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FUNDAMENTAL TONE ESTIMATES ON FINSLER MANIFOLDS

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Abstract. We study the fundamental tone of Laplacian operators on Finsler manifold M evolved by a special function $u : \Omega \subset M \to \mathbb{R}$, and provide some geometric estimates of the first eigenvalue of p-laplace and (p,q)-Laplace operators. These estimates are dependent on this function for simply connected manifolds, a class of warped product manifolds, and a class of Finsler submersions. Under a similar setting, we also study these results on a quasi-linear operator $Lu = -\Delta_p u + X|u|^{p-2}u$.
Keywords: p-Laplacian operator, nonlinear eigenvalue problem, first eigenvalue, quasi-linear operator, (p, q)-Laplacian.

1. Introduction

Let $(M, F, d\mu)$ be a compact Finsler n-manifold and denote the completion of $C^\infty(M)$ by $W^{1,p}(M)$. For a function $u \in W^{1,p}(M)$, we define its Finsler p-Laplacian as follows:

$$\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u),$$

where the equality is in the weak $W^{1,p}(M)$ sense. The Finsler p-Laplacian Δ_p is a nonlinear elliptic operator. For a function $u \in C^\infty(M)$, $\Delta_p u$ is not defined in usual sense because here ∇u is quite different from that in the Riemannian case.

Let $V \neq 0$ be a vector field on $M_u := \{u \in C^\infty(M) : du \neq 0\}$, the weighted p-Laplacian on the weighted Riemannian manifold (M, g_V) is defined by

$$\Delta_p^V u = \operatorname{div}(|\nabla^V u|^{p-2}\nabla^V u).$$

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Obviously, $\Delta_p^{\nabla u} u = \Delta_p u$.

The canonical energy functional E on $W^{1,p} \setminus \{0\}$ is defined by

$$E(u) := \frac{\int_M |\nabla u|^p d\mu}{\int_M |u|^p d\mu}.$$

Then for an arbitrary $\Phi \in C^\infty(M)$, we have the following divergence formula:

$$\operatorname{div}(\Phi |\nabla u|^{p-2} \nabla u) = \Phi \Delta_p u + |\nabla u|^{p-2} d\Phi(\nabla u).$$

If u satisfies in the Neumann boundary condition $g_\nu(\nu, \nabla u) = 0$, where ν is the outer normal vector with respect to ∂M, then we obtain

$$\int_M \Phi \Delta_p u d\mu = -\int_M |\nabla u|^{p-2} d\Phi(\nabla u), \qquad \forall \Phi \in C^\infty(M). \tag{1.1}$$

This leads to the definition of $\Delta_p u$ on the whole M in the distribution sense. Furthermore it can be shown that $E(u)$ is C^1 on $W^{1,p}(M) \setminus \{0\}$ and for any $u \in W^{1,p}(M)$ with $\int_M |u|^p d\mu = 1$, we have

$$d_u E(\Phi) = p \int_M (|\nabla u|^{p-2} d\Phi(\nabla u) - \lambda_p |u|^{p-2} u \Phi) d\mu, \quad \forall \Phi \in C^\infty(M),$$

here, $\lambda_p = E(u)$. This together with (1.1) leads to

$$d_u E(\Phi) = -p \int_M (\Delta_p u - \lambda_p |u|^{p-2} u \Phi) d\mu, \quad \forall \Phi \in C^\infty(M).$$

It follows that u satisfies $d_u E = 0$ iff $\Delta_p u = -\lambda_p |u|^{p-2} u$. Here λ_p is called Neumann eigenvalue and u is the corresponding Neumann eigenfunction of M.

