^\perp$. Using (2.2), (2.4), (3.5), (4.2) and (4.3), we get

$$\begin{aligned} g_1(\nabla_{Z_1} Z_2, V_1) &= g_1(\nabla_{Z_1}\phi Z_2, \phi V_1), \\ &= g_1(\mathcal{V}\nabla_{Z_1}\phi Z_2, \psi Q V_1) + g_1(\mathcal{T}_{Z_1}\phi Z_2, \omega Q V_1 + \omega R V_1), \\ &= g_1(\mathcal{V}\nabla_{Z_1}\phi Z_2, \psi Q V_1) + g_1(\mathcal{T}_{Z_1}\phi Z_2, \omega V_1). \end{aligned}$$

Using (3.1), (3.9) and Lemma 3.1, we get

$$g_1(\nabla_{Z_1} Z_2, V_1) = g_1(\mathcal{V}\nabla_{Z_1}\phi Z_2, \psi Q V_1) - \frac{1}{\lambda^2} g_2((\nabla\pi_*)(Z_1, \phi Z_2), \pi_*(\omega V_1)).$$

On the other hand, $\forall\ Z_1, Z_2 \in \Gamma(D\oplus < \xi >)$ and $V_2 \in \Gamma(\ker\pi_*)^\perp$, using (2.2), (2.4), (3.5), (4.3) and (4.4), we have

$$\begin{aligned} g_1(\nabla_{Z_1} Z_2, V_2) &= g_1(\nabla_{Z_1}\phi Z_2, B V_2) + g_1(\nabla_{Z_1}\phi Z_2, C V_2), \\ &= g_1(\mathcal{V}\nabla_{Z_1}\phi Z_2, B V_2) + g_1(\mathcal{T}_{Z_1}\phi Z_2, C V_2). \end{aligned}$$

Using (3.1), (3.9) and Lemma (3.1), we get

$$g_1(\nabla_{Z_1} Z_2, V_2) = g_1(\mathcal{V}\nabla_{Z_1}\phi Z_2, B V_2) - \frac{1}{\lambda^2} g_2((\nabla\pi_*)(Z_1, \phi Z_2), \pi_*(C V_2)),$$

we obtain the results of theorem. □

Proposition 5.2. *Let π is a cqhs submersion. Then D^θ is not totally geodesic.*

Proof. For $Y_1, Y_2 \in \Gamma(D^\theta)$, from equation (2.3), we have $g_1(\nabla_{Y_1} Y_2, \xi) = -g_1(Y_2, \phi Y_1)$, since $g_1(Y_2, \phi Y_1) \neq 0$, so the D^θ is not totally geodesic. □

Theorem 5.2. *Let π is a cqhs submersion. Then $D^\theta\oplus < \xi >$ is totally geodesic if and only if*

$$\begin{aligned} & g_1(\nabla_{Z_1}\omega\psi Z_2, U_1) + g_1(\mathcal{T}_{Z_1}\omega Z_2, \phi P U_1) - \eta(Z_2) g_1(Z_1, \phi P U_1) \\ = \ & \frac{1}{\lambda^2} g_2((\nabla\pi_*)(Z_1, \omega Z_2), \pi_*(\phi R U_1)), \end{aligned}$$

$$\begin{aligned} & g_1(\mathcal{T}_{Z_1}\omega Z_2, BU_2) + g_1(\mathcal{A}_{\omega Z_2}\psi Z_1, \phi CU_2) + g_1((\omega Z_1, \omega Z_2)g_1(grad \ln\lambda, \phi CU_2) \\ = & -\frac{1}{\lambda^2}g_2(\nabla_{\omega Z_2}\pi_*(\omega Z_1), \pi_*(\phi CU_2)) + \frac{1}{\lambda^2}g_2((\nabla\pi_*)(Z_1, \omega\psi Z_2), \pi_*(U_2)) + \eta(Z_2)g_1(Z_1, BU_2), \end{aligned}$$

$\forall\ Z_1, Z_2 \in \Gamma(D^\theta \oplus <\xi>)$, $U_1 \in \Gamma(D \oplus D^\perp)$ *and* $U_2 \in \Gamma(\ker\pi_*)^\perp$.