Set

$$K = \left\{ u \in W^{1,p}(M) | \int_M |u|^{p-2} u d\mu = 0, \quad \int_M u|^p = 1 \right\}.$$

The first Neumann eigenvalue of the Finsler p-Laplacian is defined by

$$\lambda_{1,p}(M) := \inf \left\{ \int_M |\nabla u|^{p-2} d\mu; u \in K \right\}.$$

Consider the following sets

$$W_0^{1,p}(M) := \left\{ u \in W^{1,p}(M) | |u|_{\partial M} = 0 \right\}, \qquad K_1 := \left\{ u \in W_0^{1,p}(M) | \int_M |u|^p = 1 \right\}.$$

Let $\Omega \subset M$ be a domain with compact closure and nonempty boundary $\partial\Omega$. The first Dirichlet eigenvalue of Finsler p-Laplacian is determined by

$$\lambda_{1,p}(\Omega) = \inf_{u \in W_0^{1,p}(\Omega) \setminus \{0\}} \left\{ \frac{\int_\Omega |\nabla u|^p d\mu}{\int_\Omega |u|^p d\mu} \right\}.$$

1.1. Eigenvalue estimate of some (p, q)-Laplacian

Let Ω be a compact domain in a complete simply connected Finsler manifold $(M, F, d\mu)$ which has a constant flag curvature k. We recall a class of (p, q)-Laplacian for $\forall u \in W = W_0^{1,p}(\Omega) \cap W_0^{1,q}(\Omega)$ introduced in [1] for Riemannian manifolds and defined for Finsler manifold in [11], as follows:

$$\Delta_p u + \Delta_q u = \operatorname{div}((|\nabla u|^{p-2} + |\nabla u|^{q-2})\nabla u), \tag{1.2}$$

where $1 < q < p < \infty$. λ is an eigenvalue of (1.2) if there is a nontrivial solution $u \in W$ for the following inequality:

$$-\Delta_p u - \Delta_q u = \lambda |u|^{p-2} u, \tag{1.3}$$

or equivalently for any $v \in W^{1,p}(\Omega) \cap W^{1,q}(\Omega)$, we have

$$\int_\Omega |\nabla u|^{p-2} \nabla u . \nabla v dv + \int_\Omega |\nabla u|^{q-2} \nabla u . \nabla v dv = \lambda \int_\Omega |u|^{p-2} u v dv. \tag{1.4}$$

Therefore the first positive eigenvalue $\lambda_{1,p,q}(\Omega)$ of (1.2) is defined as

$$\lambda_{1,p,q}(\Omega) = \inf \left\{ \int_\Omega |\nabla u|^p dv + \int_\Omega |\nabla u|^q : \int_\Omega |u|^p dv = 1 \right\}. \tag{1.5}$$

It has been long established that the eigenvalues and eigenfunctions of the Laplacian play an important role in global differential geometry since they reveal important relations between geometry of the manifold and analysis. In Riemannian manifolds most of wellknown Laplace operators such as biharmonic, p-Laplace, (p, q)-Laplace operator and so on had been studied [1, 4, 6, 7, 10, 12, 13, 23].

Generalization mathematician in last two decades have been tried to extend the eigenvalue estimation results on Riemannian manifolds to the Finsler manifolds. For instance, Yin and He in [21] obtained the Lichnerowicz-obata type estimate which depends on considering the lower bound for Ricci curvature. Lately in [22], authors generalized Cheng type, Cheeger type, Faber-Krahn type, and Mckean type inequalities for reversible Finsler manifolds which was proven in the Riemannian case [3, 5, 19].
Motivated by recent research on Riemannian manifolds in [9], we are interested to compare the results on Finsler manifolds with the same condition on Riemannian case. Here is our first result:

Theorem 1.1. *Let (M, F) be a Finsler manifold and $\Omega \subset M$ be a bounded domain. In addition assume that there is a smooth function $f : \Omega \to \mathbb{R}$ such that $F^*(df) = |\nabla f| \leq l_1$ and $\Delta_p f \geq l_2$ for some positive constant l_1, l_2. Then, the first eigenvalue of the p-Laplace operator satisfies:*

$$\lambda_{1,p}(\Omega) \geq \frac{l_2^p}{p^p l_1^{p(p-1)}},$$

and more over if we have $\Delta_q f \geq l_3$*, then the first eigenvalue of (1.2) satisfies*

$$\lambda_{1,p,q}(\Omega) \geq \left(\frac{l_2}{pl_1^{p-1}}\right)^p + \left(\frac{l_3}{ql_1^{q-1}}\right)^q.$$

As an immediate conclusion, we can state following Mckean's type theorem.

Theorem 1.2. *Let* (M^n, F) *be a forward complete Finsler manifold with flag curvature* $K \leq -\kappa^2$*. Then for any bounded domain* $\Omega \subset M$*, we have*

$$\lambda_{1,p}(\Omega) \geq \left(\frac{(n-1)\kappa}{p}\coth r - \|S\|\right)^p,$$

where $\|S\|$ *is the pointwise norm function of S-curvature, and* $R > 0$ *is the radius of such geodesic ball that contains* Ω*.*

2. Preliminaries

Let M be an n-dimensional smooth manifold and $\pi : TM \to M$ be the natural projection from the tangent bundle TM. Let (x, y) be a point of TM with $x \in M$, $y \in T_xM$, and let (x^i, y^i) be the local coordinate on TM with $y = y^i\dfrac{\partial}{\partial x^i}$. A Finsler metric on M is a function $F : TM \longrightarrow [0, \infty)$ satisfying the following properties:
(i) Regularity: F is C^∞ in $TM \setminus \{0\}$;
(ii) Positive homogeneity: $F(x, \lambda y) = \lambda F(x, y)$ for all $\lambda > 0$;
(iii) Strong convexity: the fundamental quadratic form

$$(g_{ij}) := \left([\frac{1}{2}F^2]_{y^iy^j}\right),$$

is positively definite at every point of $TM \setminus \{0\}$.

Let $X = X^i\dfrac{\partial}{\partial x^i}$ be a vector field. Then the covariant derivative of X by $v \in T_xM$ with reference vector $w \in T_xM \setminus \{0\}$ is

$$D_v^w X(x) = \left\{v^j \frac{\partial X^i}{\partial x^j}(x) + \Gamma^i_{jk}(w)v^jX^k(x)\right\}\frac{\partial}{\partial x^i},$$

where Γ^i_{jk} denote the coefficients of the Chern connection.

Given two linearly independent vectors $V, W \in T_xM\setminus\{0\}$, flag curvature $K(V, W)$ is defined as follows:

$$K(V, W) := \frac{g_V(R^V(V, W)W, V)}{g_V(V, V)g_V(W, W) - g_V(V, W)^2},$$

where R^V is the Chern curvature:

$$\mathbf{R}^V(X, Y)Z = \nabla^V_X\nabla^V_Y Z - \nabla^V_Y\nabla^V_X Z - \nabla^V_{[X,Y]}Z.$$

Then the Ricci curvature of V for (M,F) is:

$$Ric(V)=\sum_{i=1}^{n-1}K(V,e_i),$$

here $e_1,\cdots,e_{n-1},\dfrac{V}{F(V)}$ form an orthonormal basis of T_xM with respect to g_V, namely, one has $\mathrm{Ric}(\lambda V)=\mathrm{Ric}(V)$ for any $\lambda>0$.

For a given volume form $d\mu=\sigma(x)dx$ and a vector $y\in T_xM\setminus\{0\}$, the distortion of $(M,F,d\mu)$ is defined by

$$\tau(V):=\ln\frac{\sqrt{\det(g_{ij}(y))}}{\sigma}.$$

Considering the rate of changes of the distortion along geodesics, leads to the so-called S-curvature as follows

$$S(V):=\frac{d}{dt}[\tau(\gamma(t),\dot\gamma(t)]_{t=0},$$

where $\gamma(t)$ is the geodesic with $\gamma(0)=x$ and $\dot\gamma(0)=V$.

Now we can introduce the weighted Ricci curvature on the Finsler manifolds, which was defined by Ohta in [14].