Proof. $D^\theta \oplus <\xi>$ is totally geodesic foliation on N_1 if and only if $g_1(\nabla_{Z_1}Z_2, U_1) = 0$ and $g_1(\nabla_{Z_1}Z_2, U_2) = 0$, $\forall\ Z_1, Z_2 \in \Gamma(D^\theta \oplus <\xi>), U_1 \in \Gamma(D \oplus D^\perp)$ and $U_2 \in \Gamma(\ker\pi_*)^\perp$. Using (2.2), (2.4), (4.2), (4.3) and Lemma (4.4), we get

$$\begin{aligned} & g_1(\nabla_{Z_1}Z_2, U_1) \\ = & g_1(\nabla_{Z_1}\psi Z_2, \phi U_1) + g_1(\nabla_{Z_1}\omega Z_2, \phi U_1) - \eta(Z_2)g_1(Z_1, \phi PU_1), \\ = & \cos^2\theta g_1(\nabla_{Z_1}Z_2, U_1) - g_1(\nabla_{Z_1}\omega\psi Z_2, U_1) + g_1(\nabla_{Z_1}\omega Z_2, \phi U_1) - \eta(Z_2)g_1(Z_1, \phi PU_1). \end{aligned}$$

Now, using (3.6), we get

$$\begin{aligned} & \sin^2\theta g_1(\nabla_{Z_1}Z_2, U_1) \\ = & -g_1(\nabla_{Z_1}\omega\psi Z_2, U_1) + g_1(\mathcal{H}\nabla_{Z_1}\omega Z_2, \phi RU_1) + g_1(\mathcal{T}_{Z_1}\omega Z_2, \phi PU_1) - \eta(Z_2)g_1(Z_1, \phi PU_1). \end{aligned}$$

From (3.1), (3.9) and Lemma (3.1), we have

$$\begin{aligned} \sin^2\theta g_1(\nabla_{Z_1}Z_2, U_1) = & g_1(\mathcal{T}_{Z_1}\omega\psi Z_2, U_1) + g_1(\mathcal{T}_{Z_1}\omega Z_2, \phi PU_1) - \eta(Z_2)g_1(Z_1, \phi PU_1) - \\ & \frac{1}{\lambda^2}g_2((\nabla\pi_*)(Z_1, \omega Z_2), \pi_*(\phi RU_1)). \end{aligned}$$

On the other hand, using (2.2), (2.4), (3.6), (4.3), (4.4) and Lemma (4.4), we have

$$\begin{aligned} g_1(\nabla_{Z_1}Z_2, U_2) = & g_1(\nabla_{Z_1}\psi Z_2, \phi U_2) + g_1(\nabla_{Z_1}\omega Z_2, \phi U_2) - \eta(Z_2)g_1(Z_1, BU_2), \\ = & \cos^2\theta g_1(\nabla_{Z_1}Z_2, U_2) + g_1(\nabla_{Z_1}\omega\psi Z_2, U_2) + g_1(\mathcal{H}\nabla_{Z_1}\omega Z_2, CU_2) + \\ & g_1(\mathcal{T}_{Z_1}\omega Z_2, BU_2) - \eta(Z_2)g_1(Z_1, BU_2). \end{aligned}$$

Now, we have

$$\begin{aligned} & \sin^2\theta g_1(\nabla_{Z_1}Z_2, U_2) \\ = & g_1(\mathcal{H}_{Z_1}\omega\psi Z_2, U_2) + g_1(\mathcal{H}\nabla_{Z_1}\omega Z_2, CU_2) + g_1(\mathcal{T}_{Z_1}\omega Z_2, BU_2) - \eta(Z_2)g_1(Z_1, BU_2), \\ = & g_1(\mathcal{H}\nabla_{Z_1}\omega\psi Z_2, U_2) + g_1(\mathcal{T}_{Z_1}\omega Z_2, BU_2) + g_1(\mathcal{A}_{\omega Z_2}\psi Z_1, \phi CU_2) + \\ & g_1(\mathcal{H}\nabla_{\omega Z_2}\omega Z_1, \phi CU_2) - \eta(Z_2)g_1(Z_1, BU_2). \end{aligned}$$