Definition 2.1. ([14]) Let $(M,F,d\mu)$ be a Finsler n-manifold with volume form $d\mu$. Given a vector $V\in T_xM$, let $\gamma:(-\varepsilon,\varepsilon)\to M$ be a geodesic with $\gamma(0)=x$, $\dot\gamma(0)=V$, and

$$\dot S(V):=F^{-2}(V)\frac{d}{dt}[S(\gamma(t),\dot\gamma(t)]_{t=0}.$$

The weighted Ricci curvatures of M is defined as follows

$$\begin{aligned}
Ric_n(V) &:= \begin{cases} Ric(V)+\dot S(V), & for\ S(V)=0,\\ -\infty, & otherwise,\end{cases}\\
Ric_N(V) &:= Ric(V)+\dot S(V)-\frac{S(V)^2}{(N-n)F(V)^2},\quad \forall N\in(n,\infty),\\
Ric_\infty(V) &:= Ric(V)+\dot S(V).
\end{aligned}$$

For a smooth function $u:M\longrightarrow\mathbb{R}$ and any point $x\in M$, the gradient vector of u at x is defined by

$$\nabla u(x)=\begin{cases} g^{ij}(x,\nabla u)\dfrac{\partial u}{\partial x_j}\dfrac{\partial}{\partial x_i}, & du(x)\neq 0,\\ 0, & du(x)=0.\end{cases}$$

So the gradient vector field of a differentiable function f on M by the Legendre transformation $\mathcal{L}:T_xM\to T_x^*M$ is defined as

$$\nabla u:=\mathcal{L}^{-1}(du).$$

Let $\mathfrak{M} = \{x \in M : \nabla u|_x \neq 0\}$. We define the Hessian $H(u)$ of u on $\mathfrak{M}$ as follows:

$$H(u)(X,Y) := XY(u) - \nabla_X^{\nabla u} Y(u), \quad \forall X, Y \in \Gamma(TM|_{\mathfrak{M}}).$$

Fix a volume form $d\mu$, the divergence $\mathrm{div}(X)$ of X is defined as:

$$d(X\rfloor d\mu) = \mathrm{div}(X)d\mu.$$

For a given smooth function $u : M \longrightarrow \mathbb{R}$, the Laplacian Δu of u is defined by $\Delta u = \mathrm{div}(\nabla u) = \mathrm{div}(\mathcal{L}^{-1}(du))$. The Finsler p-Laplacian of a smooth function $u : M \to \mathbb{R}$ can be defined by

$$\Delta_p u := \mathrm{div}(|\nabla u|^{p-2}\nabla u).$$

Since the gradient operator ∇ is not a linear operator in general, the Finsler p-Laplacian is greatly different from the Riemannian p-Laplacian.

Given a vector field V such that $V \neq 0$ on $M_u = \{x \in M; du(x) \neq 0\}$ the weighted gradient vector and the weighted p-Laplacian on the weighted Reimannian manifold (M, g_V) are defined by

$$\triangledown^V u := \begin{cases} g^{ij}(V)\dfrac{\partial u}{\partial x^j}\dfrac{\partial}{\partial x^i}, & on\ M_u, \\ 0, & on\ M \setminus M_u, \end{cases} \qquad \Delta_p^V u := div(|\nabla^V u|^{p-2}\nabla^V u).$$

Here we note that $\nabla^V u = \nabla u$, $\Delta_p^V u = \Delta_p u$.

Definition 2.2. [16] Let $(M, F, d\mu)$ be a Finsler manifold and u be a C^2 function on M. The Shen-Laplacian Δu of u is defined as follows

$$\begin{aligned} \Delta u &= \frac{1}{\sigma_\mu(x)}\partial_k[\sigma_\mu(x)g^{kl}(x,\nabla u(x))\partial_l u] \\ &= [g^{kl}(x,\nabla u(x))\partial_k(\log(\sigma_\mu(x)) + \partial_k(g^{kl}(x,\nabla u(x)))]\partial_l u + g^{kl}(x,\nabla u(x))\partial_l\partial_k u. \end{aligned}$$