Using (3.1), (3.9) and Lemma (3.1) we have

$$\begin{aligned} & \sin^2\theta g_1(\nabla_{Z_1}Z_2, U_2) \\ = & g_1(\mathcal{T}_{Z_1}\omega Z_2, BU_2) + g_1(\mathcal{A}_{\omega Z_2}\psi Z_1, \phi CU_2) + g_1((\omega Z_1, \omega Z_2)g_1(grad \ln\lambda, \phi CU_2) \\ & +\frac{1}{\lambda^2}g_2(\nabla_{\omega Z_2}\pi_*(\omega Z_1), \pi_*(\phi CU_2)) - \frac{1}{\lambda^2}g_2((\nabla\pi_*)(Z_1, \omega\psi Z_2), \pi_*(U_2)) - \\ & -\eta(Z_2)g_1(Z_1, BU_2), \end{aligned}$$

which gives the required outcome. □

Theorem 5.3. *Let π is a cqhs submersion. Then $D^{\perp}$ is totally geodesic if and only if*

$$g_1(\mathcal{T}_{V_1}V_2, \omega\psi QW_1) = \frac{1}{\lambda^2}g_1((\nabla\pi_*)(V_1, \omega RV_2), \pi_*(\omega W_1)),$$

$$g_1(\mathcal{T}_{V_1}\omega RV_2, BW_2) - g_1(\omega RV_1, \omega RV_2)g_1(grad\ln, \phi CW_2) = \frac{1}{\lambda^2}g_2(\nabla_{\omega RV_2}\pi_*(\phi CW_2), \pi_*(\omega RV_1)),$$

$\forall\ V_1, V_2 \in \Gamma(D^{\perp}), W_1 \in \Gamma(D \oplus D^{\theta}),$ *and* $W_2 \in \Gamma(\ker\pi_*)^{\perp}$.

Proof. For $V_1, V_2 \in \Gamma(D^{\perp}), W_1 \in \Gamma(D \oplus D^{\theta})$, and $W_2 \in \Gamma(\ker\pi_*)^{\perp}$. Using (2.3), we have $g_1(\nabla_{V_1}V_2, \xi) = 0$.

Now, again using (2.2), (2.4), (4.2), (4.3) and Lemma (4.4), we have

$$\begin{aligned} g_1(\nabla_{V_1}V_2, W_1) &= g_1(\phi\nabla_{V_1}V_2, \phi PW_1 + \psi QW_1) + g_1(\nabla_{V_1}\phi V_2, \omega QW_1), \\ g_1(\nabla_{V_1}V_2, PW_1 + QW_1) &= g_1(\nabla_{V_1}V_2, PW_1) + \cos^2\theta g_1(\nabla_{V_1}V_2, QW_1) + \\ & \quad g_1(\nabla_{V_1}V_2, \omega\psi QW_1) + g_1(\nabla_{V_1}\phi V_2, \omega QW_1). \end{aligned}$$

From (3.5) and (3.6), we have

$$\sin^2\theta g_1(\nabla_{V_1}V_2, QW_1) = g_1(\mathcal{T}_{V_1}V_2, \omega\psi QW_1) + g_1(\mathcal{H}\nabla_{V_1}\omega RV_2, \omega W_1).$$

From (3.1), (3.9) and Lemma (3.1), we get

$$\sin^2\theta g_1(\nabla_{V_1}V_2, QW_1) = g_1(\mathcal{T}_{V_1}V_2, \omega\psi QW_1) - \frac{1}{\lambda^2}g_1((\nabla\pi_*)(V_1, \omega RV_2), \pi_*(\omega W_1)).$$

On the other hand (2.2), (2.4), (3.6), (4.2), (4.3) and (4.4), imply

$$\begin{aligned} g_1(\nabla_{V_1}V_2, W_2) &= g_1(\nabla_{V_1}\omega RV_2, BW_2) + g_1(\nabla_{V_1}\omega RV_2, CW_2) \\ &= g_1(\mathcal{T}_{V_1}\omega RV_2, BW_2) - g_1(\nabla_{\omega RV_2}\phi CW_1, \omega RV_1). \end{aligned}$$

Using (3.1), (3.9) and Lemma (3.1), we have

$$\begin{aligned} g_1(\nabla_{V_1}V_2, W_2) &= g_1(\mathcal{T}_{V_1}\omega RV_2, BW_2) - \frac{1}{\lambda^2}g_2(\nabla_{\omega RV_2}\pi_*(\phi CW_2), \pi_*(\omega RV_1)) \\ & \quad + \frac{1}{\lambda^2}g_2((\nabla\pi_*)(\omega RV_2, \phi CW_2), \pi_*(\omega RV_1)), \\ &= g_1(\mathcal{T}_{V_1}\omega RV_2, BW_2) - \frac{1}{\lambda^2}g_2(\nabla_{\omega RV_2}\pi_*(\phi CW_2), \pi_*(\omega RV_1)) - \\ & \quad g_1(\omega RV_1, \omega RV_2)g_1(grad\ln, \phi CW_2). \end{aligned}$$

□

Theorem 5.4. *Let π be a cqhs submersion. Then,* $(\ker\pi_*)^{\perp}$ *is not totally geodesic.*

Proof. For $Z_1, Z_2 \in \Gamma(\ker \pi_*)^\perp$, using (2.3), we have $g_1(\nabla_{Z_1} Z_2, \xi) = -g_1(Z_2, \phi Z_1)$, which is $g_1(Z_2, \phi Z_1) \neq 0$, so $(\ker \pi_*)^\perp$ is not totally geodesic. □

Proposition 5.3. *Let π is a cqhs submersion. Then,* $(\ker \pi_*)$ *is not totally geodesic.*

Proof. Suppose we have $X_1 \in (\ker \pi_*)$ and $X_2 \in (\ker \pi_*)^\perp$ by the use of (2.3), we get $g_1(\nabla_{X_1} \xi, X_2) = g_1(\phi X_1, X_2)$, since $g_1(X_2, \phi X_1) \neq 0$, so $(\ker \pi_*)$ is not totally geodesic. □

Theorem 5.5. *Let π is a cqhs submersion. Then* $(\ker \pi_*) - < \xi >$ *is a totally geodesic if and only if*

$$\begin{aligned} & \frac{1}{\lambda^2}\{g_2((\nabla \pi_*)(V_1, \omega V_2), \pi_*(CW_1)) + g_2((\nabla \pi_*)(V_1, \omega\psi Q V_2), \pi_*(W_1))\} \\ = \; & g_1(\mathcal{T}_{V_1} P V_2 + \cos^2\theta \mathcal{T}_{V_1} Q V_2, W_1) + g_1(\mathcal{T}_{V_1} \omega V_2, B W_1), \end{aligned}$$

$\forall\ V_1, V_2 \in \Gamma(\ker \pi_*) - < \xi >$ *and* $W_1 \in \Gamma(\ker \pi_*)^\perp$.

Proof. For $V_1, V_2 \in \Gamma(\ker \pi_*) - < \xi >$ and $W_1 \in \Gamma(\ker \pi_*)^\perp$, using (2.2), (2.4), (4.2), (4.3), (4.4) and Lemma (4.4), we have

$$\begin{aligned} g_1(\nabla_{V_1} V_2, W_1) &= g_1(\nabla_{V_1} P V_2, W_1) + g_1(\nabla_{V_1} Q V_2, W_1) + g_1(\nabla_{V_1} R V_2, W_1), \\ &= g_1(\nabla_{V_1} \phi P V_2, \phi W_1) + g_1(\nabla_{V_1} \phi Q V_2, \phi W_1) + g_1(\nabla_{V_1} \phi R V_2, \phi W_1), \\ &= g_1(\nabla_{V_1} P V_2 + \cos^2\theta \nabla_{V_1} Q V_2, W_1) + g_1(\nabla_{V_1} \omega\psi Q V_2, W_1) + \\ &\quad\; g_1(\nabla_{V_1}(\omega P V_2 + \omega Q V_2 + \omega R V_2), \phi W_1). \end{aligned}$$