Definition 2.3. [16] Let H_0^1 denotes the space of Hilbert functions with $\int_M u d\mu = 0$. for $u \in H_0^1$, the weak Laplacian of u is defined by

$$\int_M \Phi\Delta u d\mu = -\int_M d\Phi(\nabla u)d\mu, \qquad forall \Phi \in C_0^\infty(M).$$

3. Main results

We consider a bounded domain Ω in a n-dimensional Finsler manifold M^n, $n \geq 2$. Under some boundary assumption for $f : \Omega \to \mathbb{R}$ that stated in Theorem 1.1, we will obtain a positive lower bound for $\lambda_{1,p}$ on bounded domain Ω as follows:

Proof. [Proof of Theorem 1.1] Given $v \in C_0^\infty(\Omega)$, we have

$$\begin{aligned} l_2 \int_\Omega |v|^p d\mu &\leq \int_\Omega |v|^p \Delta_p f d\mu \\ &= -\int_\Omega < \nabla |v|^p, \|\nabla f\|^{p-2} \nabla f > d\mu \\ &= -p \int_\Omega |v|^{p-1} < \nabla |v|, \|\nabla f\|^{p-2} \nabla f > d\mu \\ &\leq p \int_\Omega |v|^{p-1} \|\nabla v\| \|\nabla f\|^{p-1} d\mu \\ &\leq p \int_\Omega |v|^{p-1} l_1^{p-1} \|\nabla v\| d\mu. \end{aligned}$$

Taking a positive constant $c > 0$ and using Young inequality, we get

$$\begin{aligned} |v|^{p-1} l_1^{p-1} \|\nabla v\| &\leq \frac{c^q |v|^{q(p-1)}}{q} + \frac{l_1^{p(p-1)} \|\nabla v\|^p}{pc^p} \\ &= \frac{(p-1) c^{p/(p-1)} |v|^p}{p} + \frac{l_1^{p(p-1)} \|\nabla v\|^p}{pc^p}. \end{aligned}$$

So

$$p \int_\Omega |v|^{p-1} l_1^{p-1} \|\nabla v\| d\mu \leq (p-1) c^{p/(p-1)} \int_\Omega |v|^p d\mu + \frac{l_1^{p(p-1)}}{c^p} \int_\Omega \|\nabla v\|^p d\mu$$

Choosing c so that $b - (p-1)c^{p/(p-1)} = \frac{b}{p}$, that is $c^p = \frac{b^{p-1}}{p^{p-1}}$. Then we infer

$$\frac{b}{p} \int_\Omega |v|^p d\mu \leq \frac{p^{p-1} l_1^{p(p-1)}}{l_2^{p-1}} \int_\Omega \|\nabla v\|^p d\mu.$$

Hence

$$\int_\Omega \|\nabla v\|^p d\mu \geq \frac{l_2^{p-1}}{p^{p-1} l_1^{p(p-1)}} \int_\Omega |v|^p d\mu.$$

Dividing both sides of the last inequality to $\int_\Omega |v|^p d\mu$, completes the proof. □

Let $(M, F, d\mu)$ be a Finsler n-dimensional manifold and $r = d_F(x, .)$ the distance function from a fixed point x. In addition, suppose that γ is a unit-speed geodesic without a conjugate point up to distance r from x and the flag curvature of M has the upper bound c. It is known that ∇r is a geodesic field such that $F(\nabla r) = 1$. Define the following function:

$$f_c(r) = \begin{cases} \frac{1}{\sqrt{-c}} sinh(\sqrt{-c}r), & if\ c < 0, \\ r, & if\ c = 0, \\ \frac{1}{\sqrt{c}} sin(\sqrt{c}r), & if\ c > 0. \end{cases} \tag{3.1}$$

So $\Delta_p r = \Delta r \geq (n-1)ct_c(r) - \|S\|$ (see Laplacian comparison in [20]), where

$$ct_c(r) = \begin{cases} \sqrt{-c}coth(\sqrt{-c}r), & if \ \ c < 0, \\ \dfrac{1}{r}, & if \ \ c = 0, \\ \sqrt{c}cot(\sqrt{c}r), & if \ \ c > 0. \end{cases}$$

Consequently, as a first application of Theorem 1.1, for the especial distance function on a geodesic ball B_r, we obtain Theorem 1.2. In particular, for Finsler manifold with Berwald metric, since S-curvature vanishes i.e. $S = 0$ (see proposition 4.3 in [17]), we get

$$\lambda_{1,p}(\Omega) \geq \left(\frac{(n-1)k}{p} \coth r\right)^p.$$

For the next result we need to recall the definition of the Busemann function in Finsler geometry from [18]:

Definition 3.1. Let (M, F) be forward complete Finsler manifold. A geodesic ray $\gamma : [0, \infty] \to M$ is called a forward ray if γ is a globally minimizing unit speed Finslerian geodesic, that means

$$d_F(\gamma(t_1), \gamma(t_2)) = t_2 - t_1, \ \ \forall t_1 < t_2, and \ F(\dot{\gamma}) = 1.$$

For a forward complete noncompact Finsler manifold M, there is always a forward ray γ with beginning point $\gamma(0) = p$. Using the Finsler distance d, we define the following function

$$b_{\gamma,t}(x) = d(x, \gamma(t)) - t, \ \ x \in M.$$

It is proven in [15] that this function is monotonically decreasing and bounded from below. So, as an immediate conclusion, the limit of $b_{\gamma,t}(x)$ exists and it is called the Busemann function to the ray γ, and denotes by

$$b_\gamma(x) := \lim_{t\to\infty} b_{\gamma,t}(x) = \lim_{t\to\infty} (d(x, \gamma(t)) - t).$$

It has been shown (see [15, 8]) that Bosemann functions are distance functions, that is $F(\nabla b_\gamma) = 1$.

Now, we recall the definition of AHF-manifolds from [15]:

Definition 3.2. A forward complete simply connected Finsler manifold $(M, F, d\mu)$ without conjugate points is an asymptotically harmonic Finsler manifold (AHF-manifold for short), if the Finsler mean curvature of horosphere is a real constant h.

Due to the definition of Busemann function b_γ, the AHF-manifold is defined in the weak sense:

Definition 3.3. Let $(M, F, d\mu)$ be a forward complete simply connected Finsler manifold without conjugate points. Then M is called an AHF-manifold in the weak sense if the weak Laplacian of Busemann function is a real constant, $\Delta b_\gamma = h$, where Δ is Shen-Laplacian.

Considerung f as a Busemann function in Theorem 1.1, it is easy to see:

Corollary 3.1. *Let Ω be a bounded domain in AHF-manifold M^n in the weak sense. Then the first eigenvalue of the p-Laplacian satisfies*

$$\lambda_{1,p}(\Omega) \geq (\frac{h}{p})^p.$$

Note that here $\Delta_p b_\gamma = \Delta b_\gamma = h$.

3.1. Quasilinear operator

Let $(M, F, d\mu)$ be a compact Finsler manifold. We can define the following quasilinear operator which was introduced in [2] on Riemannian manifolds.

Definition 3.4. Let $u \in W^{1,p}(M)$, and X is a non-negative smooth function on compact Finsler manifold M. We define

$$Lu = -\Delta_p u + X|u|^{p-2}u, \tag{3.2}$$

where $\Delta_p u$ is the p-Laplacian operator.

Corresponding to the operator L, and the definition of Dirichlet and Neumann eigenvalues for p-Laplacian on Finsler manifolds, we have:
i) Dirichlet eigenvalue of (3.2):

$$\mu_{1,p}(M) = \inf_{0 \neq u \in W_0^{1,p}(M)} \frac{\int_M (|\nabla u|^p + X|u|^p) d\mu}{\int_M |u|^p d\mu},$$

and
ii) Neumann eigenvalue of (3.2):

$$\Lambda_{1,p}(M) = \inf_{0 \neq u \in W^{1,p}(M)} \left\{ \frac{\int_M (|\nabla u|^p + X|u|^p) d\mu}{\int_M |u|^p d\mu} | \int_M |u|^{p-2} u d\mu = 0 \right\}.$$