Since $\omega P V_2 + \omega Q V_2 + \omega R V_2 = \omega V_2$ and $\omega P V_2 = 0$, we have

$$\begin{aligned} g_1(\nabla_{V_1} V_2, W_1) &= g_1(\nabla_{V_1} P V_2 + \cos^2\theta \nabla_{V_1} Q V_2, W_1) + g_1(\nabla_{V_1} \omega\psi Q V_2, W_1) + \\ &\quad\; g_1(\nabla_{V_1} \omega V_2, \phi W_1), \\ &= g_1(\mathcal{T}_{V_1} P V_2 + \cos^2\theta \mathcal{T}_{V_1} Q V_2, W_1) + g_1(\mathcal{H}\nabla_{V_1} \omega\psi Q V_2, W_1) + \\ &\quad\; g_1(\mathcal{T}_{V_1} \omega V_2, B W_1) + g_1(\mathcal{H}\nabla_{V_1} \omega V_2, C W_1). \end{aligned}$$

Using (3.1), (3.9) and Lemma (3.1) we have

$$\begin{aligned} & g_1(\nabla_{V_1} V_2, W_1) \\ = \; & g_1(\mathcal{T}_{V_1} P V_2 + \cos^2\theta \mathcal{T}_{V_1} Q V_2, W_1) - \frac{1}{\lambda^2} g_2((\nabla \pi_*)(V_1, \omega\psi Q V_2), \pi_*(W_1)) + \\ & g_1(\mathcal{T}_{V_1} \omega V_2, B W_1) - \frac{1}{\lambda^2} g_2((\nabla \pi_*)(V_1, \omega V_2), \pi_*(C W_1)). \end{aligned}$$

□

By the use of Proposition (5.2) and Theorem (5.3) one can get the following theorem.

Theorem 5.6. *Let π be a cqhs submersion. Then, π is not a totally geodesic.*

6. Example

Consider the Euclidean space R^{2k+1} whose coordinates are $\{(x^1, x^2, ..., x^k, y^1, y^2, ...y^k, z) : x^i, y^i, z \in R\}$. Let $\{E_i, E_{k+i}, \xi\}$ be the base field where $E_i = 2\frac{\partial}{\partial y^i}, E_{k+i} = 2(\frac{\partial}{\partial x^i} + y^i\frac{\partial}{\partial z})$. and $\xi = 2\frac{\partial}{\partial z}$ be the contravariant vector field. Assign LP-Sasakian structure on R^{2k+1} as follows:

$$\phi(\sum_{i=1}^{k}(X_i\frac{\partial}{\partial x^i} + Y_i\frac{\partial}{\partial y^i}) + Z\frac{\partial}{\partial z}) = -\sum_{i=1}^{k} Y_i\frac{\partial}{\partial x^i} - \sum_{i=1}^{k} X_i\frac{\partial}{\partial y^i} + \sum_{i=1}^{k} Y_i y^i\frac{\partial}{\partial z},$$

$$\xi = 2\frac{\partial}{\partial z}, \eta = -\frac{1}{2}(dz - \sum_{i=1}^{k} y^i dx^i), g_{R^{2k+1}} = -(\eta\otimes\eta) + \frac{1}{4}(\sum_{i=1}^{k} dx^i\otimes dx^i + \sum_{i=1}^{k} dy^i\otimes dy^i).$$

Then $(R^{2k+1}, \phi, \xi, \eta, g_{R^{2k+1}})$ is the LP-Sasakian manifold [24].