Here u is called the eigenfunction of L corresponding to μ (or Λ) on M. Now, we can state the following theorem for the quasilinear equation (3.2):

Theorem 3.1. *Let Ω be a bounded domain on a Finsler manifold M, and assume that there is a smooth function $f : \Omega \to \mathbb{R}$ such that satisfies $\|\nabla f\| \leq a$ and $\Delta_p f \geq b$ for some positive constants a, b, where $a > b$. Then the first Dirichlet eigenvalue of the quasilinear operator L satisfies*

$$\mu_{1,p}(\Omega) \geq \frac{b^p}{p^p a^{p(p-1)}}.$$

Proof. We first note that by density we can use smooth functions in the variational characterization of $\mu_{1,p}(\Omega)$. So given $u \in C_0^\infty(\Omega)$, based on the fact that X is positive function, we have

$$\begin{aligned} b\int_\Omega |u|^p dv &\leq \int_\Omega |u|^p(\Delta_p f + X)dv \\ &= -\int_\Omega < \nabla|u|^p, \|\nabla f\|^{p-2}\nabla f > dv + \int_\Omega |u|^p X dv \\ &= -p\int_\Omega |u|^{p-1} < \nabla|u|, \|\nabla f\|^{p-2}\nabla f > dv + \int_\Omega |u|^p X dv \\ &\leq p\int_\Omega |u|^{p-1}\|\nabla u\|\|\nabla f\|^{p-1}dv + \int_\Omega |u|^p X dv \\ &\leq p\int_\Omega |u|^{p-1}a^{p-1}\|\nabla u\|dv + \int_\Omega |u|^p X dv. \end{aligned} \tag{3.3}$$

Now considering a constant $c > 0$ and using Young inequality, we obtain

$$\begin{aligned} |u|^{p-1}a^{p-1}\|\nabla u\| &\leq \frac{c^q|u|^{q(p-1)}}{q} + \frac{a^{p(p-1)}\|\nabla u\|^p}{pc^p} \\ &= \frac{(p-1)c^{p/(p-1)}|u|^p}{p} + \frac{a^{p(p-1)}\|\nabla u\|^p}{pc^p}. \end{aligned}$$

Therefore

$$\begin{aligned} p\int_\Omega |u|^{p-1}a^{p-1}\|\nabla u\|dv + \int_\Omega |u|^p X dv &\leq (p-1)c^{p/(p-1)}\int_\Omega |u|^p dv + \frac{a^{p(p-1)}}{c^p}\int_\Omega \|\nabla u\|^p dv \\ &\quad + \int_\Omega |u|^p X dv. \end{aligned} \tag{3.4}$$

We could choose c so that $b - (p-1)c^{p/(p-1)} = \frac{b}{p}$, that is $c^p = \frac{b^{p-1}}{p^{p-1}}$. Hence with the statement in theorem $a > b$, we know

$$\frac{p^{p-1}a^{p(p-1)}}{b^{p-1}} > 1,$$

so, (3.3) and (3.4) lead to

$$\frac{b}{p}\int_\Omega |u|^p dv \leq \frac{p^{p-1}a^{p(p-1)}}{b^{p-1}}\Big(\int_\Omega \|\nabla u\|^p dv + \int_\Omega |u|^p X dv\Big).$$

Dividing both sides of the last inequality to $\int_\Omega |u|^p$, completes the proof. $\square$

Corollary 3.2. *For a bounded domain Ω of a forward complete Finsler manifold M with flag curvature $K \leq -\kappa^2$, we conclude*

$$\mu_{1,p}(\Omega) \geq \big(\frac{(n-1)\kappa}{p}\coth r - \|S\|\big)^p,$$

and

$$\Lambda_{1,p}(\Omega) \geq \big(\frac{(n-1)\kappa}{p}\coth r - \|S\|\big)^p.$$

Competing interests

The authors declare that they have no competing interests.

Consent for publication

The author declares that there is no conflict of interests regarding the publication of this paper.

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