Example 6.1. Let the R^{11} be an Euclidean space whose coordinates are $(x_1, ..., x_5, , y_1....., y_5, z)$ and $\{E_i, E_{5+i}, \xi\}$ be the base field where $E_i = 2\frac{\partial}{\partial y^i}, E_{5+i} = 2(\frac{\partial}{\partial x^i} + y^i\frac{\partial}{\partial z})$, $i = 1, ..., 5$. Let $\xi = 2\frac{\partial}{\partial z}$ be a contravariant vector field. Define an LP-Sasakian structure on R^{11} as follows:

$$\phi(\sum_{i=1}^{5}(X_i\frac{\partial}{\partial x^i} + Y_i\frac{\partial}{\partial y^i}) + Z\frac{\partial}{\partial z}) = -\sum_{i=1}^{5} Y_i\frac{\partial}{\partial x^i} - \sum_{i=1}^{5} X_i\frac{\partial}{\partial y^i} + \sum_{i=1}^{5} Y_i y^i\frac{\partial}{\partial z},$$

$$\xi = 2\frac{\partial}{\partial z}, \eta = -\frac{1}{2}(dz - \sum_{i=1}^{5} y^i dx^i), g = -(\eta\otimes\eta) + \frac{1}{4}(\sum_{i=1}^{5} dx^i\otimes dx^i + dy^i\otimes dy^i).$$

Then $(R^{11}, \phi, \xi, \eta, g)$ is LP-Sasakian manifold. Let g_{R^5} be the Riemannian metric tensor field which is defined by $g_{R^5} = \frac{1}{4e^{2\beta}}\sum_{i=1}^{5}(dv_i\otimes dv_i)$ on R^5, where $\{v_1, v_2, v_3, v_4, v_5\}$ is local coordinate system on R^5.

Let $F : R^{11} \to R^5$ be a map defined by $\pi(x_1, ..., x_5, y_1...., y_5, z) = e^{\beta}(\sin\alpha x_1 + \cos\alpha x_3, x_2, x_4, y_3, y_4)$, which is cqhs such that

$$X_1 = 2\cos\alpha(\frac{\partial}{\partial x_1} + y_1\frac{\partial}{\partial z}) - 2\sin\alpha(\frac{\partial}{\partial x_3} + y_3\frac{\partial}{\partial z}), X_2 = 2(\frac{\partial}{\partial x_5} + y_5\frac{\partial}{\partial z}),$$
$$X_3 = 2\frac{\partial}{\partial y_1}, X_4 = 2\frac{\partial}{\partial y_2}, X_5 = 2\frac{\partial}{\partial y_5}, X_6 = \xi = 2\frac{\partial}{\partial z}.$$

$$(\ker\pi_*) = (D\oplus D^\theta\oplus D^\perp\oplus <\xi>),$$

$$D =< X_2 = 2(\frac{\partial}{\partial x_5} + y_5\frac{\partial}{\partial z}), X_5 = 2\frac{\partial}{\partial y_5} >,$$

$$D^\theta =< X_1 = 2\cos\alpha(\frac{\partial}{\partial x_1} + y_1\frac{\partial}{\partial z}) - 2\sin\alpha(\frac{\partial}{\partial x_3} + y_3\frac{\partial}{\partial z}), X_3 = 2\frac{\partial}{\partial y_1} >,$$

$$D^{\perp} =< X_4 = 2\frac{\partial}{\partial y_2} >, X_6 =< \xi >=< 2\frac{\partial}{\partial z} >,$$

$$\begin{aligned}(\ker \pi_*)^{\perp} &= < V_1 = 2\sin\alpha(\frac{\partial}{\partial x_1} + y_1\frac{\partial}{\partial z}) + \cos\alpha 2(\frac{\partial}{\partial x_3} + y_2\frac{\partial}{\partial z}), V_2 = 2\frac{\partial}{\partial x_2},\\ V_3 &= 2\frac{\partial}{\partial x_4}, V_4 = 2\frac{\partial}{\partial y_3}, V_5 = 2\frac{\partial}{\partial y_4} >,\end{aligned}$$

with quasi hemi-slant angle α. Now,

$$F_*V_1 = 2e^{\beta}\frac{\partial}{\partial v_1}, F_*V_2 = 2e^{\beta}\frac{\partial}{\partial v_2}, F_*V_3 = 2e^{\beta}\frac{\partial}{\partial v_3}, F_*V_4 = 2e^{\beta}\frac{\partial}{\partial v_4}, F_*V_5 = 2e^{\beta}\frac{\partial}{\partial v_5}.$$

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